Int. J. Nonlinear Anal. Appl.
Volume 12, Special Issue, Winter and Spring 2021, 1905-1911
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5942

Fuzzy Aboodh transform for higher-order derivatives

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(Communicated by Madjid Eshaghi Gordji)

Abstract

The strongly generalized differentiability notion is used to study the fuzzy Aboodh transform formula on the fuzzy n^{th}-order differential in this paper. It is also employed in an analytic technique for fuzzy fifth-order differential equations, and the related theorems and properties are demonstrated in detail. Solving a few instances demonstrates the process.

Keywords: Fuzzy fifth-order differential equation, Fuzzy n^{th}-order differential equation, Fuzzy number, Fuzzy Aboodh transform, Strongly generalized differentiable

1. Introduction

In recent years, the field of fuzzy differential equations (FDEs) has exploded in popularity. Chang and Zadeh [10] were the first for introducing the fuzzy derivative concepts, which was followed by Dubios and Prade [11], who applied this extension principle but in their method. Puri and Ralescu [20], Goetschel and Voxman [13] have addressed several ways. The concept of FDEs was used for the analysis of fuzzy dynamical issues by Kandel [15] with Kandel and Byatt [16]. Kaleva [14], Seikkala [21], Ouyang and Wu [19], Kloeden [17], and Menda [18], as well as other researchers, thoroughly investigated the FDE while the starting of value problem concept (Cauchy problem), see Bede et al. 2006 [8]. Abbasbandy and Allahviranloo [1], 2004 [2]), Allahviranloo [5], and Ghanbari [12] presented numerical methods for solving fuzzy differential equations. Bede and Gal [9] developed the term strongly generalized differentiable. Salahshour [22] investigated the existence and the uniqueness theorem of solutions to n^{th}-order fuzzy differential equations under n^{th}-order generalized differentiability. The H-derivative is defined for a smaller class of fuzzy valued functions than the

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Received: October 2021 *Accepted:* December 2021

strongly generalized derivative, thus fuzzy differential equations can have solutions with a diminishing length of support. As a result, we apply the concept of differentiability in this study. In Allahviranloo and Barkhordari [4], Laplace transform method on fuzzy n^{th}-order derivative solved fuzzy 2^{th}-order differential equations (FTDEs), equivalent fuzzy n^{th}-order, boundary value issues and partial differential equations as well.

2. Basic concepts

This section introduces several terminology keys and basic ideas.

Definition 2.1. *[24] The mapping $\mathcal{H} : \mathcal{R} \to [0,1]$ is fuzzy number if satisfies*

i. *$\mathcal{H}$ is upper semi-continuous.*

ii. *$\mathcal{H}$ is fuzzy convex, i.e., $\mathcal{H}(\varsigma\mathfrak{t} + (1-\varsigma)\mathfrak{k}) \geq min\{\mathcal{H}(\mathfrak{t}), \mathcal{H}(\mathfrak{k})\}, forall\ \mathfrak{t}, \mathfrak{k} \in \mathcal{R}$ and $\varsigma \in [0,1]$*

iii. *$\mathcal{H}$ is normal i.e., $\exists x_0 \in \mathcal{R}$ for which $\mathcal{H}(x) = 1$.*

iv. *$Supp\ \mathcal{H} = \{x \in \mathcal{R}; \mathcal{H}(x) > 0\}$, and $cl(Supp(\mathcal{H}))$ is compact.*

Definition 2.2. *Let η and ζ are fuzzy numbers so the distance between fuzzy numbers is determined by the Hausdorff, $\Gamma : \mathcal{R}_f \times \mathcal{R}_f \to [0, +\infty]$, where $\mathcal{R}_f$ be all the fuzzy numbers set on $\mathcal{R}$:*
$\Gamma(\eta, \zeta) = \sup_{\varsigma\in[0,1]} \max\left\{|\underline{\eta}(\varsigma) - \underline{\zeta}(\varsigma)|, |\overline{\eta}(\varsigma) - \overline{\zeta}(\varsigma)|\right\}$, *where $\eta = (\underline{\eta}(\varsigma) - \overline{\eta}(\varsigma)), \zeta = (\underline{\zeta}(\varsigma), \overline{\zeta}(\varsigma))$ and $(\mathcal{R}_f, \Gamma)$ is a complete metric space and the following characteristics are well known:*

- $\Gamma(\eta \oplus \vartheta, \zeta \oplus \vartheta) = \Gamma(\eta, \zeta), \forall \eta, \zeta, \vartheta \in \mathcal{R}_f$.
- $\Gamma(\varsigma \odot \eta, \kappa \odot \zeta) = |\varsigma|\Gamma(\eta, \zeta), \forall \eta, \zeta \in \mathcal{R}_f, \varsigma \in \mathcal{R}$.
- $\Gamma(\eta \oplus \vartheta, \zeta \oplus \nu) \leq \Gamma(\eta, \zeta) + \Gamma(\vartheta, \nu), \forall \eta, \zeta, \vartheta, \nu \in \mathcal{R}_f$.

Definition 2.3. *[8] Assume that $\psi, \phi \in \mathcal{R}_f$.Where there is $\gamma \in \mathcal{R}_f$ such that$\psi = \phi + \gamma$ then ψ is known the H-differential of ψ and ϕ and it is represented by $\psi \ominus \phi$.*

Note that in this work, the sign $\ominus$ always meant the $\mathcal{H}$-difference as well as $\psi \ominus \phi \neq \psi + (-1)\phi$.

Definition 2.4. *[22] Let $\mathcal{H}(x)$ be a fuzzy valued function on $[e, r]$. Suppose that $\underline{\mathcal{H}}(x, \varsigma)$ and $\overline{\mathcal{H}}(x, \varsigma)$ are improper Riemman-integrable on $[e, r]$, then $\mathcal{H}(x)$ is an improper on $[e, r]$,and*
$\overline{(\int_e^r \mathcal{H}(y,\varsigma)dy)} = (\int_e^r \underline{\mathcal{H}(y,\varsigma)}dy)$, $\overline{(\int_e^r \mathcal{H}(y,\varsigma)dy)} = (\int_e^r \overline{\mathcal{H}(y,\varsigma)}dy)$

3. Generalization of fuzzy aboodh transform

Theorem 3.1. *[25] Let $\mathcal{H}(x)$ be a fuzzy valued function on $[e, \infty)$ embodied by $\underline{\mathcal{H}}(x, \varsigma)\overline{\mathcal{H}}(x, \varsigma)$. For any fixed $\varsigma \in [0, 1]$, let $\underline{\mathcal{H}}(x, \varsigma)\overline{\mathcal{H}}(x, \varsigma)$ are Riemann-integrals on $[e, r]$. For every $r \geq e$, if two positive functions exist $\underline{\theta}(\varsigma)$ and $\overline{\theta}(\varsigma)$ such that $\int_0^r |\underline{\mathcal{H}}(x, \varsigma)|dx \leq \underline{\theta}(\varsigma)$ and $\int_0^r |\overline{\mathcal{H}}(x, \varsigma)|dx \leq \overline{\theta}(\varsigma)$, for every $r \geq e$, then $\mathcal{H}(x)$ is said to be improper fuzzy Riemann-Liouville integrals function on $[e, \infty)$, i.e.*
$\int_0^\infty \mathcal{H}(x)dx = [\int_0^\infty \underline{\mathcal{H}}(x, \varsigma), \int_0^\infty \overline{\mathcal{H}}(x, \varsigma)dx]$

Definition 3.2. *[23] A function $\mathcal{H} : (e, r) \to \mathcal{R}_F$ and $x_0 \in (e, r)$. We say that a mapping $\mathcal{H}$ is strongly generalized differentiable of the n^{th} order at x_0. If $\mathcal{H}, \mathcal{H}', \mathcal{H}^{(2)}, \ldots, \mathcal{H}^{(s-1)}$ have been strongly generalized differentiable and there exists an element $\mathcal{H}^{(s)}(x_0) \in \mathcal{R}_F$, $\forall s = 1, 2, \ldots, n$.*

i. $\forall \tau > 0$ *sufficiently small, there exist* $\mathcal{H}^{(s-1)}(x_0+\tau) \ominus \mathcal{H}^{(s-1)}(x_0), \mathcal{H}^{(s-1)}(x_0) \ominus \mathcal{H}^{(s-1)}(x_0-\tau)$ *where* $\lim_{\tau\to 0} \frac{\mathcal{H}^{(s-1)}(x_0+\tau) \ominus \mathcal{H}^{(s-1)}(x_0)}{\tau} = \lim_{\tau\to 0} \frac{\mathcal{H}^{(s-1)}(x_0) \ominus \mathcal{H}^{(s-1)}(x_0-\tau)}{\tau} = \mathcal{H}^{(s)}(x_0)$ *or*

ii. $\forall \tau > 0$ *sufficiently small, there exist* $\mathcal{H}^{(s-1)}(x_0) \ominus \mathcal{H}^{(s-1)}(x_0+\tau), \mathcal{H}^{(s-1)}(x_0-\tau) \ominus \mathcal{H}^{(s-1)}(x_0)$ *where* $\lim_{\tau\to 0} \frac{\mathcal{H}^{(s-1)}(x_0) \ominus \mathcal{H}^{(s-1)}(x_0+\tau)}{-\tau} = \lim_{\tau\to 0} \frac{\mathcal{H}^{(s-1)}(x_0-\tau) \ominus \mathcal{H}^{(s-1)}(x_0)}{-\tau} = \mathcal{H}^{(s)}(x_0)$ *or*

iii. $\forall \tau > 0$ *sufficiently small, there exist* $\mathcal{H}^{(s-1)}(x_0+\tau) \ominus \mathcal{H}^{(s-1)}(x_0), \mathcal{H}^{(s-1)}(x_0-\tau) \ominus \mathcal{H}^{(s-1)}(x_0)$ *where* $\lim_{\tau\to 0} \frac{\mathcal{H}^{(s-1)}(x_0+\tau) \ominus \mathcal{H}^{(s-1)}(x_0)}{\tau} = \lim_{\tau\to 0} \frac{\mathcal{H}^{(s-1)}(x_0-\tau) \ominus \mathcal{H}^{(s-1)}(x_0)}{-\tau} = \mathcal{H}^{(s)}(x_0)$ *or*

iv. $\forall \tau > 0$ *sufficiently small, there exist* $\mathcal{H}^{(s-1)}(x_0) \ominus \mathcal{H}^{(s-1)}(x_0+\tau), \mathcal{H}^{(s-1)}(x_0) \ominus \mathcal{H}^{(s-1)}(x_0-\tau)$ *where* $\lim_{\tau\to 0} \frac{\mathcal{H}^{(s-1)}(x_0) \ominus \mathcal{H}^{(s-1)}(x_0+\tau)}{-\tau} = \lim_{\tau\to 0} \frac{\mathcal{H}^{(s-1)}(x_0) \ominus \mathcal{H}^{(s-1)}(x_0-\tau)}{\tau} = \mathcal{H}^{(s)}(x_0)$ *or*

Theorem 3.3. *[7] Let* $\mathcal{H}(x), \mathcal{H}^{'}(x), \mathcal{H}^{(2)}(x), \mathcal{H}^{(3)}(x), \dots, \mathcal{H}^{(n-1)}(x)$ *are differentiable fuzzy-valued functions. Moreover, we denote* ς*-cut representation of fuzzy-valued function* $\mathcal{H}(x)$ *such that:*
$\mathcal{H}(x) = [\underline{\mathcal{H}}(x,\varsigma), \overline{\mathcal{H}}(x,\varsigma)]$ *for each* $\varsigma \in [0,1]$*. Then*

$$\mathcal{H}^{(n)}(x) = \begin{cases} \left[\underline{\mathcal{H}}^{(n)}(x,\varsigma), \overline{\mathcal{H}}^{(n)}(x,\varsigma)\right] & if\ number\ of\ (ii)-differentiable\ is\ even, \\ \left[\overline{\mathcal{H}}^{(n)}(x,\varsigma), \underline{\mathcal{H}}^{(n)}(x,\varsigma)\right] & if\ number\ of\ (ii)-differentiable\ is\ odd. \end{cases}$$

Theorem 3.4. *[6] Let* $\mathcal{H}(x)$ *is the primitive of* $\mathcal{H}^{'}(x)$ *on* $[0,\infty)$ *and* $\mathcal{H}(x)$ *be an integrable fuzzy-valued function. Then:*

a. $\mathcal{H}(x)$ *is (i)-differentiable and* $\widehat{A}[\mathcal{H}^{'}(x)] = s\widehat{A}[\mathcal{H}(x)] \ominus \frac{1}{s}\mathcal{H}(0)$.

b. $\mathcal{H}(x)$ *is (ii)-differentiable and* $\widehat{A}[\mathcal{H}^{'}(x)] = (-\frac{1}{s}\mathcal{H}(0)) \ominus (-s\widehat{A}[\mathcal{H}(x)])$.

Theorem 3.5. *[3] Let* $\mathcal{H}(x)e^{-sx}, \mathcal{H}^{'}(x)e^{-sx}$ *and* $\mathcal{H}^{'}(2)e^{-sx}$ *are continuous and integrable Riemann functions on* $[0, infty)$ *so* $\mathcal{H}(x)$ *is continuous fuzzy valued function. Thus:*

a. If $\mathcal{H}(x)$ *and* $\mathcal{H}^{'}(x)$ *are (i)-differentiable, then* $\widehat{A}[\mathcal{H}^{(2)}(x)] = \{s^2\widehat{A}[\mathcal{H}(x)] \ominus \mathcal{H}(0)\} \ominus \frac{1}{s}\mathcal{H}^{'}(0)$.

b. If $\mathcal{H}(x)$ *is (i)-differentiable and* $\mathcal{H}^{'}(x)$ *is (ii)-differentiable, then*
$\widehat{A}[\mathcal{H}^{(2)}(x)] = (-\frac{1}{s}\mathcal{H}^{'}(0)) \ominus \{-s^2\widehat{A}[\mathcal{H}(x)] \ominus (-\mathcal{H}(0))\}$.

c. If $\mathcal{H}(x)$ *is (ii)-differentiable and* $\mathcal{H}^{'}(x)$ *is (i)-differentiable, then*
$\widehat{A}[\mathcal{H}^{(2)}(x)] = \{-\mathcal{H}(0) \ominus (-s^2\widehat{A}[\mathcal{H}(x)]\} \ominus \frac{1}{s}\mathcal{H}^{'}(0)$.

d. If $\mathcal{H}(x)$ *is (ii)-differentiable and* $\mathcal{H}^{'}(x)$ *is (ii)-differentiable, then*
$\widehat{A}[\mathcal{H}^{(2)}(x)] = (-\frac{1}{s}\mathcal{H}^{'}(0)) \ominus \{(\mathcal{H}(0)) \ominus s^2\widehat{A}[\mathcal{H}(x)]\}$.

Theorem 3.6. *Let $\mathcal{H}(x)e^{-sx}, \mathcal{H}'(x)e^{-sx}, \mathcal{H}^{(2)}(x)e^{-sx}, \dots, \mathcal{H}^{(n-1)}(x)e^{-sx}$ are exist, continuous and integrable Riemann functions on $[0,\infty)$ and $\mathcal{H}(x)$ is continuous fuzzy valued function. If $\mathcal{H}^{(s)}(x)$ is strongly generalized differentiable of the n^{th} order such that, there exists an element $\mathcal{H}^{(s)}(x_0) \in \mathcal{R}_F, \forall s = 0,1,\dots,n$. Then fuzzy Aboodh transform of $\mathcal{H}^{(n)}(x)$ is given by,*

$$\widehat{A}[\mathcal{H}^{(n)}(x)] = \{\{\{\dots\{\prod_{\mathbb{K}=1}^{n}\mathbb{B}(\mathbb{K})\widehat{A}[\mathcal{H}(x)] \ominus \prod_{\mathbb{K}=2}^{n}\mathbb{B}(\mathbb{K})\mathbb{E}(1)\mathcal{H}(0)\} \ominus \prod_{\mathbb{K}=3}^{n}\mathbb{B}(\mathbb{K})\mathbb{E}(2)\mathcal{H}'(0)\}$$
$$\ominus \prod_{\mathbb{K}=4}^{n}\mathbb{B}(\mathbb{K})\mathbb{E}(3)\mathcal{H}^{(2)}(0)\} \ominus \prod_{\mathbb{K}=5}^{n}\mathbb{B}(\mathbb{K})\mathbb{E}(4)\mathcal{H}^{(3)}(0)\} \ominus \dots\} \ominus \mathbb{B}(n)\mathbb{E}(n-1)\mathcal{H}^{(n-2)}(0)\} \ominus \mathbb{E}(n)\mathcal{H}^{(n-1)}(0)\},$$

where

$$\mathbb{B}(\mathbb{K}) = \begin{cases} s & if\mathcal{H}^{(k)} bei - differentiable, \\ \ominus(-s) & if\mathcal{H}^{(k)} beii - differentiable. \end{cases} \quad \mathbb{E}(\mathbb{K}) = \begin{cases} \frac{1}{s} & if\mathcal{H}^{(k)} bei - differentiable, \\ \ominus(\frac{1}{-s}) & if\mathcal{H}^{(k)} beii - differentiable. \end{cases}$$

Proof . Let $n = 1, \widehat{A}[\mathcal{H}'(x)] = \mathbb{B}(1)\widehat{A}[\mathcal{H}(x)] \ominus \mathbb{E}(1)\mathcal{H}(0)$, where

$$\mathbb{B}(\mathbb{K}) = \begin{cases} s & if\mathcal{H}^{(k)} bei - differentiable, \\ \ominus(-s) & if\mathcal{H}^{(k)} beii - differentiable. \end{cases} \quad \mathbb{E}(\mathbb{K}) = \begin{cases} \frac{1}{s} & if\mathcal{H}^{(k)} bei - differentiable, \\ \ominus(\frac{1}{-s}) & if\mathcal{H}^{(k)} beii - differentiable. \end{cases}$$

1. if $\mathcal{H}$ is (i)-differentiable then $\widehat{A}[\mathcal{H}'(x)] = s\widehat{A}[\mathcal{H}(x)] \ominus \frac{1}{s}\mathcal{H}(0)$.
2. if $\mathcal{H}$ is (i)-differentiable then $\widehat{A}[\mathcal{H}'(x)] = -\frac{1}{s}\mathcal{H}(0) \ominus -s\widehat{A}[\mathcal{H}(x)]$.

Suppose that $n = \mathbb{K}$ is true,

$$\widehat{A}[\mathcal{H}^{(\mathbb{K})}(x)] = \{\{\{\dots\{\prod_{i=1}^{\mathbb{K}}\mathbb{B}(i)\widehat{A}[\mathcal{H}(x)] \ominus \prod_{i=2}^{\mathbb{K}}\mathbb{B}(i)\mathbb{E}(1)\mathcal{H}(0)\} \ominus \prod_{i=3}^{\mathbb{K}}\mathbb{B}(i)\mathbb{E}(2)\mathcal{H}'(0)\}$$
$$\ominus \prod_{i=4}^{\mathbb{K}}\mathbb{B}(i)\mathbb{E}(3)\mathcal{H}^{(2)}(0)\} \ominus \prod_{i=5}^{\mathbb{K}}\mathbb{B}(i)\mathbb{E}(4)\mathcal{H}^{(3)}(0)\} \ominus \dots\} \ominus \mathbb{B}(\mathbb{K})\mathbb{E}(\mathbb{K}-1)\mathcal{H}^{(\mathbb{K}-2)}(0)\} \ominus \mathbb{E}(\mathbb{K})\mathcal{H}^{(\mathbb{K}-1)}(0).$$

Let $n = \mathbb{K}+1$,

$$\begin{aligned}
&\widehat{A}[\mathcal{H}^{(\mathbb{K})}(x)] = \mathbb{B}(\mathbb{K}+1)\widehat{A}[\mathcal{H}^{(\mathbb{K})}(x)] \ominus \mathbb{E}(\mathbb{K}+1)\mathcal{H}^{(\mathbb{K})}(0) \\
&= \mathbb{B}(\mathbb{K}+1)\{\{\{\dots\{\prod_{i=1}^{\mathbb{K}}\mathbb{B}(i)\widehat{A}[\mathcal{H}(x)] \ominus \prod_{i=2}^{\mathbb{K}}\mathbb{B}(i)\mathbb{E}(1)\mathcal{H}(0)\} \ominus \prod_{i=3}^{\mathbb{K}}\mathbb{B}(i)\mathbb{E}(2)\mathcal{H}'(0)\} \\
&\ominus \prod_{i=4}^{\mathbb{K}}\mathbb{B}(i)\mathbb{E}(3)\mathcal{H}^{(2)}(0)\} \ominus \prod_{i=5}^{\mathbb{K}}\mathbb{B}(i)\mathbb{E}(4)\mathcal{H}^{(3)}(0)\} \ominus \dots\} \ominus \mathbb{B}(\mathbb{K})\mathbb{E}(\mathbb{K}-1)\mathcal{H}^{(\mathbb{K}-2)}(0)\} \\
&\ominus \mathbb{E}(\mathbb{K})\mathcal{H}^{(\mathbb{K}-1)}(0)\} \ominus \mathbb{E}(\mathbb{K}+1)\mathcal{H}^{(\mathbb{K})}(0) = \{\{\{\dots\{\prod_{i=1}^{\mathbb{K}+1}\mathbb{B}(i)\widehat{A}[\mathcal{H}(x)] \ominus \prod_{i=2}^{\mathbb{K}+1}\mathbb{B}(i)\mathbb{E}(1)\mathcal{H}(0)\} \\
&\ominus \prod_{i=3}^{\mathbb{K}+1}\mathbb{B}(i)\mathbb{E}(2)\mathcal{H}'(0)\} \ominus \prod_{i=4}^{\mathbb{K}+1}\mathbb{B}(i)\mathbb{E}(3)\mathcal{H}^{(2)}(0)\} \ominus \prod_{i=5}^{\mathbb{K}+1}\mathbb{B}(i)\mathbb{E}(4)\mathcal{H}^{(3)}(0)\} \ominus \dots\} \\
&\ominus \mathbb{B}(\mathbb{K}+1)\mathbb{B}(\mathbb{K})\mathbb{E}(\mathbb{K}-1)\mathcal{H}^{(\mathbb{K}-2)}(0)\} \ominus \mathbb{B}(\mathbb{K}+1)\mathbb{E}(\mathbb{K})\mathcal{H}^{(\mathbb{K}-1)}(0) \ominus \mathbb{E}(\mathbb{K}+1)\mathcal{H}^{(\mathbb{K})}(0).
\end{aligned}$$

□

4. Illustrative example

Example: Consider the following fifth-order FIVP

$\mathcal{H}^{(5)}(x) = \beta, \mathcal{H}(0,\varsigma) = \mathcal{H}^{'}(0,\varsigma), \mathcal{H}^{(1)}(0,\varsigma), \mathcal{H}^{(2)}(0,\varsigma), \mathcal{H}^{(3)}(0,\varsigma), \mathcal{H}^{(4)}(0,\varsigma) = (\varsigma-1, 1-\varsigma)$
$\beta = (\varsigma-1, 1-\varsigma), 0 \leq \varsigma \leq .1$

Solution: Apply fuzzy Aboodh transform on both sides, to get $\widehat{A}[\mathcal{H}^{(5)}(x)] = \widehat{A}[\beta]$.

1. If $\mathcal{H}(x), \mathcal{H}^{'}(x), \mathcal{H}^{(2)}(x), \mathcal{H}^{(3)}(x)$ and $\mathcal{H}^{(4)}(x)$ are (i)-differentiable

$$\widehat{A}[\mathcal{H}^{(5)}(x)] = \{\{\{\{s^5\widehat{A}[\mathcal{H}(x)] \ominus s^3\mathcal{H}(0)\} \ominus s^2\mathcal{H}^{'}(0)\} \ominus s\mathcal{H}^{(2)}(0)\} \ominus \mathcal{H}^{(3)}(0)\} \ominus \frac{1}{s}\mathcal{H}^{(4)}(0$$
$$\{\{\{\{s^5\widehat{A}[\mathcal{H}(x)] \ominus s^3\mathcal{H}(0)\} \ominus s^2\mathcal{H}^{'}(0)\} \ominus s\mathcal{H}^{(2)}(0)\} \ominus \mathcal{H}^{(3)}(0)\} \ominus \frac{1}{s}\mathcal{H}^{(4)}(0) = A[\beta]$$
$$s^5\widehat{A}[\underline{\mathcal{H}}(x,\varsigma)] - s^3\underline{\mathcal{H}}(0,\varsigma) - s^2\underline{\mathcal{H}}^{'}(0,\varsigma) - s\underline{\mathcal{H}}^{(2)}(0,\varsigma)\} - \underline{\mathcal{H}}^{(3)}(0,\varsigma) - \frac{1}{s}\underline{\mathcal{H}}^{(4)}(0,\varsigma) = A[\underline{\beta}]$$
$$s^5\widehat{A}[\overline{\mathcal{H}}(x,\varsigma)] - s^3\overline{\mathcal{H}}(0,\varsigma) - s^2\overline{\mathcal{H}}^{'}(0,\varsigma) - s\overline{\mathcal{H}}^{(2)}(0,\varsigma)\} - \overline{\mathcal{H}}^{(3)}(0,\varsigma) - \frac{1}{s}\overline{\mathcal{H}}^{(4)}(0,\varsigma) = A[\overline{\beta}]$$
$$s^5\widehat{A}[\underline{\mathcal{H}}(x,\varsigma)] - s^3(\varsigma-1) - s^2(\varsigma-1) - s(\varsigma-1)\} - (\varsigma-1) - \frac{1}{s}(\varsigma-1) = \frac{(\varsigma-1)}{s^2}$$
$$s^5\widehat{A}[\overline{\mathcal{H}}(x,\varsigma)] - s^3(1-\varsigma) - s^2(1-\varsigma) - s(1-\varsigma)\} - (1-\varsigma) - \frac{1}{s}(1-\varsigma) = \frac{(1-\varsigma)}{s^2}$$
$$\underline{\mathcal{H}}(x,k) = (\varsigma-1)(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5).$$
$$\overline{\mathcal{H}}(x,k) = (1-\varsigma)(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5).$$

2. If $\mathcal{H}(x)$ is (i)-differentiable but $\mathcal{H}^{'}(x), \mathcal{H}^{(2)}(x), \mathcal{H}^{(3)}(x)$ and $\mathcal{H}^{(4)}(x)$ are (ii)-differentiable

$$\widehat{A}[\mathcal{H}^{(5)}(x)] = -\frac{1}{s}\mathcal{H}^{(4)}(0) \ominus \left\{\mathcal{H}^{(3)}(0) \ominus \left\{-s\mathcal{H}^{(2)}(0) \ominus \left\{s^2\mathcal{H}^{'}(0) \ominus \left\{s^5\widehat{A}[\mathcal{H}(x)] \ominus s^3\mathcal{H}(0)\right\}\right\}\right\}\right\}$$
$$-\frac{1}{s}\overline{\mathcal{H}^{(4)}}(0,\varsigma) - \underline{\mathcal{H}^{(3)}}(0,\varsigma) - s\overline{\mathcal{H}^{(2)}}(0,\varsigma) - s^2\underline{\mathcal{H}}^{'}(0,\varsigma) + s^5\widehat{A}[\underline{\mathcal{H}}(x,\varsigma)] - s^3\underline{\mathcal{H}}(0,\varsigma) = A[\underline{\beta}]$$
$$-\frac{1}{s}\underline{\mathcal{H}^{(4)}}(0,\varsigma) - \overline{\mathcal{H}^{(3)}}(0,\varsigma) - s\underline{\mathcal{H}^{(2)}}(0,\varsigma) - s^2\overline{\mathcal{H}}^{'}(0,\varsigma) + s^5\widehat{A}[\overline{\mathcal{H}}(x,\varsigma)] - s^3\overline{\mathcal{H}}(0,\varsigma) = A[\overline{\beta}]$$
$$\underline{\mathcal{H}}(x,k) = (\varsigma-1)(\frac{1}{120}x^5 + \frac{1}{6}x^3 + x + 1) + (1-\varsigma)(\frac{1}{2}x^2 + \frac{1}{24}x^4)$$
$$\overline{\mathcal{H}}(x,k) = (1-\varsigma)(\frac{1}{120}x^5 + \frac{1}{6}x^3 + x + 1) + (\varsigma-1)(\frac{1}{2}x^2 + \frac{1}{24}x^4).$$

3. If $\mathcal{H}^{'}(x)$ is (i)-differentiable but $\mathcal{H}(x), \mathcal{H}^{(2)}(x), \mathcal{H}^{(3)}(x)$ and $\mathcal{H}^{(4)}(x)$ are (ii)-differentiable

$$\widehat{A}[\mathcal{H}^{(5)}(x)] = -\frac{1}{s}\mathcal{H}^{(4)}(0) \ominus \left\{\mathcal{H}^{(3)}(0) \ominus \left\{-s\mathcal{H}^{(2)}(0) \ominus \left\{\left\{s^3\mathcal{H}(0) \ominus s^5\widehat{A}[\mathcal{H}(x)] \ominus\right\} \ominus -s^2\mathcal{H}^{'}(0)\right\}\right\}\right\}$$
$$-\frac{1}{s}\overline{\mathcal{H}^{(4)}}(0,\varsigma) - \underline{\mathcal{H}^{(3)}}(0,\varsigma) - s\overline{\mathcal{H}^{(2)}}(0,\varsigma) - s^3\underline{\mathcal{H}}(0,\varsigma) + s^5\widehat{A}[\underline{\mathcal{H}}(x,\varsigma)] - s^2\overline{\mathcal{H}^{'}}(0,\varsigma) = A[\underline{\beta}]$$
$$-\frac{1}{s}\underline{\mathcal{H}^{(4)}}(0,\varsigma) - \overline{\mathcal{H}^{(3)}}(0,\varsigma) - s\underline{\mathcal{H}^{(2)}}(0,\varsigma) - s^3\overline{\mathcal{H}}(0,\varsigma) + s^5\widehat{A}[\overline{\mathcal{H}}(x,\varsigma)] - s^2\underline{\mathcal{H}^{'}}(0,\varsigma) = A[\overline{\beta}]$$
$$\underline{\mathcal{H}}(x,k) = (\varsigma-1)(\frac{1}{120}x^5 + \frac{1}{6}x^3 + x + 1) + (1-\varsigma)(\frac{1}{2}x^2 + \frac{1}{24}x^4)$$
$$\overline{\mathcal{H}}(x,k) = (1-\varsigma)(\frac{1}{120}x^5 + \frac{1}{6}x^3 + x + 1) + (\varsigma-1)(\frac{1}{2}x^2 + \frac{1}{24}x^4).$$

4. If $\mathcal{H}^{(2)}(x)$ is (i)-differentiable but $\mathcal{H}(x), \mathcal{H}'(x), \mathcal{H}^{(3)}(x), \mathcal{H}^{(4)}(x)$ are (ii)-differentiable.

$$\widehat{A}[\mathcal{H}^{(5)}(x)] = -\frac{1}{s}\mathcal{H}^{(4)}(0) \ominus \left\{\mathcal{H}^{(3)}(0) \ominus \left\{\left\{-s^2\mathcal{H}'(0) \ominus \left\{s^3\mathcal{H}(0) \ominus s^5\widehat{A}[\mathcal{H}(x)]\right\}\right\} \ominus s\mathcal{H}^{(2)}(0)\right\}\right\}$$
$$-\frac{1}{s}\overline{\mathcal{H}^{(4)}}(0,\varsigma) - \underline{\mathcal{H}^{(3)}}(0,\varsigma) - s^2\overline{\mathcal{H}'}(0,\varsigma) - s^3\underline{\mathcal{H}}(0,\varsigma) + s^5\widehat{A}[\underline{\mathcal{H}}(x,\varsigma)] - s\underline{\mathcal{H}^2}(0,\varsigma) = A[\underline{\beta}]$$
$$-\frac{1}{s}\underline{\mathcal{H}^{(4)}}(0,\varsigma) - \overline{\mathcal{H}^{(3)}}(0,\varsigma) - s^2\underline{\mathcal{H}'}(0,\varsigma) - s^3\overline{\mathcal{H}}(0,\varsigma) + s^5\widehat{A}[\overline{\mathcal{H}}(x,\varsigma)] - s\overline{\mathcal{H}^2}(0,\varsigma) = A[\overline{\beta}]$$
$$\underline{\mathcal{H}}(x,k) = (\varsigma - 1)(\frac{1}{120}x^5 + \frac{1}{2}x^2 + 1 + \frac{1}{6}x^3) + (1 - \varsigma)(x + \frac{1}{24}x^4)$$
$$\overline{\mathcal{H}}(x,k) = (1 - \varsigma)(\frac{1}{120}x^5 + \frac{1}{2}x^2 + 1 + \frac{1}{6}x^3) + (\varsigma - 1)(x + \frac{1}{24}x^4).$$

5. If $\mathcal{H}^{(3)}(x)$ is (i)-differentiable but $\mathcal{H}(x), \mathcal{H}'(x), \mathcal{H}^{(2)}(x), \mathcal{H}^{(4)}(x)$ are (ii)-differentiable.

$$\widehat{A}[\mathcal{H}^{(5)}(x)] = -\frac{1}{s}\mathcal{H}^{(4)}(0) \ominus \left\{\left\{s\mathcal{H}^{(2)}(0) \ominus \left\{-s^2\mathcal{H}'(0) \ominus \left\{s^3\mathcal{H}(0) \ominus s^5\widehat{A}[\mathcal{H}(x)]\right\}\right\}\right\} \ominus -\mathcal{H}^{(3)}(0)\right\}$$
$$-\frac{1}{s}\overline{\mathcal{H}^{(4)}}(0,\varsigma) - s\underline{\mathcal{H}^{(2)}}(0,\varsigma) - s^2\overline{\mathcal{H}'}(0,\varsigma) - s^3\underline{\mathcal{H}}(0,\varsigma) + s^5\widehat{A}[\underline{\mathcal{H}}(x,\varsigma)] - \overline{\mathcal{H}^{(3)}}(0,\varsigma) = A[\underline{\beta}]$$
$$-\frac{1}{s}\underline{\mathcal{H}^{(4)}}(0,\varsigma) - s\overline{\mathcal{H}^{(2)}}(0,\varsigma) - s^2\underline{\mathcal{H}'}(0,\varsigma) - s^3\overline{\mathcal{H}}(0,\varsigma) + s^5\widehat{A}[\overline{\mathcal{H}}(x,\varsigma)] - \underline{\mathcal{H}^{(3)}}(0,\varsigma) = A[\overline{\beta}]$$
$$\underline{\mathcal{H}}(x,k) = (\varsigma - 1)(\frac{1}{120}x^5 + 1 + \frac{1}{2}x^2) + (1 - \varsigma)(\frac{1}{6}x^3 + x + \frac{1}{24}x^4)$$
$$\overline{\mathcal{H}}(x,k) = (1 - \varsigma)(\frac{1}{120}x^5 + 1 + \frac{1}{2}x^2) + (\varsigma - 1)(\frac{1}{6}x^3 + x + \frac{1}{24}x^4).$$

Other case are solved by the same way.

5. Conclusion

This paper presents the general formula for the fuzzy Aboodh transform, which is used to solve fuzzy n^{th}-order differential equations and we explained the using of the concept of strongly generalized differential equations. We used a fifth-order numerical example to demonstrate efficiency and quality of the method.

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