



New fixed point theorems in midconvex subgroups of abelian Banach groups

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Abstract

In this paper, using continuous, injective, and sequentially convergent mappings on a group, new generalizations of Kannan and Chatterjea's fixed points in Banach groups are presented. We generalize contractions with constants to prove some fixed point theorems in a Banach group. Moreover, non-decreasing continuous functions from the set of positive real numbers to itself are used to introduce a new extension of contractions on normed groups.

Keywords: Banach groups, Normed groups, Sequentially convergent mapping, Kannan fixed point theorem, Midconvex subgroup

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1. Introduction

Fixed point theory is one of the most useful and essential tools of nonlinear analysis with certain applications. Following the new attention evoked by Bingham and Ostaszewski [2], the authors were motivated to apply analytic concepts in algebraic structures including groups. As a first step, we investigated fixed points in structures of normed groups, hoping to show the importance of focusing on normed groups as a type of normed spaces. Since the analytical properties of the groups have not been studied fully nor coherently, normed groups were chosen as algebraic structures to reconstruct the analytical properties of normed vector spaces. Hence, convexity in normed spaces will be interpreted as midconvexity in normed groups. As a consequence of such interpretations, the concepts originated should undergo the same conversion, with results not necessarily the same. The new approach presented in this study can provide a perspective to treat groups as normed spaces.

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Let us now deal with fixed points in metric spaces and then we will take a look at the history of normed groups. In sections two and three, we will prove a number of fixed point theorems in normed groups.

The origin of fixed point theory is known as the Banach contraction principle.

Let (\mathcal{T}, d) be a metric space. A mapping $\nu : \mathcal{T} \rightarrow \mathcal{T}$, is said to be a contraction if there exists $\zeta \in [0, 1)$ such that for all $t, s \in \mathcal{T}$,

$$d(\nu(t), \nu(s)) \leq \zeta d(t, s). \quad (1.1)$$

Banach contraction principle states that any contraction on a complete metric space has a unique fixed point [1]. In 1968, Kannan introduced an extension of the Banach contraction and proved that a self-mapping ν on a complete metric space (\mathcal{T}, d) satisfying

$$d(\nu(t), \nu(s)) \leq \zeta [d(t, \nu(t)) + d(s, \nu(s))], \quad (1.2)$$

where $\zeta \in [0, \frac{1}{2})$ and $t, s \in \mathcal{T}$, has a unique fixed point [14]. A similar conclusion was also obtained by Chatterjea in 1972 [5]. The mappings satisfying (1.2) are called Kannan type mappings. The significance of Kannan's theorem showed itself in Subrahmanyam paper [27]. He showed that a metric space is complete if and only if every Kannan type mapping has a fixed point. Banach contractions do not have this property. Connell in [7] presented an example of metric space \mathcal{T} that is not complete but every Banach contraction on \mathcal{T} has a fixed point.

The Banach and Kannan fixed point theorems have been improved by various successful attempts. One such attempt was made by Koparde and Waghmode [20] who proved a fixed point theorem for a self-mapping ν on a complete metric space (\mathcal{T}, d) satisfying the Kannan type condition

$$d^2(\nu(t), \nu(s)) \leq \eta (d^2(t, \nu(t)) + d^2(s, \nu(s))),$$

for all $t, s \in \mathcal{T}$ where $0 < \eta < \frac{1}{2}$.

A large number of results on the existence and uniqueness of fixed points for Kannan mappings have been proved in Banach spaces [10, 15, 16, 17]. In 2018, using the interpolation notion, Karapinar introduced a new Kannan type contraction to maximize the rate of convergence [18]. For further details on new fixed point theorems in metric spaces, see [11, 12, 23].

Now, we can discuss normed groups, with the aim to examine the fixed points, which are in fact groups with a right-invariant metric. Although they have been introduced since the 1930s by the Birkhoff-Kakutani's theorem, the number of the results proved has not been outstanding due to the little attention paid to them. However, the field is still evolving, and has recently begun to absorb the attention of researchers, for instance probabilistic normed groups [22, 25, 26]. In 1936, Birkhoff and Kakutani separately proved a significant theorem: *A Hausdorff group \mathcal{K} is homeomorphic with a metric space if and only if, \mathcal{K} satisfies the first countability axiom* [3, 13]. They also showed that this group has a right invariant metric. The term group-norm probably first appeared in Pettis's paper in 1950 [24], however, some authors may use the term "length function" instead of "norm" for groups [21, 6]. A metric d on a semigroup \mathcal{K} is called left-invariant if $d(vx, vy) = d(x, y)$ and right-invariant if $d(xv, yv) = d(x, y)$ whenever $v, x, y \in \mathcal{K}$. The metric d is said to be invariant if it is both right and left-invariant. In 1950, V. L. Klee studied invariant metrics on groups to solve a problem of Banach [19]. See [2, 9] for a wider discussion about the history of normed groups.

Now, we should get familiar with some basic concepts in normed groups theory. Let \mathcal{J} be a group and $\vartheta : \mathcal{J} \rightarrow \mathcal{J}$ be a mapping. An element $s \in \mathcal{J}$ is called a fixed point of ϑ if $\vartheta(s) = s$. To prove

fixed points for groups, Picard iterative sequence can be used. Let $s_0 \in \mathcal{J}$ be an arbitrary element. Define the Picard iterative sequence $\{s_n\}$ in \mathcal{J} as follows

$$s_{n+1} = \vartheta(s_n), \quad (n = 0, 1, 2, \dots).$$

We note that the convergence of this sequence plays a significant role in the existence of a fixed point for mapping ϑ . Define the n^{th} iterate of ϑ as $\vartheta^0 = I$ (the identity map) and $\vartheta^n = \vartheta^{n-1} \circ \vartheta$, for $n \geq 1$.

In our further considerations, by applying sequentially convergent mappings, we will investigate fixed points in Banach groups.

2. Preliminaries

Definition 2.1. [2] Let \mathcal{J} be a group. A norm on a group \mathcal{J} is a function $\|\cdot\| : \mathcal{J} \rightarrow \mathbb{R}$ with the following properties:

1. $\|s\| \geq 0$, for all $s \in \mathcal{J}$;
2. $\|s\| = \|s^{-1}\|$, for all $s \in \mathcal{J}$;
3. $\|sq\| \leq \|s\| + \|q\|$, for all $s, q \in \mathcal{J}$;
4. $\|s\| = 0$ implies that $s = e$.

A normed group $(\mathcal{J}, \|\cdot\|)$ is a group \mathcal{J} equipped with a norm $\|\cdot\|$. By setting $d(s, q) := \|s^{-1}q\|$, it is easy to see that norms are in bijection with left-invariant metrics on \mathcal{J} .

Definition 2.2. [2] Let $(\mathcal{J}, \|\cdot\|)$ be a normed group, $s \in \mathcal{J}$ and $\{s_n\}$ be a sequence in \mathcal{J} . Then:

1. The sequence $\{s_n\}$ converges to s if for every $\epsilon \in \mathbb{R}$, $\epsilon > 0$, there exists positive integer n_0 depending on ϵ such that $\|s_n s^{-1}\| < \epsilon$ for every $n > n_0$. We denote this by $s = \lim_{n \rightarrow \infty} s_n$.
2. The sequence $\{s_n\}$ is called Cauchy sequence, if for every $\epsilon \in \mathbb{R}$, $\epsilon > 0$, there exists positive integer n_0 depending on ϵ such that $\|s_n s_m^{-1}\| < \epsilon$ for every $n, m > n_0$.
3. Normed group $(\mathcal{J}, \|\cdot\|)$ is called complete if any Cauchy sequence in \mathcal{J} converges to an element of \mathcal{J} , i.e. it has a limit in group \mathcal{J} .

Definition 2.3. [2] A Banach group is a normed group $(\mathcal{J}, \|\cdot\|)$, which is complete with respect to the metric

$$d(s, q) = \|sq^{-1}\|, \quad (s, q \in \mathcal{J}).$$

Definition 2.4. [2] A map

$$\tau : \mathcal{J} \rightarrow \mathcal{K},$$

of normed group $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$ into normed group $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$ is called continuous, if for every as small as we please $\epsilon > 0$ there exists such $\delta > 0$, that

$$\|sq^{-1}\|_{\mathcal{J}} < \delta$$

implies

$$\|\tau(s)\tau(q)^{-1}\|_{\mathcal{K}} < \epsilon.$$

Definition 2.5. [4] Let (\mathcal{T}, d) be a metric space. We call a mapping $v : \mathcal{T} \rightarrow \mathcal{T}$ is sequentially convergent if for each sequence $\{t_n\}$ that $\{v(t_n)\}$ is convergent then $\{t_n\}$ is also convergent.

3. Some fixed points in a Banach Group

In this section, applying sequentially convergent mappings, we generalize contractions with constants to prove a number of fixed point theorems in a Banach group. Moreover, some conditions will be taken as constant all through the theorems of this section. For ease of reading, they will be mentioned first.

For all results in this section, let $(\mathcal{J}, \|\cdot\|)$ be a Banach group and suppose that $\vartheta : \mathcal{J} \rightarrow \mathcal{J}$ be a map and $\tau : \mathcal{J} \rightarrow \mathcal{J}$ be a continuous, injective and sequentially convergent mapping. Now, with these assumptions, we have the following results:

Theorem 3.1. *Let Ψ be the class of all nondecreasing continuous functions $\sigma : [0, +\infty) \rightarrow [0, +\infty)$ such that $\sigma^{-1}(0) = \{0\}$. If $0 \leq \eta < \frac{1}{2}$, $\sigma \in \Psi$ and*

$$\sigma(\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\|) \leq \eta[\sigma(\|\tau(s)(\tau\vartheta(s))^{-1}\|) + \sigma(\|\tau(q)(\tau\vartheta(q))^{-1}\|)], \quad (3.1)$$

for all $s, q \in \mathcal{J}$, then, ϑ has a unique fixed point.

Proof . Since $\sigma^{-1}(0) = \{0\}$, for every $\varepsilon > 0$ we have $\sigma(\varepsilon) > 0$. Suppose that $s_0 \in \mathcal{J}$ is given and the sequence $\{s_n\}$ be defined as $s_{n+1} = \vartheta(s_n)$ for $n = 0, 1, 2, \dots$

By taking $s = s_{n-1}$ and $q = s_n$ in (3.1), we get

$$\begin{aligned} \sigma(\|\tau(s_n)\tau(s_{n+1})^{-1}\|) &= \sigma(\|\tau\vartheta(s_{n-1})(\tau\vartheta(s_n))^{-1}\|) \\ &\leq \eta[\sigma(\|\tau(s_{n-1})(\tau\vartheta(s_{n-1}))^{-1}\|) + \sigma(\|\tau(s_n)(\tau\vartheta(s_n))^{-1}\|)]. \end{aligned}$$

So,

$$\sigma(\|\tau(s_n)\tau(s_{n+1})^{-1}\|) \leq \frac{\eta}{1-\eta} \sigma(\|\tau(s_{n-1})\tau(s_n)^{-1}\|).$$

Therefore,

$$\begin{aligned} \sigma(\|\tau(s_n)\tau(s_{n+1})^{-1}\|) &\leq \frac{\eta}{1-\eta} \sigma(\|\tau(s_{n-1})\tau(s_n)^{-1}\|) \\ &\leq \left(\frac{\eta}{1-\eta}\right)^2 \sigma(\|\tau(s_{n-2})\tau(s_{n-1})^{-1}\|) \\ &\leq \dots \leq \left(\frac{\eta}{1-\eta}\right)^n \sigma(\|\tau(s_0)\tau(s_1)^{-1}\|). \end{aligned}$$

Then for every $m, n \in \mathbb{N}$ that $m > n$ we have,

$$\begin{aligned} \sigma(\|\tau(s_m)\tau(s_n)^{-1}\|) &= \sigma(\|\tau\vartheta(s_{m-1})(\tau\vartheta(s_{n-1}))^{-1}\|) \\ &\leq \eta[\sigma(\|\tau(s_{m-1})\tau(s_m)^{-1}\|) + \sigma(\|\tau(s_{n-1})\tau(s_n)^{-1}\|)] \\ &\leq \eta\left[\left(\frac{\eta}{1-\eta}\right)^{m-1} + \left(\frac{\eta}{1-\eta}\right)^{n-1}\right] \sigma(\|\tau(s_0)\tau(s_1)^{-1}\|). \end{aligned}$$

Letting $m, n \rightarrow \infty$, we get

$$\lim_{m, n \rightarrow \infty} \sigma(\|\tau(s_m)\tau(s_n)^{-1}\|) = 0.$$

Since $\sigma \in \Psi$, $\lim_{m, n \rightarrow \infty} \|\tau(s_m)\tau(s_n)^{-1}\| = 0$. From this, we conclude that $\tau(s_n)$ is a Cauchy sequence and since \mathcal{J} is a Banach group, $\tau(s_n)$ is a convergent sequence. Further, the sequence s_n is also

convergent, i.e. there exists $l \in \mathcal{J}$ such that $\lim_{n \rightarrow \infty} s_n = l$. Since τ is continuous, $\lim_{n \rightarrow \infty} \tau(s_n) = \tau(l)$. Thus,

$$\begin{aligned} \sigma(\|\tau\vartheta(l)\tau(s_{n+1})^{-1}\|) &= \sigma(\|\tau\vartheta(l)(\tau\vartheta(s_n))^{-1}\|) \\ &\leq \eta[\sigma(\|\tau(l)(\tau\vartheta(l))^{-1}\|) + \sigma(\|\tau(s_n)(\tau\vartheta(s_n))^{-1}\|)] \\ &= \eta[\sigma(\|\tau(l)(\tau\vartheta(l))^{-1}\|) + \sigma(\|\tau(s_n)\tau(s_{n+1})^{-1}\|)]. \end{aligned}$$

Since σ is continuous, letting $n \rightarrow \infty$ in the last inequality we get

$$\sigma(\|\tau\vartheta(l)\tau(l)^{-1}\|) \leq \eta[\sigma(\|\tau(l)(\tau\vartheta(l))^{-1}\|) + \sigma(0)].$$

Note that by axiom (2) of Definition 2.1, we have $\|\tau\vartheta(l)\tau(l)^{-1}\| = \|\tau(l)(\tau\vartheta(l))^{-1}\|$. Moreover, $\sigma^{-1}(0) = \{0\}$ and $0 \leq \eta < \frac{1}{2}$, in the inequality above, imply that $\|\tau\vartheta(l)\tau(l)^{-1}\| = 0$, so $\tau\vartheta(l) = \tau(l)$. Finally, τ is an injective, and thus $\vartheta(l) = l$. This means that the mapping ϑ has a fixed point. To prove uniqueness of the fixed point, let v be another fixed point of ϑ . Then by (3.1), we have

$$\begin{aligned} \sigma(\|\tau(l)\tau(v)^{-1}\|) &= \sigma(\|\tau\vartheta(l)(\tau\vartheta(v))^{-1}\|) \\ &\leq \eta[\sigma(\|\tau(l)(\tau\vartheta(l))^{-1}\|) + \sigma(\|\tau(v)(\tau\vartheta(v))^{-1}\|)] \\ &= \eta[\sigma(\|\tau(l)\tau(l)^{-1}\|) + \sigma(\|\tau(v)\tau(v)^{-1}\|)] = 0. \end{aligned}$$

The last inequality implies that $\sigma(\|\tau(l)\tau(v)^{-1}\|) = 0$, i.e. $\tau(l) = \tau(v)$ and this implies that $l = v$. \square Taking $\sigma(t) = t$ in Theorem 3.1, we get the following result:

Corollary 3.2. *Let $0 \leq \eta < \frac{1}{2}$ and*

$$\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\| \leq \eta[\|\tau(s)(\tau\vartheta(s))^{-1}\| + \|\tau(q)(\tau\vartheta(q))^{-1}\|],$$

for all $s, q \in \mathcal{J}$. Then, ϑ has a unique fixed point.

Now, let's look at the following example to show that one can find examples so as to be true in Theorem 3.1. Such examples can support other theorems as well.

Example 3.3. *Let $\mathcal{J} = \{e, a, b, c\}$ be a Klein four-group. Define $\|\cdot\|$ on group \mathcal{J} as follows:*

$$\|e\| = 0, \quad \|a\| = 3, \quad \|b\| = 4, \quad \|c\| = 1.$$

Now, define a mapping $\vartheta : \mathcal{J} \rightarrow \mathcal{J}$ with

$$\vartheta(s) = \begin{cases} a & ; s \neq e \\ c & ; s = e. \end{cases}$$

Obviously, the inequality (1.2) is not held for ϑ , for every $0 < \eta < \frac{1}{2}$. So, Kannan's theorem is not held. Defining $\tau : \mathcal{J} \rightarrow \mathcal{J}$ by

$$\tau(s) = \begin{cases} a & ; s = c \\ b & ; s = a \\ c & ; s = e \\ e & ; s = b, \end{cases}$$

we have

$$\tau\vartheta(s) = \begin{cases} b & ; s \neq e \\ a & ; s = e. \end{cases}$$

Also,

$$\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\| \leq \frac{1}{3}[\|\tau(s)(\tau\vartheta(s))^{-1}\| + \|\tau(q)(\tau\vartheta(q))^{-1}\|],$$

for all $s, q \in \mathcal{J}$. Then, ϑ has a unique fixed point. By placing as the identity function, so all the conditions of Theorem 3.1 is hold.

Similar to Theorem 3.1, a new extension of Chatterjea's theorem in Banach groups could be proved. Because of the similarities of the two, we can ignore the proof and just write it as follows:

Theorem 3.4. Let Ψ be the class of all nondecreasing continuous functions $\sigma : [0, +\infty) \rightarrow [0, +\infty)$ such that $\sigma^{-1}(0) = \{0\}$. If $0 \leq \eta < \frac{1}{2}$, $\sigma \in \Psi$ and

$$\sigma(\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\|) \leq \eta[\sigma(\|\tau(s)(\tau\vartheta(q))^{-1}\|) + \sigma(\|\tau(q)(\tau\vartheta(s))^{-1}\|)], \quad (3.2)$$

for all $s, q \in \mathcal{J}$. Then, ϑ has a unique fixed point.

Taking $\sigma(t) = t$ in Theorem 3.4, we get the following corollary:

Corollary 3.5. Let $0 \leq \eta < \frac{1}{2}$ and

$$\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\| \leq \eta[\|\tau(s)(\tau\vartheta(q))^{-1}\| + \|\tau(q)(\tau\vartheta(s))^{-1}\|], \quad (3.3)$$

for all $s, q \in \mathcal{J}$. Then, ϑ has a unique fixed point.

By the previous assumptions, and considering two constant $\eta > 0$ and $\mu \geq 0$, with $2\eta + \mu < 1$, we have the following theorem

Theorem 3.6. Let $\eta > 0$, $\mu \geq 0$, $2\eta + \mu < 1$ and

$$\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\| \leq \eta[\|\tau(s)(\tau\vartheta(s))^{-1}\| + \|\tau(q)(\tau\vartheta(q))^{-1}\|] + \mu\|\tau(s)\tau(q)^{-1}\| \quad (3.4)$$

for all $s, q \in \mathcal{J}$. Then, ϑ has a unique fixed point.

Proof. Suppose that $s_0 \in \mathcal{J}$ is given and the sequence $\{s_n\}$ be defined as the following $s_{n+1} = \vartheta(s_n)$ for $n = 0, 1, 2, \dots$. By taking $s = s_n$ and $q = s_{n-1}$ in (3.4), we get

$$\|\tau(s_{n+1})\tau(s_n)^{-1}\| \leq \eta[\|\tau(s_{n+1})\tau(s_n)^{-1}\| + \|\tau(s_n)\tau(s_{n-1})^{-1}\|] + \mu\|\tau(s_n)\tau(s_{n-1})^{-1}\|,$$

then

$$\|\tau(s_{n+1})\tau(s_n)^{-1}\| \leq \alpha\|\tau(s_n)\tau(s_{n-1})^{-1}\|, \quad (3.5)$$

for each $n = 0, 1, 2, \dots$, and $0 < \alpha = \frac{\eta+\mu}{1-\eta} < 1$. By the inequality (3.5), we have

$$\begin{aligned} \|\tau(s_n)\tau(s_m)^{-1}\| &\leq \|\tau(s_n)\tau(s_{n-1})^{-1}\| + \|\tau(s_{n-1})\tau(s_{n-2})^{-1}\| \\ &\quad + \dots + \|\tau(s_{m+1})\tau(s_m)^{-1}\| \\ &\leq [\alpha^{n-1} + \alpha^{n-2} + \dots + \alpha^m]\|\tau(s_1)\tau(s_0)^{-1}\| \\ &\leq [\alpha^m + \alpha^{m+1} + \dots]\|\tau(s_1)\tau(s_0)^{-1}\| \\ &= (\alpha)^m \frac{1}{1-\alpha} \|\tau(s_1)\tau(s_0)^{-1}\|, \end{aligned}$$

for all $n, m \in \mathbb{N}$ and $n > m$. Since $0 < \alpha < 1$, we conclude that the sequence $\{\tau(s_n)\}$ is a Cauchy sequence. Completeness of \mathcal{J} ensures that there exists $z \in \mathcal{J}$ such that $\lim_{n \rightarrow \infty} \tau(s_n) = z$. It implies that the sequence $\{s_n\}$ is also a convergent sequence, i.e. there exists $l \in \mathcal{J}$ such that $\lim_{n \rightarrow \infty} s_n = l$. Since the mapping τ is continuous, $\lim_{n \rightarrow \infty} \tau(s_n) = \tau(l)$. Therefore, we have

$$\begin{aligned} \|\tau\vartheta(l)\tau(l)^{-1}\| &\leq \|\tau\vartheta(l)\tau(s_n)^{-1}\| + \|\tau(s_n)\tau(s_{n+1})^{-1}\| + \|\tau(s_{n+1})\tau(l)^{-1}\| \\ &= \|\tau\vartheta(l)(\tau\vartheta^n(s_0))^{-1}\| + \|\tau(s_n)\tau(s_{n+1})^{-1}\| + \|\tau(s_{n+1})\tau(l)^{-1}\| \\ &\leq \eta[\|\tau(l)\tau\vartheta(l)^{-1}\| + \|\tau\vartheta^{n-1}(s_0)(\tau\vartheta^n(s_0))^{-1}\|] \\ &\quad + \mu\|\tau(l)(\tau\vartheta^{n-1}(s_0))^{-1}\| + \alpha^n\|\tau(s_1)\tau(s_0)^{-1}\| + \|\tau(s_{n+1})\tau(l)^{-1}\| \\ &= \eta\|\tau(l)\tau\vartheta(l)^{-1}\| + \eta\alpha^{n-1}\|\tau(s_1)\tau(s_0)^{-1}\| + \mu\|\tau(l)\tau(s_{n-1})^{-1}\| \\ &\quad + \alpha^n\|\tau(s_1)\tau(s_0)^{-1}\| + \|\tau(s_{n+1})\tau(l)^{-1}\|. \end{aligned}$$

Letting $n \rightarrow \infty$ in the inequality above, we have $\|\tau\vartheta(l)\tau(l)^{-1}\| \leq \eta\|\tau\vartheta(l)\tau(l)^{-1}\|$. As $0 < \eta < 1$, then $\tau\vartheta(l) = \tau(l)$ and $\vartheta(l) = l$. For the uniqueness, we suppose that ϑ has two distinct fixed points $l, l_0 \in \mathcal{J}$. Then from (3.4) we have

$$\begin{aligned} \|\tau(l)\tau(l_0)^{-1}\| &= \|\tau\vartheta(l)\tau\vartheta(l_0)^{-1}\| \\ &\leq \eta[\|\tau(l)\tau\vartheta(l)\| + \|\tau(l_0)\tau\vartheta(l_0)^{-1}\|] + \mu\|\tau(l)\tau(l_0)^{-1}\| \\ &= \mu\|\tau(l)\tau(l_0)^{-1}\|. \end{aligned}$$

The last inequality implies that $\|\tau(l)\tau(l_0)^{-1}\| = 0$. As τ is an injective we have $l = l_0$. \square

Corollary 3.7. *Let $0 < \alpha < 1$ and*

$$\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\| \leq \alpha[\|\tau(s)(\tau\vartheta(s))^{-1}\| \cdot \|\tau(q)(\tau\vartheta(q))^{-1}\| \cdot \|\tau(s)(\tau(q))^{-1}\|]^{\frac{1}{3}},$$

for all $s, q \in \mathcal{J}$. Then ϑ has a unique fixed point.

Proof . The inequality of arithmetic and geometric means implies that

$$\begin{aligned} [\|\tau(s)(\tau\vartheta(s))^{-1}\| \cdot \|\tau(q)(\tau\vartheta(q))^{-1}\| \cdot \|\tau(s)(\tau(q))^{-1}\|]^{\frac{1}{3}} &\leq \frac{1}{3}[\|\tau(s)(\tau\vartheta(s))^{-1}\| \\ &\quad + \|\tau(q)(\tau\vartheta(q))^{-1}\| \\ &\quad + \|\tau(s)(\tau(q))^{-1}\|]. \end{aligned}$$

Therefore,

$$\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\| \leq \frac{\alpha}{3}[\|\tau(s)(\tau\vartheta(s))^{-1}\| + \|\tau(q)(\tau\vartheta(q))^{-1}\| + \mu\|\tau(s)(\tau(q))^{-1}\|].$$

We now apply Theorem 3.6 with $\eta = \mu$ replaced by $\frac{\alpha}{3}$. It completes the proof. \square

Similar to Theorem 3.6 we can prove the following theorem which is another extension of Chatterjea’s theorem in Banach groups.

Theorem 3.8. *Let $\eta > 0$, $\mu \geq 0$, $2\eta + \mu < 1$ and*

$$\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\| \leq \eta[\|\tau(s)(\tau\vartheta(q))^{-1}\| + \|\tau(q)(\tau\vartheta(s))^{-1}\|] + \mu\|\vartheta(s)(\tau(q))^{-1}\|$$

for all $s, q \in \mathcal{J}$. Then ϑ has a unique fixed point.

Corollary 3.9. *Let $0 < \alpha < 1$ and*

$$\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\| \leq \alpha[\|\tau(s)(\tau\vartheta(q))^{-1}\| \cdot \|\tau(q)(\tau\vartheta(s))^{-1}\| \cdot \mu\|\tau(s)(\tau(q))^{-1}\|]^{\frac{1}{3}}.$$

for all $s, q \in \mathcal{J}$. Then there is a unique fixed point on ϑ .

Proof . The inequality of arithmetic and geometric means implies that

$$\begin{aligned} [\|\tau(s)(\tau\vartheta(q))^{-1}\| \cdot \|\tau(q)(\tau\vartheta(s))^{-1}\| \cdot \|\tau(s)(\tau(q))^{-1}\|]^{\frac{1}{3}} &\leq \frac{1}{3}[\|\tau(s)(\tau\vartheta(q))^{-1}\| \\ &+ \|\tau(q)(\tau\vartheta(s))^{-1}\| \\ &+ \|\tau(s)(\tau(q))^{-1}\|]. \end{aligned}$$

Therefore,

$$\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\| \leq \frac{\alpha}{3}[\|\tau(s)(\tau\vartheta(q))^{-1}\| + \|\tau(q)(\tau\vartheta(s))^{-1}\| + \mu\|\tau(s)(\tau(q))^{-1}\|].$$

For $\eta = \mu = \frac{\alpha}{3}$, in Theorem 3.8 the proof is completed. \square Like the previous theorem, the next theorem can be proved.

Theorem 3.10. *Let $\eta > 0, \mu \geq 0, 2\eta + \mu < 1$ and*

$$\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\|^2 \leq \eta[\|\tau(s)(\tau\vartheta(s))^{-1}\|^2 + \|\tau(q)(\tau\vartheta(q))^{-1}\|^2] + \mu\|\tau(s)(\tau(q))^{-1}\|^2 \tag{3.6}$$

for all $s, q \in \mathcal{J}$. Then ϑ has a unique fixed point.

Proof . Suppose that $s_0 \in \mathcal{J}$ is given and the sequence $\{s_n\}$ be defined as the following $s_{n+1} = \vartheta(s_n)$, for $n = 0, 1, 2, \dots$. By (3.6), we have

$$\|\tau(s_{n+1})\tau(s_n)^{-1}\|^2 \leq \eta[\|\tau(s_{n+1})\tau(s_n)^{-1}\|^2 + \|\tau(s_n)\tau(s_{n-1})^{-1}\|^2] + \mu\|\tau(s_n)\tau(s_{n-1})^{-1}\|^2.$$

Therefore,

$$\|\tau(s_{n+1})\tau(s_n)^{-1}\|^2 \leq \alpha\|\tau(s_n)\tau(s_{n-1})^{-1}\|^2, \tag{3.7}$$

for each $n = 0, 1, 2, \dots$, and $0 < \alpha = (\frac{\eta+\mu}{1-\eta})^{\frac{1}{2}} < 1$. By the inequality (3.7) we have

$$\|\tau(s_n)\tau(s_m)^{-1}\|^2 \leq \frac{\alpha^m}{1-\alpha}\|\tau(s_1)\tau(s_0)^{-1}\|^2,$$

for all $n, m \in \mathbb{N}$ where $n > m$. Since $0 < \alpha < 1$, we conclude that the sequence $\{\tau(s_n)\}$ is a Cauchy sequence and there exists $z \in \mathcal{J}$ such that $\lim_{n \rightarrow \infty} \tau(s_n) = z$.

As $\tau : \mathcal{J} \rightarrow \mathcal{J}$ is a sequentially convergent mapping, the sequence $\{s_n\}$ is also convergent, i.e. there exists $l \in \mathcal{J}$ that $\lim_{n \rightarrow \infty} s_n = l$.

Since the mapping τ is continuous, $\lim_{n \rightarrow \infty} \tau(s_n) = \tau(l)$. Now, we are going to show that l is the unique fixed point of ϑ . By (3.6), we have

$$\begin{aligned} \|\tau\vartheta(l)\tau(l)^{-1}\| &\leq \|\tau(l)\tau(s_{n+1})^{-1}\| + \|\tau(s_{n+1})(\tau\vartheta(l))^{-1}\| \\ &= \|\tau(l)\tau(s_{n+1})^{-1}\| + \|\tau\vartheta(s_n)(\tau\vartheta(l))^{-1}\| \\ &\leq \|\tau(l)\tau(s_{n+1})^{-1}\| + [\eta[\|\tau(s_n)(\tau\vartheta(s_n))^{-1}\|^2 \\ &+ \|\tau(l)(\tau\vartheta(l))^{-1}\|^2] + \mu\|\tau(s_n)\tau(l)^{-1}\|^2]^{\frac{1}{2}} \\ &= \|\tau(l)\tau(s_{n+1})^{-1}\| + [\eta[\|\tau(s_n)(\tau(s_{n+1}))^{-1}\|^2 \\ &+ \|\tau(l)(\tau\vartheta(l))^{-1}\|^2] + \mu\|\tau(s_n)\tau(l)^{-1}\|^2]^{\frac{1}{2}}, \end{aligned}$$

for each $n \in \mathbb{N}$. For $n \rightarrow \infty$, the latter is transformed as the following $\|\tau\vartheta(l)\tau(l)^{-1}\| \leq \eta^{\frac{1}{2}}\|\tau\vartheta(l)\tau(l)^{-1}\|$. This implies that $\|\tau\vartheta(l)\tau(l)^{-1}\| = 0$ and $\vartheta(l) = l$. To see the uniqueness of the fixed point of ϑ , let $l, l_0 \in \mathcal{J}$ be two fixed points on ϑ . Using (3.6), we have

$$\begin{aligned} \|\tau(l)\tau(l_0)^{-1}\|^2 &= \|\tau\vartheta(l)(\tau\vartheta(l_0))^{-1}\|^2 \\ &\leq \eta[\|\tau(l)(\tau\vartheta(l))^{-1}\|^2 + \|\tau(l_0)(\tau\vartheta(l_0))^{-1}\|^2] + \mu\|\tau(l)\tau(l_0)^{-1}\|^2 \\ &= \eta[\|\tau(l)\tau(l)^{-1}\|^2 + \|\tau(l_0)\tau(l_0)^{-1}\|^2] + \mu\|\tau(l)\tau(l_0)^{-1}\|^2 \\ &= \mu\|\tau(l)\tau(l_0)^{-1}\|^2, \end{aligned}$$

and since $0 < \mu < 1$ the latter inequality implies that $\|\tau(l)\tau(l_0)^{-1}\| = 0$ and $l = l_0$. \square

Corollary 3.11. *Let $0 < \eta < \frac{1}{2}$ and*

$$\|\tau\vartheta(s)(\tau\vartheta(q))^{-1}\|^2 \leq \eta[\|\tau(s)(\tau\vartheta(s))^{-1}\|^2 + \|\tau(q)(\tau\vartheta(q))^{-1}\|^2]$$

for all $s, q \in \mathcal{J}$. Then ϑ has a unique fixed point.

Proof . For $\mu = 0$, Theorem 3.10 implies the validity of the corollary. \square

4. Fixed points in midconvex subgroups of a Banach group

In this section, using closed midconvex subgroups of a Banach group, we will prove some fixed points through the concept of \mathbb{N} -homogeneous norms on groups. First, we need to define the concept of midconvexity and \mathbb{N} -homogeneity.

Definition 4.1. [8] *Let \mathcal{J} be a group. An element $s \in \mathcal{J}$ is said to be divisible by $n \in \mathbb{Z}$ if $s = q^n$ has a solution q in \mathcal{J} . A group \mathcal{J} is called infinitely divisible if each element in \mathcal{J} is divisible by every positive integer.*

A group-norm $\|\cdot\|$ is \mathbb{N} -homogeneous if for each $n \in \mathbb{N}$,

$$\|s^n\| = n\|s\| \quad (\forall s \in \mathcal{J}).$$

Definition 4.2. [2] *Let \mathcal{J} be a group. A subset S of \mathcal{J} is called $\frac{1}{2}$ -convex (or midconvex), if for every $s, q \in S$ there exists an element $z \in S$, denoted by $(sq)^{\frac{1}{2}}$, such that $z^2 = sq$.*

In what follows, let $(\mathcal{J}, \|\cdot\|)$ be a Banach abelian group with \mathbb{N} -homogeneous norm, S be a nonempty, closed and $\frac{1}{2}$ -convex subgroup of \mathcal{J} and let $v : S \rightarrow S$ be a mapping. With these assumptions, we will have the following results.

Theorem 4.3. *If $2 \leq \kappa < 4$ and*

$$\|sv(s)^{-1}\| + \|qv(q)^{-1}\| \leq \kappa\|sq^{-1}\|,$$

for all $s, q \in S$, then v has at least one fixed point.

Proof . Let for arbitrary element $s_0 \in S$, a sequence $(s_n)_{n=1}^\infty$ be defined by

$$s_{n+1} = (s_n v(s_n))^{\frac{1}{2}} \quad (n = 0, 1, 2, \dots).$$

Then we have

$$s_n v(s_n)^{-1} = s_n^2 s_n^{-1} v(s_n)^{-1} = s_n^2 ((s_n v(s_n))^{\frac{-1}{2}})^2 = (s_n s_{n+1}^{-1})^2,$$

and since the norm is \mathbb{N} -homogeneous, we have

$$\|s_n v(s_n)^{-1}\| = \|(s_n s_{n+1}^{-1})^2\| = 2\|s_n s_{n+1}^{-1}\|.$$

So, for $s = s_{n-1}$ and $q = s_n$, we obtain

$$2\|s_{n-1} s_n^{-1}\| + 2\|s_n s_{n+1}^{-1}\| \leq \kappa \|s_{n-1} s_n^{-1}\|.$$

Therefore, $\|s_n s_{n+1}^{-1}\| \leq \beta \|s_{n-1} s_n^{-1}\|$, where $0 \leq \beta = \frac{\kappa-2}{2} < 1$, as $2 \leq \kappa < 4$. Hence, for every $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} \|s_m s_n^{-1}\| &\leq \|s_m s_{m-1}^{-1}\| + \|s_{m-1} s_{m-2}^{-1}\| + \dots + \|s_{n+1} s_n^{-1}\| \\ &\leq [\beta^{m-1} + \beta^{m-2} + \dots + \beta^n] \|s_0 s_1^{-1}\| \\ &\leq \frac{\beta^n}{1 - \beta} \|s_0 s_1^{-1}\|. \end{aligned}$$

Since $\beta < 1$, the latter implies that the sequence $(s_n)_{n=1}^\infty$ is Cauchy sequence and hence, it converges to some $z \in S$. Since

$$\|z v(s_n)^{-1}\| \leq \|z s_n^{-1}\| + \|s_n v(s_n)^{-1}\| = \|z s_n^{-1}\| + 2\|s_n s_{n+1}^{-1}\|,$$

we have

$$\lim_{n \rightarrow \infty} v(s_n) = z.$$

Therefore, for $s = z$ and $q = s_n$, we have

$$\|z v(z)^{-1}\| + 2\|s_n s_{n+1}^{-1}\| \leq \kappa \|z s_n^{-1}\|.$$

With n , tending to infinity, we have $v(z) = z$. \square

Corollary 4.4. *If $0 \leq \iota < 2$ and*

$$\|s v(q)^{-1}\| + \|q v(s)^{-1}\| \leq \iota \|s q^{-1}\|,$$

for all $s, q \in S$. Then v has a fixed point.

Proof . For all $s, q \in S$, we have

$$\|s v(s)^{-1}\| + \|q v(q)^{-1}\| \leq \|s q^{-1}\| + \|q v(s)^{-1}\| + \|s q^{-1}\| + \|s v(q)^{-1}\|.$$

Thus,

$$\|s v(s)^{-1}\| + \|q v(q)^{-1}\| \leq \iota \|s q^{-1}\| + 2\|s q^{-1}\|.$$

Therefore, we conclude that v satisfies Theorem 4.3 with $\kappa = \iota + 2$. \square

Theorem 4.5. *If $2 \leq \kappa < 5$ and*

$$\|v(s) v(q)^{-1}\| + \|s v(s)^{-1}\| + \|q v(q)^{-1}\| \leq \kappa \|s q^{-1}\|, \tag{4.1}$$

for all $s, q \in S$, then v has at least one fixed point.

Proof . Let for arbitrary element $s_0 \in S$, a sequence $(s_n)_{n=1}^\infty$ be defined by

$$s_{n+1} = (s_n v(s_n))^{\frac{1}{2}} \quad (n = 0, 1, 2, \dots).$$

So,

$$s_n v(s_{n-1})^{-1} = (s_{n-1} v(s_{n-1}))^{\frac{1}{2}} v(s_{n-1})^{-1} = (s_{n-1} v(s_{n-1}^{-1}))^{\frac{1}{2}}.$$

Then

$$2\|s_n s_{n+1}^{-1}\| - \|s_{n-1} s_n^{-1}\| \leq \|v(s_{n-1}) v(s_n)^{-1}\|.$$

In (4.1), put $s = s_{n-1}$ and $q = s_n$, then we get

$$2\|s_n s_{n+1}^{-1}\| - \|s_{n-1} s_n^{-1}\| + 2\|s_{n-1} s_n^{-1}\| + \|s_n s_{n+1}^{-1}\| \leq \kappa \|s_{n-1} s_n^{-1}\|,$$

and so $\|s_n s_{n+1}^{-1}\| \leq \frac{\kappa-1}{4} \|s_{n-1} s_n^{-1}\|$. The sequence $(s_n)_{n=1}^\infty$ is a Cauchy sequence in S and hence, it converges to some $z \in S$. Since $v(s_n)$ also converges to z , we get $\|v(z) z^{-1}\| + \|z v(z)^{-1}\| \leq 0$ which implies $v(z) = z$. \square

Theorem 4.6. *If there exist real numbers a, b and κ such that*

$$0 \leq \kappa + |a| - 2b < 2(a + b);$$

and for all $s, q \in S$,

$$a\|v(s)v(q)^{-1}\| + b[\|sv(s)^{-1}\| + \|qv(q)^{-1}\|] \leq \kappa\|sq^{-1}\|, \tag{4.2}$$

then v has at least one fixed point.

Proof . Let for arbitrary element $s_0 \in S$, a sequence $(s_n)_{n=1}^\infty$ be defined by

$$s_{n+1} = (s_n v(s_n))^{\frac{1}{2}} \quad (n = 0, 1, 2, \dots).$$

If $a \geq 0$, by putting $s = s_{n-1}$ and $q = s_n$ in (4.2), we obtain

$$2a\|s_n s_{n+1}^{-1}\| - |a|\|s_{n-1} s_n^{-1}\| + 2b[\|s_{n-1} s_n^{-1}\| + \|s_n s_{n+1}^{-1}\|] \leq \kappa\|s_n s_{n+1}^{-1}\|.$$

If $a < 0$, by using the inequality

$$\|s_n v(s_n)^{-1}\| + \|s_n v(s_{n-1})^{-1}\| \geq \|v(s_{n-1}) v(s_n)^{-1}\|,$$

we obtain

$$\|s_n s_{n+1}^{-1}\| \leq \lambda \|s_{n-1} s_n^{-1}\|,$$

where $\lambda = \frac{|a|-2b+\kappa}{2(a+b)}$. As $0 \leq \lambda < 1$, the sequence $(s_n)_{n=1}^\infty$ is a Cauchy sequence in S and hence, it converges to some $z \in S$. With n , tending to infinity, we get

$$a\|v(z)z^{-1}\| + b\|zv(z)^{-1}\| \leq 0.$$

Then, as $a + b > 0$, it follows that $v(z) = z$. \square

As a conclusion, in this article, we attempted to prove some fixed points in normed groups to show how capable they are when analytical devices are applied on them. We proved theorems which have counterparts in complete metric spaces, and also theorems which are true specifically for normed groups.

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