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General formulas for solution some important partial differential equations using El-Zaki transform

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Abstract

Our aim in this work, general formula for solution some partial differential equations are derived, which have applications in other sciences. Elzaki transformation was used to obtain these formulas. Moreover, some examples are solved using these formulas.

Keywords: Elzaki transformation, Linear differential equations, Constants coefficients.

1. Introduction

By converting differential and integral equations to algebraic equations, the integral transform has played a significant role in solving differential and integral problems. Furthermore, various integral transformations have been applied in many of the solutions to problems that are difficult to address using traditional methods such as Laplace, Temimi, Novel, Sumudu, and so on [10, 4, 3, 2].

The Laplace transform can be used to solve differential equations which have constants or variable coefficients [13]. It's also used to solve systems of ordinary differential equations [7], and defined as follows:

$$L\left[f\left(t\right)\right]=T\left(v\right)=\int_0^\infty f\left(t\right)e^{-vt}dt,\quad t\geq 0,$$

In 2016, an integral transform known the Novel transform was introduced to solve a large number of differential equations [9, 8]. The function is defined as follows:

$$\gamma\left(s\right)=N_1(y(t))=\frac{1}{s}\int_0^\infty e^{-st}y(t)dt,\ t>0$$

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when $y(t)$ is a real function, $t > 0$, $\frac{e^{-st}}{s}$ is the kernel function.

In this study, we used Elzaki transform [6, 5, 12], for solving some partial differential equations, which have more application in other sciences as Laplace, Transport, Poisson, Heat and Telegraph equations.

Moreover, General formula for solution of these equations are derived using Elzaki transform, whereas, they homogenous or non-homogenous.

The definitions and attributes of the Elzaki transform for several functions are shown in section 2. We got the general formulas for solving the Laplace, Transport, Poisson, Heat and Telegraph equations in section 3. In the concluding section 4, we'll use these formulas to solve some problems [12, 1, 11].

2. Elzaki Transform: Fundamental Definitions and Properties

2.1. The Elzaki transform is described as follows [1]

$$E\left(\gamma(t);\mu\right)=\mu\int_0^{\infty}\gamma\left(\mathrm{t}\right)\ \mathrm{e}^{-\mathrm{t}/\mu}\ \mathrm{dt}=\mathrm{T}\left(\mu\right),\quad \mu\in\left(-\alpha_1,\alpha_2\right),\ \ \alpha_1,\alpha_2>0$$

The set ϖ is also defined as follows:

$$\varpi=\{\gamma\left(\mathrm{t}\right);\exists\mathrm{N},\alpha_1,\alpha_2>0\text{ such that }\left|\gamma\left(\mathrm{t}\right)\right|<\mathrm{N}\mathrm{e}^{|\mathrm{t}|/\alpha_{\mathrm{i}}}\text{ if }\mathrm{t}\in(-1)^{\mathrm{i}}\times[0.\infty)\},$$

α_1 and α_2 can be finite or infinite, but N is constant must be finite.

The inverse of Elzaki transform is define by: $E^{-1}(T(M)) = \gamma(t)$, where (E^{-1}) returning the transform to the original function.

2.2. Properties

The ELzaki transform of the function $\pounds(t)$ is defined as:

$$E[\pounds(t)]=T(\mu)=\mu\int_0^{\infty}\pounds\left(t\right)\mathrm{e}^{-\mathrm{t}/\mu}\mathrm{dt},\quad \mathrm{t}>0,\ \ \mu\in(-\ k_1,k_2).$$

We employ integration by parts to derive the ELzaki transform of partial derivatives as follows [6]:

$$\begin{aligned}E\left[\frac{\partial\pounds}{\partial t}\left(x,t\right)\right]&=\int_0^{\infty}\mu\frac{\partial\pounds}{\partial t}\ \mathrm{e}^{-\mathrm{t}/\mu}\mathrm{dt}=\lim_{a\to\infty}\int_0^{a}\mu\ \ \mathrm{e}^{-\mathrm{t}/\mu}\frac{\partial\pounds}{\partial t}\ dt\\&=\lim_{a\to\infty}\left[\left[\mu\mathrm{e}^{-t/\mu}\pounds\left(x,t\right)\right]\Big|_0^a-\int_0^a\mathrm{e}^{-t/\mu}\pounds\left(x,t\right)\mathrm{d}t\right]\\&=\frac{\mathrm{T}\left(x,\mu\right)}{\mu}-\mu\pounds\left(x,0\right).\end{aligned}$$

We consider $\pounds$ to be piecewise continuous and of exponential order:

$$E\left[\frac{\partial\pounds}{\partial x}\right]=\int_0^{\infty}\mu\mathrm{e}^{-t/\mu}\frac{\partial\pounds}{\partial x}\left(x,t\right)\mathrm{d}t=\frac{\partial}{\partial x}\int_0^{\infty}\mu\mathrm{e}^{-t/\mu}\pounds\left(x,t\right)\mathrm{d}t=\frac{\partial}{\partial x}\left[\mathrm{T}\left(x,\mu\right)\right],$$

and

$$E\left[\frac{\partial\pounds}{\partial x}\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{T}\left(x,\mu\right)\right].$$

Table 1: Elzaki transformation for some functions

ID	Function $\gamma(t)$	$E(\gamma(t)) = \mu \int_0^\infty \gamma(t)e^{-t/\mu}dt = T(\mu)$
1	1	μ^2
2	t^n	$n!\mu^{n+2}$
3	e^{at}	$\frac{\mu^2}{1-a\mu}$
4	$\sin(at)$	$\frac{a\mu^3}{1+a^2\mu^2}$
5	$\cos(at)$	$\frac{\mu^2}{1+a^2\mu^2}$
6	$\frac{t^{n-1}e^{at}}{(n-1)!}$, $n = 1, 2, ...$	$\frac{\mu^{n+1}}{(1-a\mu)^n}$

Also we can find:

$$\mathrm{E}\left[\frac{\partial^2 \pounds}{\partial x^2}\right] = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\mathrm{T}\left(x,\mu\right)\right].$$

To find $\mathrm{E}\left[\frac{\partial^2 \pounds}{\partial t^2}(x,t)\right]$, assume that $\frac{\partial \pounds}{\partial t} = h$. By using the equation

$$\lim_{a\to\infty}\left[\left[\mu \mathrm{e}^{-t/\mu}\pounds\left(x,t\right)\right]\Big|_0^a - \int_0^a \mathrm{e}^{-t/\mu}\pounds\left(x,t\right)dt\right] = \frac{\mathrm{T}\left(x,\mu\right)}{\mu} - \mu\pounds\left(x,0\right),$$

we have,

$$\begin{aligned}\mathrm{E}\left[\frac{\partial^2 \pounds}{\partial t^2}(x,t)\right] &= \mathrm{E}\left[\frac{\partial h}{\partial t}(x,t)\right] \\ &= \mathrm{E}\left[\frac{h(x,t)}{\mu}\right] - \mu h(x,0) \\ &= \frac{1}{\mu^2}\,\mathrm{T}\left(x,\mu\right) - \pounds\left(x,0\right) - \mu\frac{\partial \pounds}{\partial t}(x,0).\end{aligned}$$

Using the mathematical induction, we can readily extend this result to the n-th partial derivative.

3. General Formulas of Laplace, Poisson, Transport, Heat and Telegraph Equations

Formula(1): Consider the following Laplace equation:

$$Ł_{tt}(x,t) + Ł_{xx}(x,t) = 0, \tag{3.1}$$

with the initial conditions $Ł(x,0) = \eth(x)$, $Ł_t(x,0) = \mathfrak{J}(x)$ and $Ł(0,t) = Ł(1,t) = 0$. By taking Elzaki transform to the both sides, we have

$$\frac{T(x,\mu)}{\mu^2} - Ł(x,0) - Ł_t\mu(x,0) + \frac{\mathrm{d}^2}{dx^2}T(x,\mu) = 0.$$

After substituting the initial conditions:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}T(x,\mu) + \frac{\mathrm{T}\left(\mathrm{x},\mu\right)}{\mu^2} = \eth\left(\mathrm{x}\right) + \mu\,\mathfrak{J}\left(\mathrm{x}\right) \tag{3.2}$$

Note that (3.2) is an ordinary differential equation and has the solution:

$$T(x,\mu)=\left[h_1\cos\left(\frac{1}{\mu}\right)+h_2\sin\left(\frac{1}{\mu}\right)x\right]+$$
$$\left[\left(-\mu\int((\eth(x)+\mu\ \mathfrak{J}(x)))\sin\left(\frac{1}{\mu}\right)x\mathrm{d}x\right)\cos\left(\frac{1}{\mu}\right)x+\left(\mu\int((\eth(x)+\mu\mathfrak{J}(x)))\cos\left(\frac{1}{\mu}\right)x\mathrm{d}x\right)\sin\left(\frac{1}{\mu}\right)x\right]$$

Utilizing the boundary conditions, yields: $h_1=h_2=0$, so

$$T(x,\mu)=\left[\left(-\mu\int((\eth(x)+\mu\mathfrak{J}(\mathrm{x})))\sin(\frac{1}{\mu})x\mathrm{d}x\right)\cos(\frac{1}{\mu})x+\left(\mu\int((\eth(x)+\mu\mathfrak{J}(x)))\cos(\frac{1}{\mu})x\mathrm{d}x\right)\sin(\frac{1}{\mu})x\right].$$

The general solution of equation (3.1) obtained by taking the inverse of both sides is as follows:

$$T(x,t)=E^{-1}\left[(-\mu\int((\eth(x)+\mu\mathfrak{J}(x)))\sin(\frac{1}{\mu})x\mathrm{d}x)\cos(\frac{1}{\mu})x+(\mu\int((\eth(x)+\mu\mathfrak{J}(x)))\cos(\frac{1}{\mu})x\mathrm{d}x)\sin(\frac{1}{\mu})x\right]. \tag{3.3}$$

Formula(2): Consider the Poisson equation

$$Ł_{tt}(x,t)+Ł_{xx}(x,t)=\pounds(x,t). \tag{3.4}$$

Using Elzaki transform to equation (3.4) and applying formula (3.3), yields:

$$\begin{aligned}T(x,\mu)=E^{-1}\Big[&(-\mu\int((\mathrm{E}(\pounds(x,t))+\eth(x)+\mu\mathfrak{J}(x)))\sin(\frac{1}{\mu})x\mathrm{d}x)\cos(\frac{1}{\mu})x\\&+(\mu\int((E(\pounds(x,t))+\eth(x)+\mu\mathfrak{J}(x)))\cos(\frac{1}{\mu})x\mathrm{d}x)\sin(\frac{1}{\mu})x\Big].\end{aligned} \tag{3.5}$$

Formula(3): Consider the following homogeneous transport equation:

$$Ł_t+\eta Ł_x=0, \tag{3.6}$$

under the conditions $Ł(x,0)=\eth(x), Ł(0,t)=Ł(1,t)=0$, and η is constant. Using Elzaki transform to the both sides and substitution the initial conditions:

$$\frac{d}{dx}T\left(x,\mu\right)+\frac{1}{\mu\eta}T\left(x,\mu\right)=\frac{\mu}{\eta}\eth\left(\mathrm{x}\right), \tag{3.7}$$

The above equation (3.7) is an ordinary linear differential equation and has the following solution:

$$T\left(x,\mu\right)=\frac{1}{\eta}\mathrm{e}^{\frac{-1}{\eta\mu}x}\lim_{a\to-\infty}\int_a^x\mu\eth(x)\mathrm{e}^{\frac{1}{\eta\mu}\tau}\mathrm{d}\tau.$$

We get the general solution of equation (3.6), by taking the inverse of Elzaki transform as follows:

$$Ł(x,t)=\frac{\mu}{\eta}E^{-1}\left[\mathrm{e}^{\frac{-1}{\eta\mu}x}\int\eth(x)\mathrm{e}^{\frac{1}{\eta\mu}t}\mathrm{d}t\right] \tag{3.8}$$

Formula(4): Consider the following non-homogeneous transport equation:

$$Ł_t+\eta Ł_x=\pounds(x,t), \tag{3.9}$$

under the conditions $Ł(x,0) = ð(x), Ł(0,t) = Ł(1,t) = 0$, where $ð$ and η are constants. Using Elzaki transform and utilizing the pervious formula with equation (3.9), we get

$$T(x,t) = \frac{1}{\eta}E^{-1}\left[e^{\frac{-1}{\eta\mu}x}\int E(£(x,t)e^{\frac{1}{\eta\mu}t}dt\right] + \frac{\mu}{\eta}E^{-1}\left[e^{\frac{-1}{\eta\mu}x}\int ð(x)e^{\frac{1}{\eta\mu}t}dt\right]. \tag{3.10}$$

Formula(5): Consider the homogeneous heat equation:

$$Ł_t(x,t) - \eta Ł_{xx}(x,t) = 0, \tag{3.11}$$

with the initial and boundary conditions $Ł(x,0) = ð(x), Ł(0,t) = Ł(1,t) = 0$, and η is constant. We take Elzaki transform and substitute the initial conditions:

$$\frac{d^2}{dx^2}T(x,\mu) - \frac{1}{\eta\mu}T(x,\mu) = -\frac{\mu}{\eta}ð(x), \tag{3.12}$$

Equation (3.12) is an ordinary differential equation which has the following solution:

$$T(x,\mu) = \left(h_1 e^{\sqrt{\frac{1}{\eta\mu}}x} + h_2 e^{-\sqrt{\frac{1}{\eta\mu}}x}\right) + \frac{-1}{2\eta\sqrt{\frac{1}{\eta\mu}}}\left(\int \mu ð(x)e^{-\sqrt{\frac{1}{\eta\mu}}x}dx\right)e^{\sqrt{\frac{1}{\eta\mu}}x}$$
$$+ \frac{1}{2\eta\sqrt{\frac{1}{\eta\mu}}}\left(\int \mu ð(x)e^{\sqrt{\frac{1}{\eta\mu}}x}dx\right)e^{-\sqrt{\frac{1}{\eta\mu}}x}$$

Since $T(x,\mu)$ is bounded, we have $h_1 = h_2 = 0$ and so

$$T(x,\mu) = \frac{-1}{2\eta\sqrt{\frac{1}{\eta\mu}}}\left(\int \mu ð(x)e^{-\sqrt{\frac{1}{\eta\mu}}x}dx\right)e^{\sqrt{\frac{1}{\eta\mu}}x} + \frac{1}{2\eta\sqrt{\frac{1}{\eta\mu}}}\left(\int \mu ð(x)e^{\sqrt{\frac{1}{\eta\mu}}x}dx\right)e^{-\sqrt{\frac{1}{\eta\mu}}x}.$$

We get the general solution of equation (3.11) by taking the inverse of both sides as follows:

$$T(x,t) = E^{-1}\left[\frac{-1}{2\eta\sqrt{\frac{1}{\eta\mu}}}\left(\int \mu ð(x)e^{-\sqrt{\frac{1}{\eta\mu}}x}dx\right)e^{\sqrt{\frac{1}{\eta\mu}}x} + \frac{1}{2\eta\sqrt{\frac{1}{\eta\mu}}}\left(\int \mu ð(x)e^{\sqrt{\frac{1}{\eta\mu}}x}dx\right)e^{-\sqrt{\frac{1}{\eta\mu}}x}\right]. \tag{3.13}$$

Formula(6): Consider the following non-homogeneous heat equation:

$$Ł_t(x,t) - \eta Ł_{xx}(x,t) = £(x,t), \tag{3.14}$$

with the conditions $Ł(x,0) = ð(x), Ł(0,t) = Ł(1,t) = 0$, where $ð$ and η are constants. Using Elzaki transform with equation (3.14), we have

$$T(x,t) = E^{-1}\left[\frac{-1}{2\eta\sqrt{\frac{1}{\eta\mu}}}\left(\int [E(£(x,t)) + \mu ð(x)]e^{-\sqrt{\frac{1}{\eta\mu}}x}dx\right)e^{\sqrt{\frac{1}{\eta\mu}}x}\right.$$
$$\left. + \frac{1}{2\eta\sqrt{\frac{1}{\eta\mu}}}\left(\int [E(£(x,t)) + \mu ð(x)]e^{\sqrt{\frac{1}{\eta\mu}}x}dx\right)e^{-\sqrt{\frac{1}{\eta\mu}}x}\right] \tag{3.15}$$

Formula(7): Consider the following homogeneous telegraph equation:

$$Ł_{tt}(x,t)+2\partial Ł_t(x,t)+\eta^2 Ł(x,t)=Ł_{xx}(x,t), \tag{3.16}$$

with the conditions $Ł(x,0)=ð(x), Ł_t(x,0)=\mathfrak{J}(x), Ł(0,t)=Ł(1,t)=0$, such that η and ∂ are constants. By applying Elzaki transform to the both sides and substitute the initial conditions, imply that

$$\frac{d^2}{dx^2}\mathrm{T}(x,\mu)-\frac{\mathrm{T}(x,\mu)}{\mu^2}-2\partial\frac{\mathrm{T}(x,\mu)}{\mu}-\eta^2\mathrm{T}(x,\mu)=-ð(x)-\mu\mathfrak{J}(x)-2\partial\mu ð(x). \tag{3.17}$$

Equation (3.17) is an ordinary differential equation and has the following general solution:

$$\begin{aligned}\mathrm{T}(x,\mu)=&h_1(x)e^{\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}+h_2(x)e^{-\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}\\&+\frac{1}{2(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})}\left[\int[(-2\partial\mu-1)ð(x)-\mu\mathfrak{J}(x)]e^{-\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}dx\right]e^{\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}\\&-\frac{1}{2(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})}\left[\int[(-2\partial\mu-1)ð(x)-\mu\mathfrak{J}(x)]e^{\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}dx\right]e^{-\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}\end{aligned}$$

Since $T(x,\mu)$ is bounded, we have $h_1=h_2=0$ and so

$$\begin{aligned}\mathrm{T}(x,t)=&E^{-1}\left[\frac{1}{2(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})}\left[\int[(-2\partial\mu-1)ð(x)-\mu\mathfrak{J}(x)]e^{-\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}dx\right]e^{\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}\right.\\&\left.-\frac{1}{2(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})}\left[\int[(-2\partial\mu-1)ð(x)-\mu\mathfrak{J}(x)]e^{\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}dx\right]e^{-\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}\right].\end{aligned}$$

The general solution of equation (3.16) by the inverse Elzaki transform is as follows:

$$\begin{aligned}\mathrm{T}(x,t)=&E^{-1}\left[\frac{1}{2(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})}\left[\int[(-2\partial\mu-1)ð(x)-\mu\mathfrak{J}(x)]e^{-\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}dx\right]e^{\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}\right.\\&\left.-\frac{1}{2(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})}\left[\int[(-2\partial\mu-1)ð(x)-\mu\mathfrak{J}(x)]e^{\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}dx\right]e^{-\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}\right]\end{aligned} \tag{3.18}$$

Formula(8): Consider the following non- homogeneous Telegraph equation:

$$Ł_{tt}(x,t)+2\partial Ł_t(x,t)+\eta^2 Ł(x,t)=Ł_{xx}+(£(x,t)), \tag{3.19}$$

with the conditions $Ł(x,0)=ð(x), Ł_t(x,0)=\mathfrak{J}(x), Ł(0,t)=Ł(1,t)=0$, η and ∂ are constant. By applying Elzaki transform to the pervious formula, we get

$$\mathrm{T}(x,t)=$$

$$E^{-1}\left[\frac{1}{2(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})}\left[\int[(-2\partial\mu-1)\eth(x)-\mu\mathfrak{J}(x)-E(\pounds(x,t))]\mathrm{e}^{-\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}\mathrm{d}x\right]\mathrm{e}^{\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}\right.$$

$$\left.-\frac{1}{2(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})}\left[\int[(-2\partial\mu-1)\eth(x)-\mu\mathfrak{J}(x)-E(\pounds(x,t))]\mathrm{e}^{\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}\mathrm{d}x\right]\mathrm{e}^{-\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}\right]. \tag{3.20}$$

4. Applications

In this section the transform efficiency in finding solution of partial differential equations by applying the previous formulas in solving the following examples is shown:

Example 4.1. *Consider the Laplace equation:*

$$Ł_{tt}(x,t)+Ł_{xx}(x,t)=0 \tag{4.1}$$

with the conditions $Ł(x,0)=0, Ł_t(x,0)=x$ *and* $Ł(0,t)=Ł(1,t)=0$. *Using the equation* (3.3) *in the formula* (4.1), *we have*

$$\begin{aligned}\mathrm{T}(x,\mu)=&E^{-1}\left[\left(-\mu\int((\eth(x)+\mu\mathfrak{J}(x)))\sin(\frac{1}{\mu})x\mathrm{d}x\right)\cos(\frac{1}{\mu})x\right.\\&\left.+\left(\mu\int((\eth(x)+\mu\mathfrak{J}(x)))\cos(\frac{1}{\mu})x\mathrm{d}x\right)\sin(\frac{1}{\mu})x\right]\end{aligned}$$

$$\begin{aligned}\pounds(x,t)&=\mathrm{T}(x,t)=E^{-1}\left[\mu^3x\cos^2(\frac{1}{\mu})x-\mu^4\sin(\frac{1}{\mu})x\cos(\frac{1}{\mu})x+\mu^3x\sin^2(\frac{1}{\mu})x+\mu^4\sin(\frac{1}{\mu})x\cos(\frac{1}{\mu})x\right]\\&=xE^{-1}(\mu^3)=xt.\end{aligned}$$

which represents the particular solution of equation (4.1).

Example 4.2. *Consider the following Poisson equation*

$$\pounds_{tt}(x,t)+\pounds_{xx}(x,t)=\mathrm{e}^t \tag{4.2}$$

with the conditions $Ł(x,0)=0, Ł_t(x,0)=1$ *and* $Ł(0,t)=Ł(1,t)=0$. *Then by applying the equation* (3.5) *on the formula* (4.2), *we obtain that*

$$\begin{aligned}\mathrm{T}(x,\mu)=&E^{-1}\left[\left(-\mu\int((E(\pounds(x,t))+\eth(x)+\mu\mathfrak{J}(x)))\sin(\frac{1}{\mu})x\mathrm{d}x\right)\cos(\frac{1}{\mu})x\right.\\&\left.+\left(\mu\int((E(\pounds(x,t))+\eth(x)+\mu\mathfrak{J}(x)))\cos(\frac{1}{\mu})x\mathrm{d}x\right)\sin(\frac{1}{\mu})x\right]\\\pounds(x,t)=&\mathrm{T}(x,t)=E^{-1}\left[\frac{\mu^2}{1-\mu}\cos^2(\frac{1}{\mu})x+\frac{\mu^2}{1-\mu}\sin^2(\frac{1}{\mu})x\right]\\=&E^{-1}\left[\frac{\mu^2}{1-\mu}\right]=\mathrm{e}^t\end{aligned}$$

which represents the solution of equation (4.2).

Example 4.3. *Consider the homogeneous transport equation*

$$u_x + 3u_t = 0 \tag{4.3}$$

with the conditions $Ł(x,0) = \mathrm{e}^x$ *and* $Ł(0,t) = Ł(1,t) = 0$. *By employing the equation* (3.8) *to the formula* (4.3), *we have*

$$\mathrm{T}(x,t) = \frac{\mu}{\eta}E^{-1}\left[\mathrm{e}^{\frac{-1}{\eta\mu}x}\int ð(x)\mathrm{e}^{\frac{1}{\eta\mu}t}\mathrm{d}t\right]$$
$$\mathrm{T}(x,t) = E^{-1}\left[\frac{3\mu^2}{3+\mu}\mathrm{e}^x\right]$$
$$u(x,t) = \mathrm{T}(x,t) = \mathrm{e}^x\mathrm{e}^{\frac{-1}{3}t}$$

which represents the particular solution of equation (4.3).

Example 4.4. *Consider the non- homogeneous transport equation*

$$2u_x + u_t = 6 \tag{4.4}$$

with the conditions $u(x,0) = \mathrm{e}^{2x}$ *and* $u(0,t) = u(1,t) = 0$. *Employing equation* (3.10) *to* (4.4) *implies that*

$$u(x,t) = \frac{1}{\eta}E^{-1}\left[e^{\frac{-1}{\eta\mu}x}\int E(£(x,t))\mathrm{e}^{\frac{1}{\eta\mu}t}\mathrm{d}t\right] + \frac{\mu}{\eta}E^{-1}\left[\mathrm{e}^{\frac{-1}{\eta\mu}x}\int ð(x)\mathrm{e}^{\frac{1}{\eta\mu}t}\mathrm{d}t\right] = 6t + \mathrm{e}^{2x}\mathrm{e}^{-4t}$$

which represents the particular solution of equation (4.4).

Example 4.5. *Consider the homogeneous heat equation*

$$Ł_t(x,t) - Ł_{xx}(x,t) = 0 \tag{4.5}$$

with the conditions $Ł(x,0) = \mathrm{e}^{5x}, Ł(0,t) = Ł(1,t) = 0$. *Then by applying the equation* (3.13) *on* (4.5), *we have*

$$\mathrm{T}(x,t) = E^{-1}\left[\frac{-1}{2\eta\sqrt{\frac{1}{\eta\mu}}}\left(\int \mu ð(x)\mathrm{e}^{-\sqrt{\frac{1}{\eta\mu}}x}\mathrm{d}x\right)\mathrm{e}^{\sqrt{\frac{1}{\eta\mu}}x} + \frac{1}{2\eta\sqrt{\frac{1}{\eta\mu}}}\left(\int \mu ð(x)\mathrm{e}^{-\sqrt{\frac{1}{\eta\mu}}x}\mathrm{d}x\right)\mathrm{e}^{-\sqrt{\frac{1}{\eta\mu}}x}\right]$$
$$Ł(x,t) = \mathrm{T}(x,t) = E^{-1}\left[\frac{\mu^2}{1-25\mu}\mathrm{e}^{5x}\right] = \mathrm{e}^{5x}\mathrm{e}^{25t}.$$

which represents the particular solution of equation (4.5).

Example 4.6. *Consider the non-homogeneous heat equation*

$$Ł_t(x,t) - Ł_{xx}(x,t) = \mathrm{e}^{x+t} \tag{4.6}$$

with the conditions $Ł(x,0) = \mathrm{e}^x, Ł(0,t) = Ł(1,t) = 0$. *By applying* (3.15) *on* (4.6), *we get*

$$\mathrm{T}(x,t) = E^{-1}\left[\frac{\mathrm{e}^{-\sqrt{\frac{1}{\eta\mu}}x}}{2\eta\sqrt{\frac{1}{\eta\mu}}}\left(-\int[E(£(x,t)) + \mu ð(x)]\mathrm{d}x + \int[E(£(x,t)) + \mu ð(x)]\mathrm{e}^{\sqrt{\frac{1}{\eta\mu}}x}\mathrm{d}x\right)\right]$$
$$Ł(x,t) = \mathrm{T}(x,t) = E^{-1}\left[\frac{-\mu^3\mathrm{e}^x}{-(1-\mu)(1-\mu)} + \frac{\mu^2\mathrm{e}^x}{1-\mu}\right] = \mathrm{e}^x(t\mathrm{e}^t) + \mathrm{e}^x\mathrm{e}^t.$$

which represents the particular solution of equation (4.6).

Example 4.7. *Consider the homogeneous Telegraph equation:*

$$Ł_{tt}(x,t) + 2\partial Ł_t(x,t) + \eta^2 Ł(x,t) = Ł_{xx} \tag{4.7}$$

with the conditions $Ł(x,0) = \mathrm{e}^x, Ł_t(x,0) = e^x$ *and* $Ł(0,t) = Ł(1,t) = 0$. *Using the equation* (3.20) *on* (4.7), *where* $\partial = -\frac{1}{2}$ *and* $\eta = 1$, *we have*

$$\begin{aligned}
\mathrm{T}(x,t) = E^{-1}\Bigg[&\frac{1}{2(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})}\left[\int[(-2\partial\mu-1)\eth(x) - \mu\mathfrak{J}(x)]\mathrm{e}^{-(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})x}\mathrm{d}x\right]\mathrm{e}^{(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})x} \\
&- \frac{1}{2(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})}\left[\int[(-2\partial\mu-1)\eth(x) - \mu\mathfrak{J}(x)]\mathrm{e}^{(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})x}\mathrm{d}x\right]\mathrm{e}^{-(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})x}\Bigg] \\
Ł(x,t) = \mathrm{T}(x,t) = E^{-1}&\left[\frac{\mu^2}{1-\mu}\mathrm{e}^x\right] = \mathrm{e}^x\mathrm{e}^t.
\end{aligned}$$

which represents the particular solution of equation (4.7).

Example 4.8. *Consider the non-homogeneous telegraph equation:*

$$Ł_{tt}(x,t) + 2\partial Ł_t(x,t) + \eta^2 Ł(x,t) = Ł_{xx} + \mathrm{e}^t \sinh(x) \tag{4.8}$$

with the conditions $Ł(x,0) = 0, Ł_t(x,0) = \sinh(x)$ *and* $Ł(0,t) = Ł(1,t) = 0$. *Applying the equation* (3.18) *on* (4.8), *where* $\partial = -\frac{1}{2}$ *and* $\eta = 1$ *imply that*

$$\begin{aligned}
\mathrm{T}(x,t) = E^{-1}\Bigg[&\frac{\mathrm{e}^{\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}}{2(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})}\left[\int[(-2\partial\mu-1)\eth(x) - \mu\mathfrak{J}(x) - E(\pounds(x,t))]\mathrm{e}^{-(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})x}\mathrm{d}x\right] \\
&- \frac{\mathrm{e}^{-(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})x}}{2(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2})}\left[\int[(-2\partial\mu-1)\eth(x) - \mu\mathfrak{J}(x) - E(\pounds(x,t))]\mathrm{e}^{\left(\sqrt{\frac{1}{\mu^2}+\frac{2\partial}{\mu}+\eta^2}\right)x}\mathrm{d}x\right]\Bigg] \\
Ł(x,t) = \mathrm{T}(x,t) = E^{-1}&\left[\frac{\mu^3}{(1-\mu)^2}\sinh(x)\right] = t\mathrm{e}^t(\sinh(x)),
\end{aligned}$$

which represents the particular solution of equation (4.8).

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