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Wavelet analytical method on the Heston option pricing model

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Abstract

In this paper, the Heston partial differential equation option pricing model is considered and the Legendre wavelet method (LWM) is used to solve this equation. The attributes of Legendre wavelets are used to reduce the PDEs problem into the solution of the ODEs system. The wavelet base is used in approximation due to its simplicity and efficiency. The method of creating Legendre wavelets and their main properties were briefly mentioned. Some numerical schemes have been compared with the LWM in the result.

Keywords: partial and stochastic differential equation, Heston model, Legendre wavelet method
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1 Introduction

Many problems in real world are modeled as stochastic form of partial differential equation or integral differential equation. Since finding the solution of these equations is complicated, in recent years a lot of attention has been devoted by researchers to find numerical solution of these equations. In financial topics, Heston (1993)[11], introduced one of popular stochastic volatility option pricing model. Which is known after his name, by Heston model. This model is based on the stock price and variance dynamics,

$$\begin{cases} \dfrac{dS(t)}{S(t)} = rd(t) + \sqrt{V(t)}d\widehat{W}_1(t) \\ \dfrac{dV(t)}{S(t)} = (a - bV(t))d(t) + \sigma\sqrt{V(t)}d\widehat{W}_2(t). \end{cases} \tag{1.1}$$

where r is rate of interests a,b are parameters and σ positive constant, and $d\widehat{W}_1(t)$, $d\widehat{W}_2(t)$ are correlated Brownian motions under the risk-neutral measure with the correlation coefficient $\rho \in (-1, 1)$[2, 3].

Derivative products models in mathematical finance usually begin with a system of stochastic differential equations that correspond to state variables same stock, interest rate and volatility. A SV model with associated price and volatility innovations can address both experiential stylized realities. The SV option pricing model was extended with a series of participations from Johnson and Shanno (1987), Wiggins (1987), Hull and White (1987, 1988), Scott (1987), Stein and Stein (1991) and Heston (1993). It was in Heston (1993) that a semi-closed model solution was derived based on the characteristic function of the price distribution [4, 5, 6].

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In this paper, the Heston PDE model that has derived from the stochastic model was applied with the Legendre wavelets method. The Heston SDE model has converted to a partial differential equation with an essential lemma in the stochastic differential equation which is called Ito lemma that includes derivatives and integration in stochastic arithmetic. Considering the properties and using the structure of wavelets for a given problem leads to a decrease the time.

The LWM converts a boundary value problem (BVP) into a system of algebraic equations [16]. In this problem, the vector whose components are the decomposition coefficients of the BVP solution into Legendre wavelets basis is an unknown parameter and we use the Legendre wavelets method to solve a partial differential equation. Specifically, the coefficients of this parsing will depend on the temporal variable. Hence, via this technique, the solution of a partial differential equation is reduced to the solution of a time-dependent differential equation.

2 Stochastic Volatility Model

If Black-Scholes (BS) [1], is the correct option pricing model, then there can only be one BS implied volatility regardless of the strike price of the option. This raises the basic concepts about the relationship between BS implied volatility and true volatility. In the stochastic volatility option pricing models Heston (1993), is the most important and motivated by the widespread evidence that volatility is stochastic and that the distribution of risky asset changes has a tail(s) longer than that of a normal distribution. An SV model with correlated price and volatility innovations can address both experimental stylized realities. The Heston formula has two bases of randomness, the bivariate Itos lemma is used to derive the basic partial differential equation. The levels that consist of the derivation of the Heston option pricing equation are the same as those in the no-arbitrage derivation for the Black-Scholes formula except that two derivative assets are required to obtain a risk-neutral portfolio. Let us rewrite Eq (1.1) with call option C together with positions in δ units of the underlying asset and γ units of a second derivative C_1 writing in the same underlying. C_1 differs from C with their maturity or strike price[11, 13].

$$\begin{cases} dS = \mu_s dt + \sigma_s dZ_1 \\ dV = \mu_v dt + \sigma_v dZ_2 \end{cases} \tag{2.1}$$

which $C(S, V, t)$ denote the price of a call option, from the bivariate Ito's lemma dynamic C may be written as

$$dC = [\frac{1}{2}\sigma_S^2 \frac{\partial^2 C}{\partial S^2} + \rho\sigma_S\sigma_V \frac{\partial^2 C}{\partial S \partial V} + \frac{1}{2}\sigma_V^2 \frac{\partial^2 C}{\partial V^2} + \mu_S \frac{\partial C}{\partial S} + \mu_V \frac{\partial C}{\partial V} + \frac{\partial C}{\partial t} + \sigma_S \frac{\partial C}{\partial S} dZ_1 + \sigma_V \frac{\partial C}{\partial V} dZ_2. \tag{2.2}$$

Value of portfolio

$$W = C - \delta S - \gamma C_1.$$

$$dW = dC - \delta dS - \gamma dC = \tag{2.3}$$

$$[\frac{1}{2}\sigma_S^2 \frac{\partial^2 C}{\partial S^2} + \rho\sigma_S\sigma_V \frac{\partial^2 C}{\partial S \partial V} + \frac{1}{2}\sigma_V^2 \frac{\partial^2 C}{\partial V^2} + \mu_S \frac{\partial C}{\partial S} + \mu_V \frac{\partial C}{\partial V} + \frac{\partial C}{\partial t} - \delta\mu_S]dt -$$

$$\gamma[\frac{1}{2}\sigma_S^2 \frac{\partial^2 C_1}{\partial S^2} + \rho\sigma_S\sigma_V \frac{\partial^2 C_1}{\partial S \partial V} + \frac{1}{2}\sigma_V^2 \frac{\partial^2 C_1}{\partial V^2} + \mu_S \frac{\partial C_1}{\partial S} + \mu_V \frac{\partial C_1}{\partial V} + \frac{\partial C_1}{\partial t}]dt +$$

$$[\sigma_S \frac{\partial C}{\partial S} - \delta\sigma_S - \gamma\sigma_S \frac{\partial C_1}{\partial S}]dZ_1 + [\sigma_V \frac{\partial C}{\partial V} - \delta\sigma_V - \gamma\sigma_V \frac{\partial C_1}{\partial V}]dZ_2.$$

The coefficients of dZ_1 and dZ_2 to obtain risk neutrality, must be zero. This means that

$$\frac{\partial C}{\partial S} = \delta + \gamma \frac{\partial C_1}{\partial S}, \tag{2.4}$$

$$\frac{\partial C}{\partial V} = \gamma \frac{\partial C_1}{\partial V}.$$

With conditions of (2.4), the change of the value of the portfolio must be equal to the return on a risk-free investment. Otherwise, there will be an arbitrage opportunity. Then,

$$dW = r[C - \delta S - \gamma C_1]dt. \tag{2.5}$$

If we equate (2.3) and (2.5), and substitute the values of and from (2.4), hence

$$[\frac{1}{2}\sigma_S^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma_S\sigma_V\frac{\partial^2 C}{\partial S\partial V} + \frac{1}{2}\sigma_V^2\frac{\partial^2 C}{\partial V^2} + \mu_S\frac{\partial C}{\partial S} + \mu_V\frac{\partial C}{\partial V} + \frac{\partial C}{\partial t} - rC]/\frac{\partial C}{\partial V} \tag{2.6}$$
$$[\frac{1}{2}\sigma_S^2\frac{\partial^2 C_1}{\partial S^2} + \rho\sigma_S\sigma_V\frac{\partial^2 C_1}{\partial S\partial V} + \frac{1}{2}\sigma_V^2\frac{\partial^2 C_1}{\partial V^2} + \mu_S\frac{\partial C_1}{\partial S} + \mu_V\frac{\partial C_1}{\partial V} + \frac{\partial C_1}{\partial t} - rC_1]/\frac{\partial C_1}{\partial V}.$$

The same equation must hold for any type of call option of any maturity and strike price. Each side will be equal to some function $\lambda(S, V, t)$ that depends on S and V. This function may be explained as a volatility risk premium. Replacing the parameters by their actual units, we see that the fundamental partial differential equation is now

$$0 = \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho\eta V\frac{\partial^2 C}{\partial S\partial V} + \frac{1}{2}\eta^2 V\frac{\partial^2 C}{\partial V^2} + rS\frac{\partial C}{\partial S}[k(\theta - V) - \tag{2.7}$$
$$\lambda(S, V, t)]\frac{\partial C}{\partial V} - rC + \frac{\partial C}{\partial t}.$$

Heston considered the assumption that the volatility risk premium is a linear function of V_t, such that $\lambda(S, V, t)$. Moreover, introducing $x = \ln S$ or $S = e^x$. Substitute the results into Eq (2.7), then

$$0 = \frac{1}{2}V_t\frac{\partial^2 C}{\partial x^2} + \rho\eta V_t\frac{\partial^2 C}{\partial x\partial V} + \frac{1}{2}\eta^2 V_t\frac{\partial^2 C}{\partial V^2} + [k(\theta - V_t) - \lambda V_t]\frac{\partial C}{\partial V} + \tag{2.8}$$
$$(r - \frac{1}{2}V_t)\frac{\partial C}{\partial x} - rC + \frac{\partial C}{\partial t}.$$

Point view different of Eq (2.7) and Eq (2.8) is that the coefficient of the partial derivatives does not contain S (or x) making the PDE a lot easier to solve. In the case of a European Call option, we have the following boundary conditions:

$$C(S_T, V, T) = max(S_T - K, 0), \tag{2.9}$$
$$C(0, V, t) = 0,$$
$$\frac{\partial C}{\partial S_t}(\infty, V, t) = 1.$$

3 Legendre Wavelets Method

In this part, some necessary mathematical preliminaries which are used in the Legendre wavelet are given. The well-known Legendre polynomials are defined on the interval $[-1, 1]$ and it can be determined with the following recurrent formula [15, 14, 12].

$$(m + 1)L_{m+1}(t) = (2m + 1)L_m(t) - (m)L_{m-1}(t), \quad m = 1, 2, 3, ... \tag{3.1}$$

where $P_n(x) = 1$ and $P_t(x) = 2x - 1$. The wavelet basis is constructed from a single function, which is called the mother wavelet. These basis functions are called wavelets and they are an orthonormal set. One of the most important wavelets is Legendre wavelets. The Legendre wavelets are obtained from Legendre polynomials. In the past decade, special attention has been given to applications of wavelets. The main characteristic of the Legendre wavelet is that it reduces to a system of an algebraic equation.

The function $\psi(x) \in L^2(R)$ is a mother wavelet and the $\psi_{u,v}(x) = |\ u|^{\frac{-1}{2}} \psi(\frac{x-v}{u})$ which $u, v \in R$ and $u \neq 0$, are the family continuous wavelets. If we choose the dilation parameter $u = a^{-n}$ and the translation parameter $v = ma^{-n}b$, where $a > 1$ and $b > 0$ and n, m are positive integer, let us consider the following set of discrete orthogonal wavelets

$$\{\psi_{n,m}(x) = |\ u|^{\frac{n}{2}} \psi(a^n x - mb) : m, n \in Z\}.$$

The Legendre wavelet is constructed from the Legendre function. The Legendre functions satisfy the Legendre differential equation [10, 16]. One dimension Legendre wavelets over the interval $[0, 1]$ defined as

$$\psi_{n,m}(x) = \begin{cases} \sqrt{(m+\frac{1}{2})} 2^{\frac{k}{2}} P_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}} \\ 0, & o.w \end{cases} \tag{3.2}$$

with $n = 1, 2, ..., 2k - 1, m = 0, 1, 2, ..., M - 1$. In Eq. (3.2) $\{P_m\}$'s are ordinary Legendre functions of order m is defined over the interval $[-1, 1]$. The Legendre wavelet is an orthonormal set as

$$\int_0^1 \psi_{n,m}(x)\psi_{n',m'}(x)dx = \delta_{n,n'}\delta_{m,m'}. \tag{3.3}$$

Any element $f \in L^2([0, 1])$, may be expanded as

$$f(x) \cong \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x), \tag{3.4}$$

where the approximation coefficients are entirely determined by $C_{n,m} = \langle h, \Psi_{n,m}\rangle$ in which $\langle ., .\rangle$ denotes the inner product of $L^2[0, 1]$. Since the series (3.4) converges on [0, 1], the function h can be approximated as

$$f(x) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T \Psi(x), \tag{3.5}$$

where C and Ψ are 2^{j-1} dimension vectors given by

$$C = [C_{1,0}, C_{1,1}, ..., C_{1,nc-1}, C_{2,0}, ..., C_{2^{j-1},0}, ..., C_{2^{j-1},nc-1}]^T, \tag{3.6}$$

$$\Psi(x) = [\psi_{1,0}, \psi_{1,1}, ..., \psi_{1,nc-1}, \psi_{2,0}, ..., \psi_{2^{j-1},0}, ..., \psi_{2^{j-1},nc-1}]^T.$$

Let us consider the space

$$H = L^2([0, T] : L^2([0, 1])). \tag{3.7}$$

As the function $f(t, .)$ belongs to $L^2([0, 1])$, then by (3.4), we have

$$f(x, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{n,m}(t)\psi_{n,m}(x)dx, \tag{3.8}$$

where the coefficients $C_{n,m}(t)$ depending on the variable t are defined by

$$C_{n,m}(t) = \langle f(t, .), \psi_{n,m}\rangle = \int_0^1 f(t, x)\psi_{n,m}(x)dx. \tag{3.9}$$

The functions $f(t, .)$ and $\psi_{n,m}$, being both in $L^2([0, 1])$, their product is in $L^1([0, 1])$, (according to Cauchy Schwartz inequality), which allows us to conclude that the coefficients $C_{n,m}(t)$ are well defined, for all $t \in [0, T]$. Consequently, the relation (3.9) is justified.

Lemma 3.1. If $f \in C([0, 1].L^2([0, 1]))$, then the function coefficients $C_{n,m}(t)$ are continuous in $[0, T]$.

Proof . It arises from the fact that the inner product is a continuous function of its both arguments. □

Lemma 3.2. If $f \in C(]0,T[.L^2([0,1]))$, then the function coefficients $C_{n,m}(t)$ belong to$C^1(]0,1[)$. Furthermore, if $\frac{\partial f}{\partial t} \in L^2(]0,T].L^2([0,1]))$, then

$$\frac{dC_{n,m}(t)}{dt} = \int_0^1 \frac{\partial f(t,x)}{\partial t}\psi_{n,m}(x)dx. \tag{3.10}$$

Proof . Lemma 3.2 is based on

$$\frac{C_{n,m}(t+\Delta t) - C_{n,m}(t)}{\Delta t} = \int_0^1 \frac{f(t+\Delta t,x) - f(t,x)}{\Delta t}\psi_{n,m}(x)dx, \tag{3.11}$$

and

$$\frac{\partial f(t+\Delta t,x) - f(t,x)}{\Delta t} = \frac{\partial f}{\partial t}(t,x) + \varepsilon(t,\Delta t,x), \tag{3.12}$$

whit $\lim_{\Delta t \to \infty} \varepsilon(t,\Delta t,x) = 0$. □

4 Operational Matrix of Integration

In this section the operational matrix of integration [9], will be obtained. The integration into $[0,x]$, where $x \in (0,x]$ of the vector $\psi(x)$ can be written as

$$\int_0^x \psi(t)dt = P\psi(x), \tag{4.1}$$

where

$$P = \frac{1}{2^j}\begin{bmatrix} L & F & F & \cdots & F \\ 0 & L & F & \cdots & F \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L \end{bmatrix}, \tag{4.2}$$

is the $(2^{j-1} - nc) \times (2^{j-1} - nc)$ operational matrix of integration, F and L are $nc \times nc$ matrices given by

$$F = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{4.3}$$

$$L = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & \cdots & 0 \\ \frac{-\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

5 Application on Heston Equation

In this methodology section, the Heston partial differential equation pricing model has been solved using Legendre wavelets, within Eq. (2.7). First, we consider

$$\frac{\partial u}{\partial t} = \alpha_1 \frac{\partial^2 u}{\partial s^2} + \alpha_2 \frac{\partial^2 u}{\partial v^2} + \alpha_3 \frac{\partial^2 C}{\partial s \partial v} + \alpha_4 \frac{\partial u}{\partial s} + \alpha_5 \frac{\partial u}{\partial v} - \alpha_6 u. \tag{5.1}$$

with boundary condition $u(s,v,0) = \beta_1(s,v,t)$ and $\frac{\partial u(s,v,0)}{\partial t} = \beta_2(s,v,t)$. That coefficients of α_i, the same parameters and variables which be in Eq. (2.7), in practice they will be calculate with this technique in follow

$$\frac{\partial u}{\partial t} = C_1^T(s,v,t)\psi(s,v,t). \tag{5.2}$$

Integrating (5.1) with respect to the second variable over $[0,t]$, we get

$$u(s,v,t) = C_1^T(s,v,t)P\psi(s,v,t) + \beta_2(u,s,t). \tag{5.3}$$

As well as

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{dC^T(s,v,t)}{ds}P\psi(s,v,t) + \frac{d\beta_2(u,s,t)}{ds},\\ \frac{\partial u}{\partial v} &= \frac{dC^T(s,v,t)}{dv}P\psi(s,v,t) + \frac{d\beta_2(u,s,t)}{dv}.\end{aligned} \tag{5.4}$$

Therefore

$$\begin{aligned}\frac{\partial^2 u}{\partial s^2} &= \frac{d^2C^T(s,v,t)}{ds^2}P\psi(s,v,t) + \frac{d^2\beta_2(u,s,t)}{ds^2},\\ \frac{\partial^2 u}{\partial v^2} &= \frac{d^2C^T(s,v,t)}{dv^2}P\psi(s,v,t) + \frac{d^2\beta_2(u,s,t)}{dv^2},\\ \frac{\partial^2 u}{\partial s\partial v} &= \frac{d}{ds}\frac{dC^T(s,v,t)}{dv}P\psi(s,v,t) + \frac{d}{ds}\frac{d\beta_2(u,s,t)}{dv}.\end{aligned} \tag{5.5}$$

Substituting (5.2) to (5.5) in (5.1), we obtain

$$\begin{aligned}C^T(s,v,t) &= \alpha_1(\frac{\partial^2C^T(s,v,t)}{\partial s^2}P\psi(s,v,t) + \frac{d^2\beta_2(u,s,t)}{ds^2}d^T\psi)+\\ &\alpha_2(\frac{\partial^2C^T(s,v,t)}{\partial v^2}P\psi(s,v,t) + \frac{d^2\beta_2(u,s,t)}{dv^2}d^T\psi) + \alpha_3(\frac{d}{ds}\frac{dC^T(s,v,t)}{dv}P\psi(s,v,t) + \frac{d}{ds}\frac{d\beta_2(u,s,t)}{dv}d^T\psi)\\ &+\alpha_4(\frac{\partial C^T(s,v,t)}{\partial s}P\psi(s,v,t) + \frac{d\beta_2(u,s,t)}{ds}d^T\psi) + \alpha_5(\frac{\partial C^T(s,v,t)}{\partial v}P\psi(s,v,t) + \frac{d\beta_2(u,s,t)}{dv}d^T\psi)\\ &-\alpha_6(C^T(s,v,t)P\psi + \beta_2 d^T\psi)\end{aligned} \tag{5.6}$$

which $1 = d^T\psi(s,v,t)$. This system can be solved for unknown coefficients of the vector, in this case, Adomian decomposition method have used, [16-18]. Consequently, the solution can be calculated $C(s,v,t)$.

Table 1: The values of parameters

ρ	δ	η	a	b	k	T
0.06	0.04	0.12	0.2	0.05	0.1	1
0.1	0.9	0.2	0.5	0.1	0.1	1

Applying the same technique in Heston PDE model and also Adomian decomposition method as a compare result bring in Figure 2.

6 Conclusion

In this work the Heston PDE model has gained from the stochastic differential equation by applying one of the important Lemma, Ito, to construct the PDE model. This paper shows that the Legendre wavelet method is an effective approach to reduce partial differential equations, hence instead of solving a complicated system, an ordinary differential equation is used to solve. This method has been tested under different values of parameters in the model. In addition, the Adomian decomposition method is a semi-analytic and powerful tool to solve partial differential equations, integral equations, etc. That has been used for solving the Heston PDE model and results of this comparison have been brought finally.

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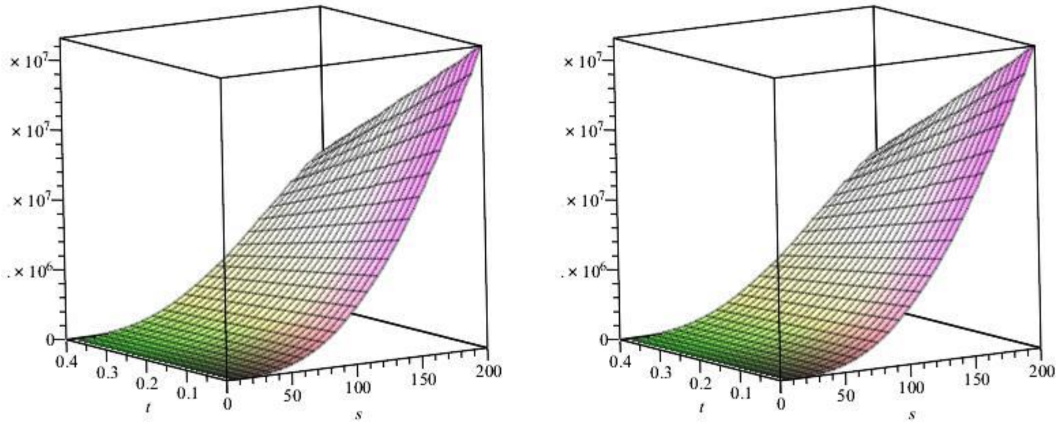

Figure 1: Solution of Heston PDE mode by LWM in various time

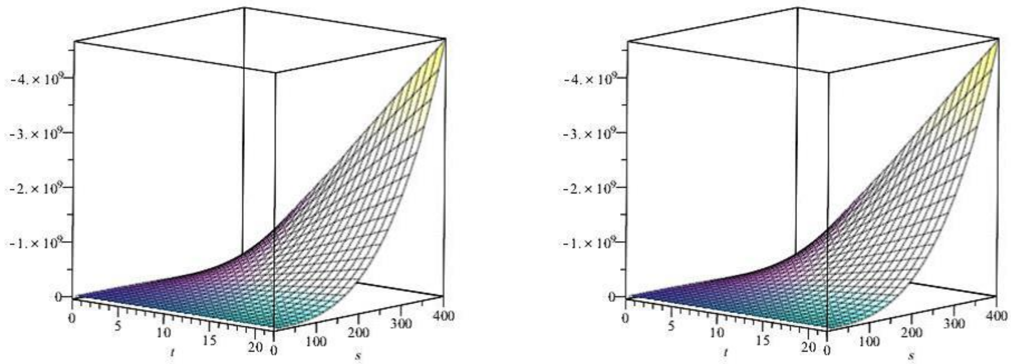

Figure 2: Solution of Heston PDE mode by LWM and ADM

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