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Determining the practical frontier for decision-making units by developing a new additive model in the DEA

A. Monzeli[a], B. Daneshian[a,*], G. Tohidi[a], M. Sanei[a], S. Razaveian[b]

[a]*Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran*

[b]*Department of Mathematics, South Tehran Branch, Islamic Azad University, Tehran, Iran*

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Abstract

Data envelopment analysis (DEA) assigns a score to each unit of the decision-making units being analyzed indicating the efficiency or inefficiency of that unit over the other units. However, in the early DEA models, there is no strategy to improve the efficiency of the efficient units. Therefore, in Paradi & Solati's (2004) practical boundary theory, they tried to expand these models to increase the efficiency for the efficient decision-making units. They had a basis for improving performance to a certain extent, thus, they presented the P-DEA linear programming model to extend the efficiency of the efficient units. Because of the staff management in organizations, it is important to increase the efficiency units in order to improve the organization based on the possible changes in the level of input and output of decision-making units. This is done to produce new advanced based on the efficiency of these new units. In this research, after studying the P-DEA model thoroughly, we identified its drawbacks and proposed a new method for determining the practical boundary by developing an additive model using an example.

Keywords: Data envelopment analysis, Decision-making unit, Linear programming, practical boundary
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1 Introduction

Data envelopment analysis (DEA) is a powerful technique in productivity management. It is a linear programming based on the methodology for measuring the relative efficiency of decision-making units (DMUs) (introduced by Charnes et al. [3]). A DEA analysis provides a variety of valuable information. It assigns a single score to each DMU making the comparison easy.

The method has the ability to handle multiple inputs and outputs simultaneously without requiring any judgments on their relative importance. Consequently, it does not need a parametrically driven input and the output produced function. It establishes the best practice boundary among the units based on a comparison process. The units are efficient units with an efficiency score of 1.0 on this boundary and the rest of the units are deemed inefficient. The level of inefficiency is measured by the unit's distance from this boundary.

*Corresponding author

Email addresses: abbas_monzeli@yahoo.com (A. Monzeli), daneshian@azad.ac.ir (B. Daneshian), gas.tohidi@iauctb.ac.ir (G. Tohidi), m_sanei@iauctb.ac.ir (M. Sanei), sh_razavyan@azad.ac.ir (S. Razaveian)

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One of the important advantages of DEA is its ability to identify the performance targets for inefficient units and indicate what improvements could be made to achieve Pareto-efficiency [3, 4]. In the real world, it might not be possible to adjust all of the inputs and outputs of the inefficient units based on the DEA results; therefore, we need to adjust all of the inputs and outputs for these DMUs, which are feasible in practice, to change their efficiencies.

For efficient units further improvement can not be considered based on DEA. As increasing performance is very important to management even for the best performers, specifying targets for efficient units is interesting for operations analysts, management, and industrial engineers. We have shown in our research that if the inputs and outputs of an efficient unit change within a range, it is possible to find another combination of inputs and outputs within such constraints and define an artificial DMU that is more efficient compared to the DEA efficient unit from which it is derived. Although the "theoretical" boundary is not known, it is possible to define a "practical" one.This new boundary envelops or touches the empirical boundary. The idea of introducing artificial ("unobserved") DMUs was used in [11] to capture value judgments in DEA.

To overcome the issues related to complete flexibility weights in DEA [11], they used unobserved DMUs as the alternative approach to weight restrictions. These units were constructed by varying the input-output levels of real DMUs in order to extend the production possibility set. In this paper, artificial DMUs were created using a linear programming model. This was done in a way that the new boundary identifies the adjusted efficiency measures for DMUs and indicates targets for empirically efficient units.

The rest of the paper is organized as follows: Section 2 presents the proposed model and methodology. Data Envelopment Analysis (DEA) is a mathematical programming method for evaluating decision-making units (DMUs) and one of the most popular methods for determining performance.

Charnes, Cooper and Rhodes [3] proposed the first model in DEA to evaluate the performance of decision units based on the resources (inputs) and products (outputs). DMUs that provide the best input-output relationships are the boundaries of system performance. This model, introduced in 1978, was called the CCR. A few years later, Banker, Charans, & Cooper (1984) introduced the BCC model with variable-scale returns [3].

Linear programming identifies efficient boundaries and uses them to determine productivity. Both output and input-based are used in this method. Although these two models are not the only ones used, they are still the most popular models of DEA. Many researchers use the DEA method to determine the boundary of efficiency and evaluate the efficiency [7]. In natural problems, the theory boundary or function diagram is usually unknown. DEA method gives us the experimental boundary while taking into account the experimental data on which the efficient units are placed. Several different studies have been carried out using efficient units to obtain better boundaries.

Therefore, an artificial DMU can be defined within the specified range with another combination of inputs and outputs that is more efficient than the DEA operating unit. In this case, it is possible that some of them are at the same level of efficiency, but its input is reduced compared to previous state or the efficiency of the unit does not change at all. Although "practical boundaries " are not known, it is possible to produce and define them. This new boundary surrounds the experimental boundary and its surface is smoother than the experimental boundary [11].

In the classical DEA models, it is assumed that all of the outputs can be expanded. For example, a change in one DMU output does not affect the other outputs. However, this hypothesis is proved when all the outputs are constant; whereas, cases of DMUs with total fixed-size outputs, such as sports games where medals are fixed outputs, often exists in the real world [5, 13].

In reviewing the related literature, it is clear that the common boundary is easily obtained between the efficiency of decision-making units; however, this boundary may not be unique. To solve this problem, Young et al. highlighted the importance of this matter in his future researches in [15]. In [17], Qingyuan Zhu et al. introduced a unique performance boundary with a set of fixed outputs in data envelopment analysis. Carbon reduction technologies such as renewable energy, nuclear energy, and CCS (Carbon capture and storage) technology for the power industry play a significant role in achieving low-carbon development goals. In 2017, Nannan Wang et al. employed a meta boundary DEA approach to evaluate the carbon reduction efficiency of technologies on the project level. The sample consists of several groups such as nuclear energy, hydro-electric energy, wind energy, solar energy, biomass energy, and CCS technology in power plants [12]. In 2020, Daniel Adelman used a performance boundary approach to score and rank hospital performance [1].

In 2020, Qiang Cui reviewed a data comparison based on the airline's five adverse output approaches to environmental productivity. In this study, it is stated that the carbon emission rate of airlines is growing rapidly. Therefore, in assessing the environmental efficiency of the airlines, carbon emissions should be considered as an undesirable output [10].

In 2020, Lim examined the impact of oil price shocks on the production boundary using reverse data envelopment analysis for operational planning. This method is a useful planning tool, especially when it is accompanied by boundary changes, it reflects reality correctly. In this paper, a reverse optimization model for operational planning is proposed by considering the boundary changes in relation to the environmental factors. The aim was providing a computational method of a new measure to investigate the effective changes in the performance boundary. It also shows the previous boundary changes to provide insight into the future performance boundary estimates [8]. In 2020, Ojo and his colleague proposed an endogenous modified random boundary model that showed the impact of climate change adaptation strategies on rice productivity in southwestern Nigeria [9].

A modified endogenous random boundary model was used in this paper as well. The results of the study show that the adoption of adaptation strategies is determined endogenously by rice productivity. Therefore, lack of endogenous calculation of efficiency estimates is a contradictory parameter. In 2020, Gianfranco et al. conducted a comparative study using CCR and BCC data envelopment analysis models to assess road safety in urban road networks. The social cost of accidents was used as the only output indicator for the first time in this study [6]. In 2020, Lei Chen et al. used the cross-functional approach to evaluate the performance of the cross-boundary analysis [4].

In this research, first, we have further studied the initial DEA models and the P-DEA model. Then we have proposed a new method for determining the practical boundary and improving efficient and inefficient units by changing the input and output factors.

2 Model DEA

The data envelopment analysis (DEA) estimates the relative efficiency size of each decision-making unit according to comparison with other units. The BCC model form to determine the effectiveness of the decision - making units in DEA is defined as follows:

$$
\begin{aligned}
\text{Max} \quad & h_0 = \frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{i0}}, \\
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{ij}} \le 1 \quad \forall j, \\
& u_r \ge \epsilon \quad \forall r, \\
& v_i \ge \epsilon \quad \forall i, \\
& u_0 \qquad \text{free.}
\end{aligned}
$$

In the above model x_{ij} and y_{rj} are the inputs and outputs of the jth DMU; u_r and v_i are the output and input weights, respectively. The objective is to obtain those weights that maximize the efficiency of the unit under evaluation, DMU_0, while the efficiency of all DMU_s must not exceed 1.0. The efficiency score and input- output weights are the variables of the BCC model. The inputs and outputs of DMU_0 are known. If DMU_0 is efficient then $h_0 = 0.1$.

This model can be linearized as follows

$$
\begin{aligned}
\text{Max} \quad & \sum_{r=1}^{s} u_r y_{rj} + u_0, \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{i0} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \le 0 \quad \forall j, \\
& u_r \ge \epsilon \quad \forall r, \\
& v_i \ge \epsilon \quad \forall i, \\
& u_0 \qquad \text{free,}
\end{aligned}
$$

where v_i, u_r are the weights that depend on the inputs and outputs corresponding to the model variables. They can be interpreted as the price of a normalized shadow. Therefore, the input and output price of the decision-making unit under the evaluation shown is the best possible price. In addition, a linear equation multiplied by a non-zero scalar is stable, so to remove this source there is an uncertainty constraint $\sum_{i=1}^{m} v_i x_{i0} = 1$, we call it the normalization constraint.

The dual of the mentioned model is as follows:

$$\begin{aligned}
\text{Min} \quad & \theta - \epsilon \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right), \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{i0}, \quad \forall i, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{r0}, \quad \forall r, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0 \quad j = 1, 2, ..., n.
\end{aligned}$$

Another model used to examine the concept of efficiency in data envelopment analysis is the additive model introduced by Charans et al. (1985).

$$\begin{aligned}
\text{Max} \quad & \sum_{i=1}^{m} S_i^- + \sum_{r=1}^{s} S_r^+, \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + S_i^- = x_{i0}, \quad \forall i, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - S_r^+ = y_{r0}, \quad \forall r, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0 \quad j = 1, 2, ..., n.
\end{aligned}$$

The objective function of the model is a superline that expresses norm L_1. In this model, the objective function calculates the maximum distance of the unit under evaluation from the performance units that dominate itself. The dual of the above model is as follows:

$$\begin{aligned}
\text{Min} \quad & -\sum_{r=1}^{s} u_r y_{r0} + \sum_{i=1}^{m} v_i x_{i0} - u_0, \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \leq 0 \quad \forall j, \\
& u_r \geq 0 \quad \forall r, \\
& v_i \geq 0 \quad \forall i, \\
& u_0 \qquad \text{free.}
\end{aligned}$$

In the additive model, the decision-making unit under evaluation is efficient if the objective function be zero in optimization ($S^{-*} = 0$ and $S^{+*} = 0$).

3 Model: practical DEA (P-DEA)

In the efficiency evaluation by DEA models, a hypothetical unit as a specific goal to improve their performance can be considered for inefficient units. while further improvement can not be determined for efficient units based on DEA analysis. While increasing performance can be very important even for the best executives in management.

In the P-DEA model, the input and output have high and low limits. they can be changed in a certain range. The goal is to find new inputs and outputs for decision-making units within the specified range, while one level has a higher efficiency than the unit under evaluation. The purpose of P-DEA is to generate new decision-making units by the results of different inputs and outputs according to the limited characteristic by management, so the P-DEA model is defined as follows:

$$
\begin{aligned}
\text{Max} \quad & h_0 = \frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{i0}}, \\
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1 \quad \forall j, \\
& u_r \geq \epsilon \quad \forall r, \\
& v_i \geq \epsilon \quad \forall i, \\
& u_0 \qquad \text{free.}
\end{aligned}
$$

In the above model x_ij and y_rj are the inputs and outputs of the $j-$th DMU; and u_r and v_i are the output and input weights, respectively. The objective is to obtain those weights that maximize the efficiency of the unit under evaluation, DMU_0, while the efficiency of all DMU_s must not exceed 1.0. The efficiency score and input-output weights are the variables of the BCC model. The inputs and outputs of DMU_0 are known. If DMU_0 is efficient then $h_0 = 1.0$, [12].

In the real world, some of the factors (inputs and outputs) are fixed, and it is not possible to vary their values, e.g. a store's floor space. However, changes in other factors are permitted within certain ranges, i.e., $L_{x_{i0}} \leq x_{i0} \leq U_{x_{i0}}$ and $L_{y_{r0}} \leq y_{r0} \leq U_{y_{r0}}$.

Furthermore, some factors may have a specific relationship with other factors. This information about inputs and outputs can be obtained from management. Suppose that there are upper and lower bounds for some or all inputs and outputs. Our goal is to look for the inputs and outputs of a new DMU within the specified range, but one that has an efficiency score greater than that of DMU_0, which is, at present, 1.0. We are attempting to create new DMU_s by adjusting the already efficient DMU_s input and output variables according to the limits determined by management. This should produce units that could be used as models for the efficient DMU_s from which they were derived. In the following models $\tilde{x_{i0}}$(inputs of the artificial DMU), $\tilde{y_{io}}$ (outputs of the artificial DMU), u_r and v_i are variables.

The model then becomes[14]:

$$
\begin{aligned}
\text{Max} \quad & \frac{\sum_{r=1}^{s} u_r \tilde{y}_{rj} + u_0}{\sum_{i=1}^{m} v_i \tilde{x}_{i0}}, \\
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1 \quad \forall j, \\
& 1 \leq \frac{\sum_{r=1}^{s} u_r \tilde{y}_{r0} + u_0}{\sum_{i=1}^{m} v_i \tilde{x}_{i0}} \leq 1 + \delta, \\
& L_{x_{i0}} \leq \tilde{x}_{i0} \leq U_{x_{i0}}, \quad \forall r, \\
& L_{y_{i0}} \leq \tilde{y}_{i0} \leq U_{y_{i0}}, \quad \forall r, \\
& u_r \geq \epsilon \quad \forall r, \\
& v_i \geq \epsilon \quad \forall i, \\
& u_0 \qquad \text{free.}
\end{aligned}
$$

Using the P-DEA model, find x_{i0} and y_{i0} as new inputs and outputs for the efficient unit, which are the weights of the selected efficient units.

In this model, the goal is to increase the performance of the artificial decision-making unit. For this purpose, a limit above δ is considered efficiency for the artificial decision-making unit, otherwise the model will be infinite. In fact, in this model, the efficiency possible value of an experimental efficient unit increases by δ which can be determined by the management, if there is no progress possibility for some other data, the artificial decision-making unit will be on the experimental frontier.

Figure (1) shows the empirical, practical, and theoretical frontiers for the input and output factors. In this figure, the curved line shows the theoretical frontier, which, of course, is not known in any analysis [12]. The relative P-DEA

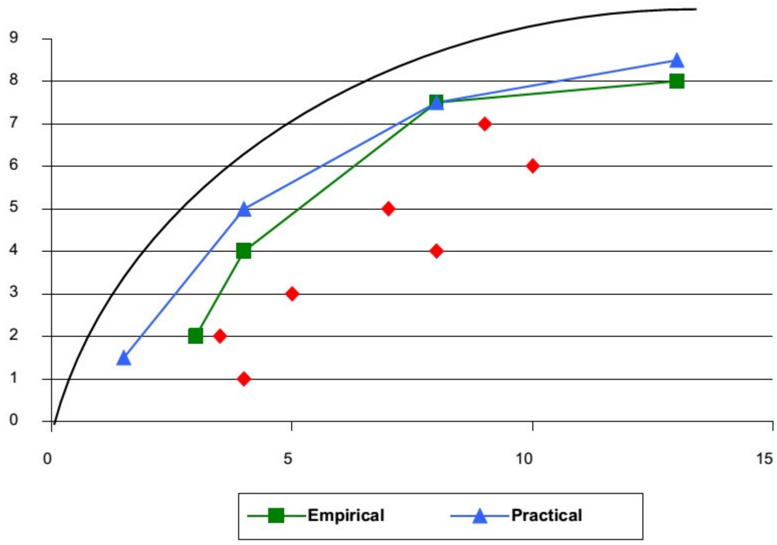

Figure 1: The theoretical, practical, and empirical frontiers.

model can be converted to a linear programming model by changing the following variables:

$$p_r = \tilde{y}_{r0} u_r, q_i = \tilde{x}_{i0} v_i.$$

The above formula will be obtained by substituting P_r and q_r in constraints $L_{x_{i0}} \le \tilde{x}_{i0} \le U_{x_{i0}}, L_{y_{i0}} \le \tilde{y}_{i0} \le U_{y_{i0}}, v_i L_{x_{i0}} \le q_i \le v_i U_{x_{i0}}, u_r L_{y_{i0}} \le p_r \le u_r U_{y_{i0}}$.

Therefore, the relative model can be turned into a linear programming problem as follows:

$$\begin{aligned}
\text{Max} \quad & \sum_{r=1}^{s} p_r + u_0, \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{r0} - \sum_{i=1}^{m} u_i x_{i0} + u_0 \le 1 \quad \forall j, \\
& -\sum_{r=1}^{s} p_r + \sum_{i=1}^{m} q_i - u_0 \le 0, \\
& \sum_{r=1}^{s} p_r - \sum_{i=1}^{m} (1+\delta) q_i + u_0 \le 0, \\
& v_i L_{x_{i0}} \le q_i \le v_i U_{x_{i0}}, \quad \forall r, \\
& u_r L_{y_{i0}} \le p_r \le u_r U_{y_{i0}}, \quad \forall r, \\
& u_r \ge \epsilon \quad \forall r, \\
& v_i \ge \epsilon \quad \forall i, \\
& u_0 \qquad \text{free.}
\end{aligned}$$

By solving the above model p_r^* and q_i^*, and considering the relations $q_i = \tilde{x}_{i0} v_i$ and $p_r = \tilde{y}_{r0} u_r$, the variables $\tilde{x}_{i0}$ and $\tilde{y}_{r0}$ can be calculated, which are the size of the new inputs and outputs of the artificial unit. Practical frontier is defined as the frontier formed by artificial decision-making units.

4 P-DEA model bug analysis

Suppose the following set of production possibilities is formed by real efficient decision-making units, i.e.:

$$T^e = \left\{ (X,Y) | X \geq \sum_{j=1}^{k} \lambda_j X_j, Y \leq \sum_{j=1}^{k} \lambda_j Y_j, \sum_{j=1}^{k} \lambda_j = 1, \lambda_j \geq 0, j = 1, ..., k \right\}.$$

Here (x_j, y_j) is assumed that the observed decision-making units are efficient. the following production possibility set is formed by efficient real decision-making units and artificial decision-making units, i.e.:

$$T^p = \left\{ (X,Y) | X \geq \sum_{j=1}^{2k} \lambda_j X_j, Y \leq \sum_{j=1}^{2k} \lambda_j Y_j, \sum_{j=1}^{2k} \lambda_j = 1, \lambda_j \geq 0, j = 1, ..., 2k \right\}.$$

Here (x_{n+j}, y_{n+j}) is assumed to be an artificial decision-making unit corresponding to (x_j, y_j). It is clear that it is always $T^e \subset T^p$. In fact, the frontiers of sets T^e and T^p are experimental and practical frontiers.

One might expect that real efficient units should be inefficient relative to the set of production possibilities. The article presented by Paradi and Solati in introducing the practical frontier pointed to the real efficient decision-making units which are also efficient in relation to T^p. Their interpretation of these real efficient decision-making units was that these decision-making units are fully efficient and their performance will not increase with the expansion of the production possibility set. Here we show that their interpretation of these decision-making units is wrong.

Theorem 4.1. Assume that the actual decision-making unit (X_k, Y_k) is efficient. In this case an artificial decision-making unit can be created by P-DEA model that (x_k, Y_k) is inefficient in relation to the production possibility set.

Proof . Assume that (U^*, V^*, u^*) is the optimal solution in solving the P-DEA model to obtain the artificial decision-making unit corresponding to the decision-making unit (X_0, Y_0). The artificial decision-making unit must meet the following limitations.

$$\begin{aligned}
&\sum_{r=1}^{s} \tilde{y}_{r0} u_r^* + u_0 - (1+\delta) \sum_{i=1}^{m} \tilde{x}_{i0} v_i^* = 0, \\
&\sum_{i=1}^{m} \tilde{x}_{i0} v_i^* = 1, \\
&L_{y_{i0}} \leq \tilde{y_{r0}} \leq U_{y_{i0}}, , \\
&L_{x_{i0}} \leq \tilde{x_{i0}} \leq U_{x_{i0}}.
\end{aligned}$$

On the other hand, we have:

$$\sum_{r=1}^{s} y_{r0} u_r^* + u_0 - \sum_{i=1}^{m} x_{i0} v_i^* \leq 0, \Rightarrow \sum_{r=1}^{s} y_{r0} u_r^* + u_0 - (1+\delta) \sum_{i=1}^{m} x_{i0} v_i^* < 0,$$

It is clear that

$$\sum_{r=1}^{s} U_{y_{r0}} u_r^* + u_0 - (1+\delta) \sum_{i=1}^{m} L_{x_{i0}} v_i^* > 0,$$

Otherwise it will be impossible. So we can say that there is a convex combination of 1 and 2 so that it is active on the following page i. e.:

$$\exists \lambda \in (0,1), \quad \text{s.t} \quad \sum_{r=1}^{s} (\lambda U_{y_{r0}} + (1-\lambda) y_{r0}) u_r^* + (1+\delta) \sum_{i=1}^{m} (\lambda L_{x_{i0}} + (1-\lambda) x_{i0}) v_i^* = 0.$$

This decision-making unit dominates the unit under evaluation. means:

$$\begin{pmatrix} -(\lambda L_{x_0} + (1-\lambda)x_0) \\ \lambda U_{y_0} + (1-\lambda)y_0 \end{pmatrix} \geq \begin{pmatrix} -x_0 \\ y_0 \end{pmatrix}$$

But $\begin{pmatrix} -(\lambda L_{x_0} + (1-\lambda)x_0) \\ \lambda U_{y_0} + (1-\lambda)y_0 \end{pmatrix}$ can be an artificial decision-making unit for (X_0, Y_0) that dominates. So the proof is all. □

This case shows that their analysis of such decision-making units is unreasonable. Paradi and Solati in the P-DEA model claimed that the performance of the artificial decision-making unit will be increased δ size more than the empirical efficient unit. This is conditional when at least one of the constraints corresponding to the actual decision-making units is active in P-DEA model. while their model never guarantees this case. But this condition is met in our proposed model.

5 Proposed model

Note that in this model, unlike the standard DEA model,the inputs and outputs are also variables. The objective function is to maximize the efficiency of the artificial DMU, while the weights must be feasible for all other units. The factors can vary within the specified ranges. To have an improved unit, the efficiency score of the artificial unit is set to be greater than or equal to 1.0. DEA models that result in an efficiency score of more than 1.0 have been reported in the literature.

Andersen and Petersen [2] developed modified versions of the DEA models for ranking efficient units in which the unit, a super efficient unit, could obtain an efficiency score of more than 1.0 by excluding such unit from the analysis[2].

In this paper, an upper limit, $(1+\delta)$, is considered in the model for the efficiency of the new unit, otherwise the model would be unbounded. The amount of possible increase in the efficiency of an empirical efficient unit, designated as, can be specified by management (for example 5%). The upper and lower bounds for factors and the possible improvement in the efficiency of an empirically efficient unit δ can be local or global based on the application; for example for comparing different branches of the same bank the information can be global, while it can be local if different banks are compared. According to the explanations of the previous section, by solving the P-DEA model, the optimal value of the objective function was to be obtained by $(1+\delta)$, which was proved to have a contradiction. In fact, the purpose of solving this model is not to obtain the optimal value of the objective function which is active in the clause related to the artificial decision-making unit, it is enough that its optimization guarantees the answer, so we consider D_0 corresponding to DMU_0 as follows:

$$D_0 = \{(x, y) | L_{x_0} \leq x \leq U_{x_0}, L_{y_0} \leq y \leq U_{y_0}\}.$$

In the proposed model, the performance of constraint has been considered $(1+\delta)$ by activating the constraint related to the artificial decision-making unit. In the objective function, we are looking for an artificial unit that by adding it to the production possibility set, the maximum distance of real decision-making unit corresponding to the efficient frontier be as small as possible in relation to L_1 norm.

$$\begin{aligned} \text{Min}_{(\tilde{x},\tilde{y})}\text{Min} \quad & -\sum_{r=1}^{s} u_r y_{r0} + \sum_{i=1}^{m} v_i x_{i0} - u_0, \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \leq 0 \quad \forall j, \\ & \frac{\sum_{r=1}^{s} u_r \tilde{y}_{r0} + u_0}{\sum_{i=1}^{m} v_i \tilde{x}_{i0}} = 1 + \delta, \\ & u_r \geq 1 \quad \forall r, \\ & v_i \geq 1 \quad \forall i, \\ & u_0 \qquad \text{free.} \end{aligned}$$

In fact, we have developed the additive model. In this model, we have tried to create artificial decision-making units. while their performance has increased by δ compared to their corresponding real decision-making units, we have tried to create as little expansion as possible in the practical frontier. By merging the two problems of minimization of the above model and converting the problem of minimization to maximization, we have:

$$
\begin{aligned}
\text{Max} \quad & \sum_{r=1}^{s} u_r y_{r0} - \sum_{i=1}^{m} v_i x_{i0} + u_0, \\
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{ij}} \le 1 \quad \forall j, \\
& \frac{\sum_{r=1}^{s} u_r \tilde{y}_{r0} + u_0}{\sum_{i=1}^{m} v_i \tilde{x}_{i0}} = 1 + \delta, \\
& L_{x_{i0}} \le \tilde{x}_{i0} \le U_{x_{i0}}, \quad \forall r, \\
& L_{y_{i0}} \le \tilde{y}_{i0} \le U_{y_{i0}}, \quad \forall r, \\
& u_r \ge 1 \quad \forall r, \\
& v_i \ge 1 \quad \forall i, \\
& u_0 \quad \text{free.}
\end{aligned}
$$

The ratio model, (2), can be transformed into a linear fractional programming model by substituting $\tilde{y}_{r0}u_r$ and $\tilde{x}_{i0}v_i$ with new variables p_r and q_i, respectively, and replacing $L_{x_{i0}} \le \tilde{x}_{i0} \le U_{x_{i0}}$ and $L_{y_{i0}} \le \tilde{y}_{i0} \le U_{y_{i0}}$ with $v_i L_{x_{i0}} \le q_i \le v_i U_{x_{i0}}$ and $u_r L_{y_{i0}} \le p_r \le u_r U_{y_{i0}}$, correspondingly. Then the linear fractional program can be transformed to a linear program [12], which is shown in (3) so that the linear programming method can be applied to solve the case. The process is relatively straightforward.

Therefore, the model can be turned into a linear programming problem as follows:

$$
\begin{aligned}
\text{Max} \quad & \sum_{r=1}^{s} u_r y_{r0} - \sum_{i=1}^{m} v_i x_{i0} + u_0, \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \le 0 \quad \forall j, \\
& \sum_{r=1}^{s} p_r - \sum_{i=1}^{m} (1+\delta) q_i + u_0 = 0, \\
& v_i L_{x_{i0}} \le q_i \le v_i U_{x_{i0}}, \quad \forall r, \\
& u_r L_{y_{i0}} \le p_r \le u_r U_{y_{i0}}, \quad \forall r, \\
& u_r \ge 1 \quad \forall r, \\
& v_i \ge 1 \quad \forall i, \\
& u_0 \quad \text{free.}
\end{aligned}
$$

By solving the above model, we get p_r^* and q_i^*, and by considering the relations $q_i = \tilde{x}_{i0}v_i$ and $p_r = \tilde{y}_{r0}u_r$, the variables $\tilde{x}_{i0}$ and $\tilde{y}_{r0}$ can be calculated. This is the size of the new inputs and outputs of the artificial unit. practical frontier is defined as a frontier produced by the artificial decision-making units.

The objective function in optimization is always zero, which means that the performance of the artificial decision-making unit is $1 + \delta$, while the performance of the real decision-making unit corresponding to it is one. The P-DEA model did not guarantee this.

Example 5.1. Consider the characteristics of 7 decision-making units with an input and output as follows: Solve these units with the BCC model and the proposed model and obtain their results according to Table (1) in which by changing the input and output factors (Artificial unit creation), we have changed their efficiency to a maximum of a certain amount $\delta = 0.2$, (according to the system administrator). We have also shown the improved frontier in Figure (1).

In the above table, x^* and y^* are the modified (artificial) units by using the proposed model. After creating this data, we have calculated their efficiency with the mentioned model. It is clear that the efficiency of all units, both efficient and inefficient, has been improved and a new frontier has been set.

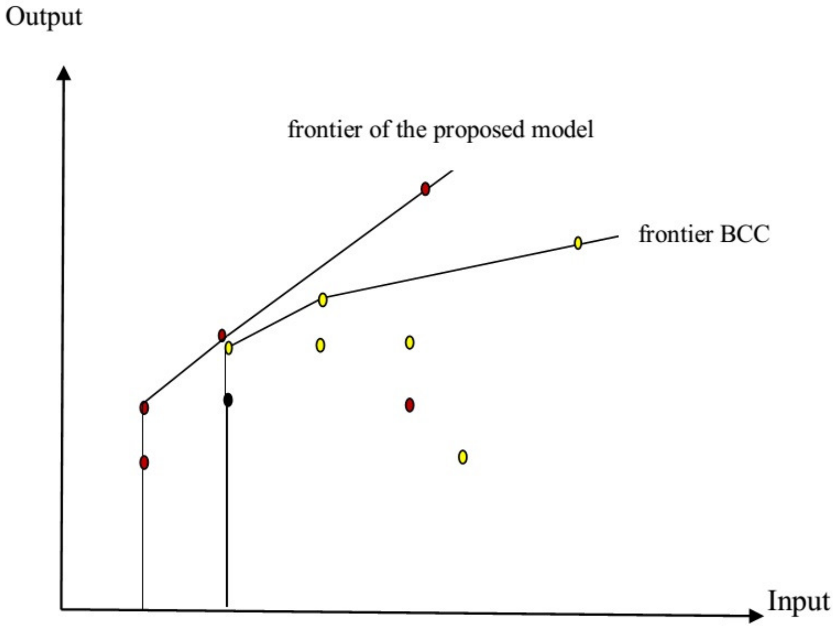

Figure 2: Comparison of the efficiency frontier produced in the proposed model.

Table 1: Comparing the efficiency of the proposed model with the BCC model.

DMU_0	I	O	EFF	V_I	U_O	x^*	y^*	$\tilde{EFF}$
1	2	5	1.00	0.50	0.20	1	4	1.02
2	4	5	0.50	0.25	0.00	2	4	0.65
3	3	5	66.67	0.33	0.00	2	4	0.80
4	3	5	1.00	0.33	0.33	2	5.10	1.02
5	2	4	1.00	0.50	0.00	1	3	1.02
6	5	3	0.40	0.20	0.00	4	4	0.45
7	6	7	1.00	0.166	0.50	4	8.49	1.02

6 Research innovation

We have presented a new model to determine the practical frontier by the additive model development. The new frontier is produced by created artificial units. It is higher than the other frontier produced in the previous models and surrounds them. Therefore, the innovation of this research according to the proposed model has raised the efficiency frontier and brought it closer to the theory or function diagram. Also, organizations and companies can make efficient units more efficient using proposed changes, in addition to strategic planning to increase the efficiency of inefficient units, in order to change from static to dynamic.

7 Suggestions based on research results

- Due to the unavailability of the real production function in multi-output mode, further investigation can be done about the effect of endogenous factors in estimating the production function.
- In this research, a new frontier was defined by upgrading the efficiency frontier using an artificial variable, but this frontier is also far from the real frontier of the production function, so a new research is needed to find the real frontier.

- " The present study was conducted by determining the upper and lower limits for input and output factors, but the correlation coefficient between input and output index factors must be studied with new research in order to do this work with the lowest cost in the real system. The actual frontier should be estimated by changes in these factors (not in all inputs and outputs) and their decrease or increase.

8 Conflict of Interest

As the author of this paper, I declare that there is no conflict of interest.

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