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# Weakly semi-primary submodules

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**Abstract**

Let $R$ be a commutative ring with identity, and let $W$ be a unitary $R$-module. In this paper, we introduced the concept of a weakly semi-primary submodule as a generalization of the primary submodule, where a submodule $X$ of $W$ is called weakly semi-primary if the $\mathrm{Rad}(X : W) = \sqrt{(X : W)}$ is a weakly prime ideal of $R$, and from this work, we have provided some characteristics of weakly semi-primary submodule.

Keywords: Weakly semi-primary, weakly prime, multiplication module, Weakly semi-primary ring, local ring and finitely generated.
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## 1 Introduction

Throughout this paper, every ring is associative with identity and every module is unitary and we use the notion of any two submodule X and Y by $(X : Y) = \{x \in R : xY \subseteq X\}$, and annihilator of a submodule X in R is denoted $\mathrm{ann}_R(X) = \{s \in R : sX = 0\}$. In [6] aproper submodule G of W is called primary submodule of W if for all, $g \in R \; and \;\; m \in W, \;\; gm \in G$ and $m \notin G$ imply that $g^n W \subseteq G$, for some n is positive integer. And in [5] an ideal I of R is called a semi-primary ideal if $\sqrt{I}$ is prime ideal. Anderson and smith in [1], a proper ideal $I$ of R is called weakly prime if for $0 \neq ij \in I$, then either $i \in I$ or $j \in I$, and it was show that a proper ideal $I$ of R is weakly prime if and only if $0 \neq PJ \subseteq I$, where $P, J$ are two ideals of R, implies that either $P \subseteq I$ or $J \subseteq I$. Where an R-submodule K of W is called prime R-submodule if and only if $K \neq W$ and $kx \in K, \; for \; k \in R, \; and \; x \in W$, then either $k \in (K : W)$ or $x \in K$ [9]. Anderson and Smith have shown that a weakly prime ideal is not prime ideal satisfies, $I^2 = 0$ and $I\sqrt{0} = 0$. The weakly prime radical $H(X)$of $X$ in W is defined the intersection of all weakly prime submodule $H$ of W such that $X \subseteq H \; i.e.H(X) = \cap\{H \subseteq W : H \; is \; weakly \; prime \; and \; X \subseteq H\}$. In this paper present a new concept which is a generalization of primary submodule, is weakly semi-primary submodule, where an R-submodule X of W is called weakly semi-primary submodule if $\sqrt{(X : W)}$ is a weakly prime ideal of R. And we proved in this work if R is weakly semi-primary ring, then R is a local ring. Finally we finish this paper in section four, we have proven if W is a faithful- multiplication R-module so W is weakly semi-primary module if and only if R is weakly semi-primary ring. Also a module multiplication contains finitely generated weakly semi-primary sub-module, is finitely generated.

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## 2 Weakly Semi-Primary Submodules

**Definition 2.1.** An proper R-submodule $X$ of W is called weakly semi-primary submodule if $\sqrt{(X:W)}$ is a weakly prime ideal of R. It is clear that every primary submodule is weakly semi-primary submodule. A ring R is called weakly semi-primary ring if R is weakly semi-primary module as an R-module.

**Remark 2.2.** Let (0) be an submodule of a module W. Then (0) is weakly semi-primary if and only if ann ( W ) is a weakly semi-primary ideal of R.

Before we give other characteristics of weakly semi-primary submodule, we need to present this lemma.

**Lemma 2.3.** Let A and B be two submodules of $W$, and let I be an ideal of $R$. If $IB \subseteq A$, then $\sqrt{I} \cap \sqrt{(B:W)} \subseteq \sqrt{(A:W)}$.

**Proof .** Let $x \in \sqrt{I} \cap \sqrt{(B:W)} \subseteq \sqrt{(A:W)}$, then there exists $k$ is positive integer such that $x^k \in I \cap (B:W)$ , thus $x^k W \subseteq B$ and $x^{2k} W \subseteq x^k B \subseteq A$. Therefore $x \in \sqrt{(A:W)}$. □

**Proposition 2.4.** Let $X$ be an $R$-submodule of W, and $I$ be an ideal of $R$. Then $Y$ is weakly semi-primary R-submodule if and only if $IX \subseteq Y$ implies that $\sqrt{I} \subseteq \sqrt{(Y:W)}$ or $\sqrt{(X:W)} \subseteq \sqrt{(Y:W)}$.

**Proof .** Let $Y$ be a weakly semi-primary submodule of W, assume that $IX \subseteq Y$ then by Lemma 2.3, $\sqrt{I} \cap \sqrt{(X:W)} \subseteq \sqrt{(Y:W)}$, thus $\sqrt{I}\ .\sqrt{(X:W)} \subseteq \sqrt{(Y:W)}$, then either $\sqrt{I} \subseteq \sqrt{(Y:W)}$ or $\sqrt{(X:W)} \subseteq \sqrt{(Y:W)}$.
Converse, let $,y \notin \sqrt{(Y:W)}$ , where $x, y \in R$ and $xy \in \sqrt{(Y:W)}$. Thus $x^n y^n W \subseteq Y$, for some n is positive integer, so either $\sqrt{(x^n)} \subseteq \sqrt{(Y:W)}$ or $\sqrt{(y^n W:W)} \subseteq \sqrt{(Y:W)}$. If $\sqrt{(x^n)} \subseteq \sqrt{(Y:W)}$, then $x \in \sqrt{(Y:W)}$ this is contradiction. Now if $\sqrt{(y^n W:W)} \subseteq \sqrt{(Y:W)}$, then $y \in \sqrt{(Y:W)}$, this is contradiction. Therefore $xy \notin \sqrt{(Y:W)}$. □

**Corollary 2.5.** Let $X$ be an submodule of M and $q \in R$. Then Y is weakly semi-primary submodule of W if and only if $qX \subseteq Y$ implies that $\sqrt{q} \subseteq \sqrt{(Y:W)}$ or $\sqrt{(X:W)} \subseteq \sqrt{(Y:W)}$.

**Proposition 2.6.** Let $W_1$ and $W_2$ be tow R-modules and $g: W_1 \to W_2$ be an epimorphism. Then $X$ is weakly semi-primary submodule of $W_2$ if and only if $g^{-1}(X)$ weakly semi-primary submodule of $W_1$.

**Proof .** It is straightforward. □

The homomorphic image of weakly semi-primary submodule need not to be weakly semi-primary submodule as the following example shows:

**Example 2.7.** Let $Z[n]$ as a Z-module and $K = (6+6n)$ is a submodule of $Z[n]$, but $(K : Z[n]) = 0$, and since $Z$ is an integral domain, then $\sqrt{(K : Z[n])} = 0$ is weakly prime, that mean $K$ is a weakly semi-primary. Now let $g : Z[n] \to Z$ such that $\left(b_m n^m + b_{m-1} n^{m-1} + \cdots + b_0\right) = b_0$ , where $b_m, b_{m-1}, \cdots, b_0 \in Z$, since g is homomorphism and $g(K) = 6z$, but $\sqrt{6z}$ is not weakly prime.

A condition is given under which a homomrphic image of primary ideal is a primary ideal [10]. Similar when adding a condition to the following proposition, it gives us the homomorphic image of weakly semi-primary submodule is a weakly semi-primary submodule.

**Proposition 2.8.** Let $W_1$ and $W_2$ be tow R-modules and $g: W_1 \to W_2$ be an epimorphism, such that $\ker g \subseteq X$, where $X$ is a submodule of $W_1$. Then $X$ is weakly semi-primary submodule of $W_1$ if and only if $g(X)$ weakly semi-primary submodule of $W_2$.

**Proof .** It is clear by direct calculations. □

**Corollary 2.9.** Let $X$ and $Y$ be two submodule of $W$, such that, $X \subseteq Y$, then $\frac{Y}{X}$ is weakly semi-primary submodule of $\frac{W}{X}$ if $Y$ is weakly semi-primary submodule of W.

**Proof .** Let $h : W \to \frac{W}{X}$ be the natural homomorphism, then

$$Y = h^{-1}\left(\frac{Y}{X}\right)$$

and hence the conclusion follows by Proposition 2.8. □

**Proposition 2.10.** Let $W = U_1\oplus U_2$ be a direct summend of an $R$-module and $X$ be a submodule of $W$,

(1) Assume that $(X : U_1) \subseteq ann(U_2)$, then $X$ is weakly semi-primary submodule of M if and only if $X$ is weakly semi-primary submodule of $U_1$.

(2) If $X$ is weakly semi-primary submodule of $U_1$, then $X\oplus U_2$ is weakly semi-primary submodule of W.

**Proof .** (1) It is clear $\sqrt{(X : W)}$ is weakly prime ideal of $R$, that we can notice that

$$\sqrt{(X : W)} = \sqrt{(X : U_1) \cap (X : U_2)} = \sqrt{(X : U_1)}.$$

Accordingly, the evidence has become clear.

(2) We can notice that $\sqrt{(X \oplus U_2 : W)} = \sqrt{(X \oplus U_2 : U_1) \cap (X \oplus U_2 : U_2)} = \sqrt{(X : U_1)}$, that mean $\sqrt{(X \oplus U_2 : W)}$ is weakly prime ideal of R. Therefore $X\oplus U_2$ is weakly semi-primary submodule of M. □

In the next proposition we give a condition which $X\oplus Y$ is a weakly semi-primary submodule of $U_1 \oplus U_2$ , where $X$ is weakly semi-primary submodule of $U_1$, and $Y$ is weakly semi-primary submodule of $U_2$. But before that we need to present the following lemma.

**Lemma 2.11.** Let $X$ be a submodule of an R-module $U_1$, and $Y$ is a submodule of an R-module $U_2$. Then $\sqrt{(X \oplus Y : U_1 \oplus U_2)} = \sqrt{(X : U_1)} \cap \sqrt{(Y : U_2)}$

**Proof .** Let $x \in \sqrt{(X \oplus Y : U_1 \oplus U_2)}$, then $x^k \in (X \oplus Y : U_1 \oplus U_2)$, for some positive integer $k$. Then $x^k(U_1 \oplus U_2) \subseteq (X \oplus Y)$, thus $x^k(U_1 \oplus 0) \subseteq (X \oplus 0)$ and $x^k(0 \oplus U_2) \subseteq (0 \oplus Y)$, then $x^k \in (X : U_1)$ and $x^k \in (Y : U_2)$. Therefore $x \in \sqrt{(X : U_1)} \cap \sqrt{(Y : U_2)}$.

Conversely, Llet $x \in \sqrt{(X : U_1)} \cap \sqrt{(Y : U_2)}$, that is $x^k \in (X : U_1) \cap (Y : U_2)$, for some positive integer $k$. Then $x^k(U_1 \oplus U_2) \subseteq (X \oplus Y)$ and hence $x \in \sqrt{(X \oplus Y : U_1 \oplus U_2)}$. □

**Proposition 2.12.** Let $W_1$ and $W_2$ be an $R$-module, and let $X_1$ and $X_2$ be submodules of $W_1$ and $W_2$, respectively. Then $\sqrt{(X_1 : W_1)} \cap \sqrt{(X_2 : W_2)}$ is a weakly prime if and only if $X_1 \oplus X_2$ is a weakly semi-primary submodule of $W_1 \oplus W_2$.

**Proof .** The proof is straightforward from the Lemma 2.11 □

**Proposition 2.13.** Let U be an R-module such that $U = X_1 \oplus X_2$ then $\sqrt{(X_1 : X_2)}$ is a weakly prime ideal of R if and only if $X_1$ is weakly semi-primary submodule of U.

**Proof .** We can notice that $\sqrt{(X_1 : U)} = \sqrt{(X_1 : X_1 \oplus X_2) \cap (X_1 : X_2)} = \sqrt{(X_1 : X_2)}$. Accordingly, the evidence has become clear. □

**Definition 2.14.** An R-module W is called weakly semi-primary R-module if every proper submodule of W is a weakly semi-primary submodule.

**Proposition 2.15.** Every module over a valuation ring is a weakly semi-primary module.

**Proof .** Let $X$ be a submodule of an R-module W, since R is a valuation ring then by [6] $\sqrt{(X : W)}$ is a weakly prime ideal of R. Therefore $X$ is weakly semi-primary submodule of W. □

## 3 Weakly Semi-Primary Rings

In this section we prove that every ideal of a ring R is a weakly semi-primary, and we offer the relationship between a weakly semi-primary ideal and local ring. Also the correspondence between the weakly semi-primary ideals of R and R[y] is studied. And recall that prufer domains were defined in [8] as domains in which every finitely generated ideal is invertible.

**Proposition 3.1.** The ideal $I$ is weakly semi-primary of R if and only if for all $i, j \in \frac{R}{I}$ such that $ij = 0$, either $i^n = 0$ *or* $j^n = 0$, for some n is positive integer.

**Proof .** Evident. □

**Proposition 3.2.** The (0) is a weakly semi-primary ideal of a ring R, then (0) and (1) are the only idempotent elements of R.

**Proof .** Let $i$ be an idempotent elements in R, since (0) is a weakly semi-primary ideal then by Proposition 3.1, $i\,(1-i)=0$, so either $i^n=0$ or $(1-i)^n=0$. Hence either $i=0$ or $(1-i)=0$. □

**Corollary 3.3.** Let (0) is a weakly semi-primary ideal of a ring R, then R is an indecomposable ring.

**Proof .** Let $I$ and $J$ be an ideal of R, such that $R=I\oplus J$, then $i+j=1$, thus $i=i^2+ij$ and $i=i^2$, hence by Proposition 3.2, either $i=0$ $i=1$ or. If $i=0$, then $R=J$ and if $i=1$ then $R=I$. □

**Corollary 3.4.** Let $R$ be a ring such that every cyclic ideal of $R$ is projective. If (0) is a weakly semi-primary ideal, then $R$is an integral domain.

**Proof .** Let $a\in R$, and $a\neq 0$, think about the following exact sequence:

$$\begin{array}{ccc} 0 & \longrightarrow & ann(a) \\ \uparrow & & \downarrow \\ Ra & \longleftarrow & R \end{array}$$

Since $Ra$is projective, $R=Ra\oplus ann(a)$. Thus by Coro.llary 3.3, either $Ra=0\ \ or\ ann\,(a)=0$. But $Ra\neq 0$, then $\mathrm{ann}\,(a)=0$. Therefore $R$ is integral domain. □

Recall that a ring R with unit element is said semi-hereditary if any finitely generated ideal of R is projective [3].

**Corollary 3.5.** Let $R$be a semi-hereditary ring. If (0) is a weakly semi-primary ideal of $R$, then $R$ is a prufer domain.

**Proof .** It is clear. □

**Theorem 3.6.** If $R$ be a ring, and (0) is weakly semi-primary, then $R$ is local ring.

**Proof .** Let $y\in R$ and $y$ be not a unit element. Then $R=(y)+(1-y)$, thus $(y)\cap(1-y)=0$ or $(y)\cap(1-y)\neq 0$. If $(y)\cap(1-y)=0$, then $R=(y)\oplus(1-y)$. Since (0) is a weakly semi-primary ideal, $R$ is an indecomposable ring, by Corollary 3.3. That is either $R=(y)$ or $R=(1-y)$, but $(y)$ is not a unit element, then $R=(1-y)$. Therefore $(1-y)$ is a unit element. Now if $(y)\cap(1-y)\neq 0$, assume that there exists amaximal ideal $I$ of R, such that, $(1-y)\in I$, so $y\,(1-y)\in(y)\cap(1-y)$, then $y\,(1-y)\in\sqrt{(y)\cap(1-y)}$, so either $y\in\sqrt{(y)\cap(1-y)}$ or $(1-y)\in\sqrt{(y)\cap(1-y)}$. If $(1-y)\in\sqrt{(y)\cap(1-y)}$, then $(1-y)^m\in(y)\cap(1-y)$, for some m is positive integer. Since $y$ is not a unit element, there exists amaximal ideal J of R, such that, $(y)\subseteq J$, that mean $(1-y)^m\in(y)\subseteq J$, then $(1-y)\in J$, thus $J=R$. This is a contradiction. Hence $y\in\sqrt{(y)\cap(1-y)}$, so $y^n\in(y)\cap(1-y)$, for some n is positive integer, where $y^n\in(1-y)\subseteq I$, that mean $I=R$. Therefore $(1-y)$ is a unit element in R. □

**Corollary 3.7.** Let R be a ring whose proper ideals are linearly ordered with respect to inclusion, then R is a weakly semi-primary ring and is a local ring.

The converse of Theorem 3.6 is not true, the following example illustrates this.

**Example 3.8.** Let R be a set of real-value continuous functions on the interval $[0,1]$, where we define $(h+q)\,(y)=h\,(y)+q(y)$ and (h.q) $(y)=h\,(y)\,.q(y)$, where $0,1$ is constant function. Then $(R,\ +,\ \ .\ \ ,\ 0,1)$, is a commutative ring. Now let $I=\left\{h:h\in R,\ \ h\left(\frac{1}{2}\right)=0\right\}$, where $I$ is a maximal ideal of R. Let:

$$h\,(y)=\begin{cases}0 & for \quad 0\leq y\leq \frac{1}{2}\\ y-\frac{1}{2} & for \quad \frac{1}{2}\leq y\leq 1\end{cases}$$

$$q\,(y)=\begin{cases}-y+\frac{1}{2} & for \quad 0\leq y\leq \frac{1}{2}\\ 0 & for \quad \frac{1}{2}\leq y\leq 1\end{cases}$$

Thus $h\,(y)\,.\mathrm{q}\,(y)=0$, but $h(y)_I\neq 0\ ,\ \ q(y)_I\neq 0$ in $R_I$ and $h(y)_I.\ q(y)_I\in\sqrt{0}$, then $h(y)_I^m\neq 0$ and $q(y)_I^m\neq 0$, for some m is positive integer. Hence (0) is not a weakly semi-primary ideal in $R_I$.

In the following remark we give description for integral domains using a weakly semi-primary concept.

**Remark 3.9.** The ring R is integral domain if and only if (0) is a weakly semi-primary ideal of R and $\sqrt{0} = 0$.

**Proof .** It is evident. □

If $X$ is a weakly semi-primary of W and $I = \sqrt{(X, W)}$, we say that $X$ is $I-$weakly semi-primary ideal.

**Proposition 3.10.** If $J$ is $I$- weakly semi-primary ideal of a ring R, then $J[y]$ is $I[Y]$- weakly semi-primary ideal of $R[y]$.

**Proof .** Let $g \in I[y]$, such that $g = b_0 + b_1 y + \cdots + b_n y^n$ ,$b_i \in I$, where $0 \leq i \leq n$, since $I = \sqrt{J}$ then for all i there exists $n_i$ is positive integer, such that $b_i^{n_i} \in J$. Now let $h = m_0 + m_1 + \cdots + m_n$ then $b_i^h \in J$ for all $i$. Then $g^h \in J[y]$, hence $I[y] \subseteq J[y]$. Since $J \subseteq I$, $J[y] \subseteq I[y]$, that is $\sqrt{J[y]} \subseteq I[y]$. Hence $\sqrt{J[y]} = I[y]$. □

## 4 Weakly Semi-Primary Submodules of Multiplication Modules

In this section, we study the properties of multiplication module which contains a weakly semi-primary submodule, where an R-module W is called a multiplication module if every submodule $K$ of W is of the from $K = JW$ for some ideal $J$ of R [2]. And we also proved if W is a multiplication faithful R-module, then W is a weakly semi-primary R-module if and only if R is a weakly semi-primary ring. And we prove that if a multiplication R-module which contains a finitely generated weakly semi-primary submodule so the module is finitely generated.

**Theorem 4.1.** Let W be a multiplication R-module the following statement are equivalent for a proper sub-module $X$ of W:

(1) $X$ is a weakly semi-primary submodule.
(2) $(X : W)$ is a weakly semi-primary ideal of R.
(3) $X = JW$ for some weakly semi-primary ideal $J$of R with $ann\ (W) \subseteq J$

**Proof .**
(1) $\Longrightarrow$ (2) It is evident by definition.
(2) $\Longrightarrow$ (3) Put $J = (X : W)$.
(3) $\Longrightarrow$ (1) Let $X = JW$ for some weakly semi-primary ideal $J$of R with $ann\ (W) \subseteq J$, since W is multiplication module, so $X = (X : W)\, W$, we assume that $\sqrt{(X : W)} = \sqrt{J}$. But JW is a proper submodule of W, then by [4] JW is contained a maximal submodule H of W. That is $I\,(\text{JW}) = \sqrt{J}W \neq W$, but $J$ is weakly semi-primary, then $\sqrt{J}$ is weakly prime. Thus $(X : W)\, W = JW \subseteq \sqrt{J}W$ and $\sqrt{J}W \neq JW$, then there exists $j \in W$ and $j \notin \sqrt{J}W$ such that $j\,(X : W) \subseteq \sqrt{J}W$. Then by [4] $(X : W) \subseteq \sqrt{J}$, hence $\sqrt{(X : W)} \subseteq \sqrt{J}$. Since $X = JW$, implies that $J \subseteq (X : W)$, thus $\sqrt{J} \subseteq \sqrt{(X : W)}$. Therefore X is weakly semi-primary submodule. □

**Corollary 4.2.** Let U be a multiplication faithful R-module, then U is a weakly semi-primary R-module if and only if R is a weakly semi-primary ring.

**Corollary 4.3.** Let U be a multiplication faithful R-module weakly semi-primary R-module, then U is a cyclic R-module, and isomorphic to R.

**Proof .** Let U be a weakly semi-primary R-module, then R is weakly semi-primary ring. Hence by (Theorem 3.6) R is local ring, then U is cyclic and isomorphic to R by [2] □

**Lemma 4.4.** Let W be a multiplication R-module the following statement are equivalent for a proper submodule $X$ of W:

(1) $X$ is a weakly prime submodule.
(2) ann $\left(\frac{W}{X}\right)$ is a weakly prime ideal of R.
(3) $X = AW$, where $A$ is weakly prime ideal of R, which is a maximal with respect to this property (i.e., $JW \subseteq X$ *implies that* $J \subseteq A$).

**Proof .** Straightforward. □

**Proposition 4.5.** Let $W$be a multiplication $R$-module which contains finitely generated weakly semi-primary submodule, then $W$ is a finitely generated R-module.

**Proof .** Let $X$ be a weakly semi-primary submodule of $W$, then $\sqrt{(X:W)}$ is weakly prime ideal of $R$, and thus $A = \sqrt{(X:W)}W$ is a weakly prime submodule of $W$, by Lemma 4.4. Since $W$ is multiplication, we have $\frac{W}{A}$ is multiplication and $\operatorname{ann}\left(\frac{W}{A}\right)$ is weakly prime. Then $\frac{W}{A}$ is finitely generated R-module [7]. So $W = R_{y_1} + R_{y_2} + \cdots + R_{y_n} + A$, where $y_1 + y_2 + \cdots + y_n \in W$. We claim that $W = R_{y_1} + R_{y_2} + \cdots + R_{y_n} + X$. Assume that, $R_{y_1} + R_{y_2} + \cdots + R_{y_n} + X \neq W$, then there exists a maximal submodule $B$ of W such that $R_{y_1} + R_{y_2} + \cdots + R_{y_n} + X \subseteq B$ by [4]. That mean $H(X) = \sqrt{(X:W)}W \subseteq B$, thus $W \subseteq B$, that is contradiction. Therefore $W = R_{y_1} + R_{y_2} + \cdots + R_{y_n} + X$. □

**Corollary 4.6.** If $\operatorname{ann}(W)$ is a weakly semi-primary ideal of R and W is a multiplication R-module, then W is a finitely generated R-module.

**Proof .** $\operatorname{ann}(W)W = (0:W)W = (0)$, thus (0) is weakly semi-primary (Theorem 4.1). Then by (Prop. 4.5) W is a finitely generated R-module. □

**Corollary 4.7.** Let X be a weakly semi-primary submodule of a multiplication R-module W, such that $\frac{W}{X} \equiv Y$, where $Y$ is a submodule of W. If $\sqrt{ann(Y)} = \sqrt{ann(W)}$, then W is a finitely generated R-module.

**Proof .** Let $\frac{W}{X} \equiv Y$, then $\operatorname{ann}\left(\frac{W}{X}\right) = (X:W) = ann(Y)$. But X is weakly semi-primary submodule and $\sqrt{ann(Y)} = \sqrt{ann(W)}$, then $ann(W)$ is weakly semi-primary ideal of R. Then by Corollary 4.6, $W$is a finitely generated R-module. □

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