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Kamal transform technique for solving system of linear Volterra integro-differential equations of the second kind

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Abstract

Many advanced problems in science and engineering can represent mathematically in the form of linear Volterra integro-differential equations or their system. In this paper, we present the solution of the linear system of Volterra integro-differential equations of the second kind by Kamal transform. Some numerical problems have been given and solved by Kamal transform for illustrating the applicability of the Kamal transform. Results of numerical problems assert that the Kamal transform is very effective for obtaining the exact solution of the linear system of Volterra integro-differential equations of the second kind.

Keywords: Volterra Integro-Differential Equation, Kamal Transform, Convolution, Inverse Kamal Transform.
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1 Introduction

The linear system of Volterra integro-differential equations of the second kind is given by

$$\begin{cases} v_1^{(m)}(s) = h_1(s) + \int_0^s K_{11}(s-t)v_1(t)dt + \int_0^s K_{12}(s-t)v_2(t)dt + \ldots + \int_0^s K_{1n}(s-t)v_n(t)dt \\ v_2^{(m)}(s) = h_2(s) + \int_0^s K_{21}(s-t)v_1(t)dt + \int_0^s K_{22}(s-t)v_2(t)dt + \ldots + \int_0^s K_{2n}(s-t)v_n(t)dt \\ \vdots \\ v_n^{(m)}(s) = h_n(s) + \int_0^s K_{n1}(s-t)v_1(t)dt + \int_0^s K_{n2}(s-t)v_2(t)dt + \ldots + \int_0^s K_{nn}(s-t)v_n(t)dt \end{cases}$$

with the initial conditions

$$\begin{cases} v_1^{(l)}(\cdot) = a_{1l}, \quad l = 0,1,2,\ldots,m-1 \\ v_2^{(l)}(\cdot) = a_{2l}, \quad l = 0,1,2,\ldots,m-1 \\ \vdots \\ v_n^{(l)}(\cdot) = a_{nl}, \quad l = 0,1,2,\ldots,m-1; \end{cases}$$

where the unknown functions $v_1(t), v_2(t), \ldots, v_n(t)$ which shall be determined, appear only inside the integral sign whilst the derivatives of $v_1(t), v_2(t), \ldots, v_n(t)$ mostly occur outside the integral sign. The Kernels $K_{ji}(s,t)$, and the

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function $h_j(s)$ for $i = 1, 2, \ldots, n$ are given real-valued functions. There are many phenomena and processes in different areas of engineering and applied science where integro-ordinary differential equation plays an important role (like in circuit analysis, glass forming process, wave propagation, nuclear reactors, nano-hydrodynamics, biological population, visco-elasticity, optimal control system, geophysics, population genetics, radiation optimization, hereditary phenomena in biology and physics, magnetism and electricity, medicine, kinetic theory of gases, communication theory, quantum mechanics. The Kamal transform of the function $W(t)$ for all $t \geq 0$ is defined as [14]

$$K(W(t)) = \int_0^\infty W(t) e^{\frac{-t}{q}} dt = G(q),$$

where K is Kamal transform operator.

The Kamal transform of the function $W(t)$ for $t \geq 0$ exist if $W(t)$ is piecewise continuous and of exponential order. They are sufficient conditions for the existence of Kamal transform of the function $W(t)$.

In recent years some researchers for solving systems of Fredholm and Volterra integro-differential equations have used several techniques. Kamal transform has been applied to determined exact solutions to many problems in many important physics and mathematics areas.

In [4] authors used Laplace-Carson transform for solving the linear system of Volterra integro-differential equations of the second kind. [13] solved the system of Volterra-Fredholm Integro-differential equations of the second kind by modified decomposition method. [10] by applying Linear Programming problem demonstrated the numerical solution of mixed Volterra-Fredholm integral equations. In [11] the Aitken method is applied for solving Volterra-Fredholm integral equations of the second kind with the homotopy perturbation technique. [8] showed an efficient modification of the Adomian decomposition method to solve the non-linear system of Fredholm and Volterra integro-differential equations. [12] used power functions to determine the solution of the system of Fredholm integral equation of the second kind.

If we concentrate on the Kamal transform, [1] used Kamal transform for solving linear Volterra integro- differential equations of the second kind. [6] used Kamal transform to determine the solution of Abel's integral equation. [2] determined the solution of decay problems and population growth by kamal transform. [5] applied Kamal transform to solve linear Volterra integral equations of the first kind.

The main purpose of this paper is to determine the solution of the linear system of Volterra integro-differential equations of the second kind by applying Kamal transform.

Table 1: Useful Properties of Kamal Transform [7], [14], [9]

S.N	Name of Property	Mathematical Form
1	Linearity	$K\{cW_1(t) + dW_2(t)\} = cK\{W_1(t)\} + dK\{W_2(t)\}$
2	Change of Scale	$K\{W(ct)\} = \frac{1}{c}G(cq)$
3	Shifting	$K\{e^{ct}W(t)\} = G(\frac{q}{1-cq})$
4	First Derivative	$K\{W'(t)\} = \frac{1}{q}G(q) - W(.)$
5	Second Derivative	$K\{W''(t)\} = \frac{1}{q^2}G(q) - \frac{1}{q}W(.) - W'(.)$
6	nth Derivative	$K\{W^{(n)}(t)\} = \frac{1}{q^n}G(q) - \frac{1}{q^{n-1}}W(.) - \frac{1}{q^{n-2}}W'(.) - \ldots - W^{(n-1)}(.)$
7	Convolution	$K\{W_1(t) \times W_2(t)\} = K(W_1(t)) \times K(W_2(t))$

Table 2: Kamal Transform Of Useful Functions [3]

S.N	$W(t)$	$K\{W(t)\} = G(q)$
1	1	q
2	t	q^2
3	t^2	$2!q^3$
4	$t^n, n \geq 0$	$n!q^{n+1}$
5	e^{ct}	$\frac{q}{1-cq}$
6	$\sin ct$	$\frac{cq^2}{1+c^2q^2}$
7	$\cos ct$	$\frac{q}{1+c^2q^2}$
8	$\sin hct$	$\frac{cq^2}{1-c^2q^2}$
9	$\cos hct$	$\frac{q}{1-c^2q^2}$

Table 3: Invers Kamal Transform Of Useful Functions [3]

S.N	$G(q)$	$W(t) = K^{-1}(G(q))$
1	q	q
2	q^2	t
3	q^3	$\frac{t^2}{2!}$
4	$q^{n+1}, n \geq 0$	$\frac{t^n}{n!}$
5	$\frac{q}{1-cq}$	e^{ct}
6	$\frac{q^2}{1+c^2q^2}$	$\frac{\sin ct}{c}$
7	$\frac{q}{1+c^2q^2}$	$\cos ct$
8	$\frac{q}{1-c^2q^2}$	$\frac{\sin hct}{c}$
9	$\frac{q}{1-c^2q^2}$	$\cos hct$

2 Kamal Transform to Solve System of Linear Voltera Integro-Differential Equations of the Second Kind

The general system of linear Volterra integro-ordinary differential equations of second kind is given by [15],

$$\begin{cases} v_1^{(m)}(s) = h_1(s) + \int_0^s K_{11}(s-t)v_1(t)dt + \int_0^s K_{12}(s-t)v_2(t)dt + \ldots + \int_0^s K_{1n}(s-t)v_n(t)dt \\ v_2^{(m)}(s) = h_2(s) + \int_0^s K_{21}(s-t)v_1(t)dt + \int_0^s K_{22}(s-t)v_2(t)dt + \ldots + \int_0^s K_{2n}(s-t)v_n(t)dt \\ \vdots \\ v_n^{(m)}(s) = h_n(s) + \int_0^s K_{n1}(s-t)v_1(t)dt + \int_0^s K_{n2}(s-t)v_2(t)dt + \ldots + \int_0^s K_{nn}(s-t)v_n(t)dt \end{cases} \tag{2.1}$$

with

$$\begin{cases} v_1^{(l)}(\cdot) = a_{1l}, \quad l = 0,1,2,\ldots,m-1 \\ v_2^{(l)}(\cdot) = a_{2l}, \quad l = 0,1,2,\ldots,m-1 \\ \vdots \\ v_n^{(l)}(\cdot) = a_{nl}, \quad l = 0,1,2,\ldots,m-1; \end{cases} \tag{2.2}$$

Taking Kamal transform's operator on system (2.1) and using convolution theorem of Kamal transform, we get

$$\begin{aligned} K(v_1^{(m)}(s)) &= K(h_1(s))\Big[K(K_{11}(s))K(v_1(s)) + K(K_{12}(s))K(v_2(s)) + \ldots + K(K_{1n}(s))K(v_n(s))\Big] \\ K(v_2^{(m)}(s)) &= K(h_2(s))\Big[K(K_{21}(s))K(v_1(s)) + K(K_{22}(s))K(v_2(s)) + \ldots + K(K_{2n}(s))K(v_n(s))\Big] \\ &\ldots \\ K(v_n^{(m)}(s)) &= K(h_n(s))\Big[K(K_{n1}(s))K(v_1(s)) + K(K_{n2}(s))K(v_2(s)) + \ldots + K(K_{nn}(s))K(v_n(s))\Big] \end{aligned} \tag{2.3}$$

Using the property "Kamal transforms of derivatives" on system (2.3), we get

$$\begin{aligned} &\frac{1}{q^m}K(v_1(s)) - \frac{1}{q^{m-1}}v_1(.) - \frac{1}{q^{m-2}}v_1'(.) - \ldots - v_1^{(m-1)}(.) = K(h_1(s)) + \Big[K(K_{11}(s))K(v_1(s)) + \ldots + K(K_{1n}(s))K(v_n(s))\Big] \\ &\frac{1}{q^m}K(v_2(s)) - \frac{1}{q^{m-1}}v_2(.) - \frac{1}{q^{m-2}}v_2'(.) - \ldots - v_2^{(m-1)}(.) = K(h_2(s)) + \Big[K(K_{21}(s))K(v_1(s)) + \ldots + K(K_{2n}(s))K(v_n(s))\Big] \\ &\vdots \\ &\frac{1}{q^m}K(v_n(s)) - \frac{1}{q^{m-1}}v_n(.) - \frac{1}{q^{m-2}}v_n'(.) - \ldots - v_n^{(m-1)}(.) = K(h_n(s)) + \Big[K(K_{n1}(s))K(v_1(s)) + \ldots + K(K_{nn}(s))K(v_n(s))\Big] \end{aligned} \tag{2.4}$$

substituting equation (2.2) in system (2.4), we obtain

$$\begin{aligned}
&\frac{1}{q^m}K(v_1(s)) - \frac{1}{q^{m-1}}a_{10} - \frac{1}{q^{m-2}}a_{11} - \ldots - a_{1(m-1)} = K(h_1(s)) + \Big[K(K_{11}(s))K(v_1(s)) + \ldots + K(K_{1n}(s))K(v_n(s))\Big]\\
&\frac{1}{q^m}K(v_2(s)) - \frac{1}{q^{m-1}}a_{20} - \frac{1}{q^{m-2}}a_{21} - \ldots - a_{2(m-1)} = K(h_2(s)) + \Big[K(K_{21}(s))K(v_1(s)) + \ldots + K(K_{2n}(s))K(v_n(s))\Big]\\
&\ldots\\
&\frac{1}{q^m}K(v_n(s)) - \frac{1}{q^{m-1}}a_{n0} - \frac{1}{q^{m-2}}a_{n1} - \ldots - a_{n(m-1)} = K(h_n(s)) + \Big[K(K_{n1}(s))K(v_1(s)) + \ldots + K(K_{nn}(s))K(v_n(s))\Big]
\end{aligned} \tag{2.5}$$

After simplification system (2.5), we get

$$\begin{aligned}
&\Big[\frac{1}{q^m} - K(K_{11}(s))K(v_1(s)) + \ldots + K(K_{1n}(s))K(v_n(s))\Big] = K(h_1(s)) + \frac{1}{q^{m-1}}a_{10} + \frac{1}{q^{m-2}}a_{11} + \ldots + a_{1(m-1)}\\
&\Big[- K(K_{21}(s))K(v_1(s)) + (\frac{1}{q^m}K(K_{22}(s))) - \ldots - K(K_{2n}(s))K(v_n(s))\Big] = K(h_2(s)) + \frac{1}{q^{m-1}}a_{20} + \frac{1}{q^{m-2}}a_{21} + \ldots + a_{2(m-1)}\\
&\ldots\\
&\Big[- K(K_{n1}(s))K(v_1(s)) - K(K_{n2}(s))K(v_2(s)) - \ldots + (\frac{1}{q^m} - K(K_{nn}(s)))K(v_n(s))\Big] = K(h_n(s)) + \frac{1}{q^{m-1}}a_{n0} + \frac{1}{q^{m-2}}a_{n1} + \ldots + a_{n(m-1)}
\end{aligned} \tag{2.6}$$

The solution of system (2.6) is given as

$$K(v_1(s)) = \frac{\begin{bmatrix}(\frac{1}{q^m} - K(K_{11}(s)))\Big(K(h_1(s)) + \frac{1}{q^{m-1}}a_{10} + \frac{1}{q^{m-2}}a_{11} + \ldots + a_{1(m-1)}\Big) - \ldots - K(K_{1n}(s))\\ - K(K_{21}(s))\Big(K(h_2(s)) + \frac{1}{q^{m-1}}a_{20} + \frac{1}{q^{m-2}}a_{21} + \ldots + a_{2(m-1)}\Big) - \ldots - K(K_{2n}(s))\\ \ldots\\ - K(K_{n1}(s))\Big(K(h_n(s)) + \frac{1}{q^{m-1}}a_{n0} + \frac{1}{q^{m-2}}a_{n1} + \ldots + a_{n(m-1)}\Big) - \ldots - (\frac{1}{q^m} - K(K_{nn}(s)))\end{bmatrix}}{\begin{bmatrix}(\frac{1}{q^m} - K(K_{11}(s))) - K(K_{12}(s)) - \ldots - K(K_{1n}(s))\\ - K(K_{21}(s))\Big(\frac{1}{q^m} - K(K_{22}(s))\Big) - \ldots - K(K_{2n}(s))\\ \ldots\\ - K(K_{n1}(s)) - K(K_{n2}(s)) - \ldots - (\frac{1}{q^m} - K(K_{nn}(s)))\end{bmatrix}}$$

$$K(v_2(s)) = \frac{\begin{bmatrix}(\frac{1}{q^m} - K(K_{11}(s)))\Big(K(h_1(s)) + \frac{1}{q^{m-1}}a_{10} + \frac{1}{q^{m-2}}a_{11} + \ldots + a_{1(m-1)}\Big) - \ldots - K(K_{1n}(s))\\ - K(K_{21}(s))\Big(K(h_2(s)) + \frac{1}{q^{m-1}}a_{20} + \frac{1}{q^{m-2}}a_{21} + \ldots + a_{2(m-1)}\Big) - \ldots - K(K_{2n}(s))\\ \ldots\\ - K(K_{n1}(s))\Big(K(h_n(s)) + \frac{1}{q^{m-1}}a_{n0} + \frac{1}{q^{m-2}}a_{n1} + \ldots + a_{n(m-1)}\Big) - \ldots - (\frac{1}{q^m} - K(K_{nn}(s)))\end{bmatrix}}{\begin{bmatrix}(\frac{1}{q^m} - K(K_{11}(s))) - K(K_{12}(s)) - \ldots - K(K_{1n}(s))\\ - K(K_{21}(s))\Big(\frac{1}{q^m} - K(K_{22}(s))\Big) - \ldots - K(K_{2n}(s))\\ \ldots\\ - K(K_{n1}(s)) - K(K_{n2}(s)) - \ldots - (\frac{1}{q^m} - K(K_{nn}(s)))\end{bmatrix}}$$

$$K(v_n(s)) = \frac{\begin{bmatrix}(\frac{1}{q^m} - K(K_{11}(s))) - K(K_{12}(s)) \ldots \Big(K(h_1(s)) + \frac{1}{q^{m-1}}a_{10} + \frac{1}{q^{m-2}}a_{11} + \ldots + a_{1(m-1)}\Big)\\ - K(K_{21}(s))(\frac{1}{q^m} - K(K_{22}(s)))\Big(K(h_2(s)) + \frac{1}{q^{m-1}}a_{20} + \frac{1}{q^{m-2}}a_{21} + \ldots + a_{2(m-1)}\Big)\\ \ldots\\ - K(K_{n1}(s)) - K(K_{2n}(s))\Big(K(h_n(s)) + \frac{1}{q^{m-1}}a_{n0} + \frac{1}{q^{m-2}}a_{n1} + \ldots + a_{n(m-1)}\Big)\end{bmatrix}}{\begin{bmatrix}(\frac{1}{q^m} - K(K_{11}(s))) - K(K_{12}(s)) - \ldots - K(K_{1n}(s))\\ - K(K_{21}(s))\Big(\frac{1}{q^m} - K(K_{22}(s))\Big) - \ldots - K(K_{2n}(s))\\ \ldots\\ - K(K_{n1}(s)) - K(K_{n2}(s)) - \ldots - (\frac{1}{q^m} - K(K_{nn}(s)))\end{bmatrix}}$$

After simplification of above equations, we have the values of $K(v_1(s)), K(v_2(s)), \ldots, K(v_n(s))$. After taking the inverse Kamal transform on these values, we get the required values of $v_1(s), v_2(s), \ldots, v_n(s)$.

3 Numerical problems

In this part of the paper, some numerical problems have been considered for explaining the complete methodology.

Problem 1: Consider the following system of linear Volterra integro-ordinary differential equations of the second kind.

$$\begin{aligned} v_1^{'}(s) &= 2s^2 + \int_0^s (s-t)v_1(t)dt + \int_0^s (s-t)v_2(t)dt \\ v_2^{'}(s) &= -3s^2 - \frac{1}{10}s^5 + \int_0^s (s-t)v_1(t)dt - \int_0^s (s-t)v_2(t)dt \end{aligned} \tag{3.1}$$

with

$$v_1(0) = 1, \quad v_2(0) = 1 \tag{3.2}$$

Operating Kamal transform on system (3.1) and using convolution theorem of Kamal transform, we have

$$\begin{aligned} K(v_1^{'}(s)) &= 2K(s^2) + K(s)K(v_1(s)) + K(s)K(v_2(s)) \\ K(v_2^{'}(s)) &= -3K(s^2) - \frac{1}{10}K(s^5) + K(s)K(v_1(s)) - K(v_2(s)) \end{aligned} \tag{3.3}$$

Using the property "Kamal transforms of derivatives" on system (3.3), we have

$$\begin{aligned} \frac{1}{q}K(v_1(s)) - v_1(0) &= (2)2!q^3 + q^2K(v_1(s)) + q^2K(v_2(s)) \\ \frac{1}{q}K(v_2(s)) - v_2(0) &= (-3)2!q^3 - \frac{1}{10}5!q^6 + q^2K(v_1(s)) - q^2K(v_2(s)) \end{aligned} \tag{3.4}$$

Using equation (3.2) in system (3.4), we get

$$\begin{aligned} \frac{1}{q}K(v_1(s)) - 1 &= 4q^3 + q^2K(v_1(s)) + q^2K(v_2(s)) \\ \frac{1}{q}K(v_2(s)) - 1 &= -6q^3 - \frac{1}{10}5!q^6 + q^2K(v_1(s)) - q^2K(v_2(s)). \end{aligned} \tag{3.5}$$

After simplification system (3.5), we have

$$\begin{aligned} &(\frac{1}{q} - q^2)K(v_1(s)) - q^2K(v_2(s)) = 1 + 4q^3 \\ &- q^2K(v_1(s)) + (\frac{1}{q} + q^2)K(v_2(s)) = 1 - 6q^3 - 12q^6. \end{aligned} \tag{3.6}$$

The solution of system (3.6) is given by

$$\begin{aligned} K(v_1(s)) &= \frac{\begin{vmatrix} (1+4q^3) & -q^2 \\ (1-6q^3-12q^6) & (\frac{1}{q}+q^2) \end{vmatrix}}{\begin{vmatrix} (\frac{1}{q}-q^2) & -q^2 \\ -q^2 & (\frac{1}{q}+q^2) \end{vmatrix}} = q + 3!q^4 \\ K(v_2(s)) &= \frac{\begin{vmatrix} (\frac{1}{q}-q^2) & 1+4q^3 \\ -q^2 & (1-6q^3-12q^6) \end{vmatrix}}{\begin{vmatrix} (\frac{1}{q}-q^2) & -q^2 \\ -q^2 & (\frac{1}{q}+q^2) \end{vmatrix}} = q - 3!q^4. \end{aligned} \tag{3.7}$$

Operating inverse Kamal transforms on system (3.7), we get the required solution of system (3.1) with (3.2) as

$$\begin{aligned} v_1(s) &= K^{-1}(q + 3!q^4) = K^{-1}(q) + K(3!q^4) = 1 + s^3 \\ v_2(s) &= K^{-1}(q - 3!q^4) = K^{-1}(q) - K(3!q^4) = 1 - s^3. \end{aligned}$$

Problem 2: Consider the following system of linear Volterra integro-ordinary differential equations of the second kind.

$$
\begin{aligned}
v_1^{''}(s) &= -s^3 - s^4 + \int_0^s 3v_2(t)dt + \int_0^s 4v_3t)dt \\
v_2^{''}(s) &= 2 + s^2 - s^4 + \int_0^s 4v_3(t)dt - \int_0^s 2v_1(t)dt \\
v_3^{''}(s) &= 6 - s^2 + s^3 + \int_0^s 2v_1(t)dt - \int_0^s 3v_2(t)dt
\end{aligned} \tag{3.8}
$$

with

$$
v_1(0) = 0, \;\; v_1^{'}(0) = 1, \; v_2(0) = 0, \; v_2^{'}(0) = 0, \; v_3(0) = 0, \; v_3^{'}(0) = 0. \tag{3.9}
$$

Operating Kamal transform on system (3.8) and using convolution theorem of Kamal transform, we have

$$
\begin{aligned}
K(v_1^{''}(s)) &= -K(s^3) - K(s^4) + K(3)K(v_2(s)) + K(4)K(v_3(s)) \\
K(v_2^{''}(s)) &= K(2) + K(s^2) - K(s^4) + K(4)K(v_3(s)) - K(2)K(v_1(s)) \\
K(v_3^{''}(s)) &= 6K(s) - K(s^2) + K(s^3) + K(2)K(v_1(s)) - K(3)K(v_2(s)).
\end{aligned} \tag{3.10}
$$

Using the property "Kamal transforms of derivatives" on above system, we have

$$
\begin{aligned}
\frac{1}{q^2}K(v_1(s)) - \frac{1}{q}v_1(0) - v_1^{'}(0) &= -3!q^4 - 4!q^5 + 3qK(v_2(s)) + 4qK(v_3(s)) \\
\frac{1}{q^2}K(v_2(s)) - \frac{1}{q}v_2(0) - v_2^{'}(0) &= 2q + 2!q^3 - 4!q^5 + 4qK(v_3(s)) - 2qK(v_1(s)) \\
\frac{1}{q^2}K(v_3(s)) - \frac{1}{q}v_3(0) - v_3^{'}(0) &= 6q^2 - 2!q^3 + 3!q^4 + 2qK(v_1(s)) - 3qK(v_2(s))
\end{aligned} \tag{3.11}
$$

Using equation (3.9) in system (3.11), we get

$$
\begin{aligned}
\frac{1}{q^2}K(v_1(s)) - \frac{1}{q}(0) - 1 &= -3!q^4 - 4!q^5 + 3qK(v_2(s)) + 4qK(v_3(s)) \\
\frac{1}{q^2}K(v_2(s)) &= 2q + 2!q^3 - 4!q^5 + 4qK(v_3(s)) - 2qK(v_1(s)) \\
\frac{1}{q^2}K(v_3(s)) &= 6q^2 - 2!q^3 + 3!q^4 + 2qK(v_1(s)) - 3qK(v_2(s)).
\end{aligned} \tag{3.12}
$$

After simplification system (3.12), we have

$$
\begin{aligned}
\frac{1}{q^2}K(v_1(s)) - 3qK(v_2(s)) - 4qK(v_3(s)) &= 1 - 3!q^4 - 4!q^5 \\
2qK(v_1(s)) + \frac{1}{q^2}K(v_2(s)) - 4qK(v_3(s)) &= 2q + 2!q^3 - 4!q^5 \\
-2qK(v_1(s)) + 3qK(v_2(s)) + \frac{1}{q^2}K(v_3(s)) &= 6q^2 - 2!q^3 + 3!q^4.
\end{aligned} \tag{3.13}
$$

The solution of system (3.13) is given by

$$
K(v_1(s)) = \frac{\begin{vmatrix} (1 - 3!q^4 - 4!q^5) & -3q & -4q \\ (2q + 2!q^3 - 4!q^5) & \frac{1}{q^2} & -4q \\ (6q^2 - 2!q^3 + 3!q^4) & 3q & \frac{1}{q^2} \end{vmatrix}}{\begin{vmatrix} \frac{1}{q^2} & -3q & -4q \\ 2q & \frac{1}{q^2} & -4q \\ -2q & 3q & \frac{1}{q^2} \end{vmatrix}} = q^2 \tag{3.14}
$$

$$K(v_2(s)) = \frac{\begin{vmatrix} \frac{1}{q^2} & (1-3!q^4-4!q^5) & -4q \\ 2q & (2q+2!q^3-4!q^5) & -4q \\ -2q & (6q^2-2!q^3+3!q^4) & \frac{1}{q^2} \end{vmatrix}}{\begin{vmatrix} \frac{1}{q^2} & -3q & -4q \\ 2q & \frac{1}{q^2} & -4q \\ -2q & 3q & \frac{1}{q^2} \end{vmatrix}} = 2!q^3 \tag{3.15}$$

$$K(v_3(s)) = \frac{\begin{vmatrix} \frac{1}{q^2} & -3q & (1-3!q^4-4!q^5) \\ 2q & \frac{1}{q^2} & (2q+2!q^3-4!q^5) \\ -2q & 3q & (6q^2-2!q^3+3!q^4) \end{vmatrix}}{\begin{vmatrix} \frac{1}{q^2} & -3q & -4q \\ 2q & \frac{1}{q^2} & -4q \\ -2q & 3q & \frac{1}{q^2} \end{vmatrix}} = 3!q^4 \tag{3.16}$$

taking inverse Kamal transforms on equations (3.14), (3.15), and (3.16) we obtain the required solution of the system (3.8) with (3.9) as

$$v_1(s) = K^{-1}(q^2) = s$$
$$v_2(s) = K^{-1}(2!q^3) = s^2$$
$$v_3(s) = K^{-1}(3!q^4) = s^3$$

4 Conclusions

In this paper, we have successfully discussed the Kamal transform for solving the system of linear Volterra integro-ordinary differential equations of the second kind and the methodology is completely explained by two numerical problems. The results of these problems show that the Kamal transform is a very effective and beneficial integral transform to determine the exact solution of a linear system of Volterra integro-ordinary differential equations of the second kind. The proposed scheme can be applied to a nonlinear system of Volterra-integral equations.

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