Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 803–813
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.17800.1964

Estimation of returns to scale with reduced computational complexity in Data Envelopment Analysis

Maryam Jamshidy Naeiny[a], Mohsen Rostamy Malkhalifeh[a,*], Farhad Hosseinzadeh Lotfi[a], Razieh Mehrjoo[b]

[a] *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*

[b] *Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran*

(Communicated by Madjid Eshaghi Gordji)

Abstract

The estimating returns to scale (RTS) and identifying the area in which the type of returns to the scale of a single decision maker unit (DMU) remains sustainable helps managers to have more accurate predictions about the size variations of a DMU. Contrary to theoretical studies and attitudinal surveys, the RTS-based econometric estimates of the panel data are derived from the time changes observed in inputs and outputs. Therefore the description of returns-to-scale based on the panel data is closer to those that are related to changes in demand and changes in the policy of managers during the sample period. it's clear that the use of the panel data, in addition to estimating returns to scale closer to the view of the managers, leads to a significant increase in the number of inputs and outputs for each DMU. Since we know that the DEA models have computational complexity for the great practical issues, use particularly when there are large numbers of outputs and inputs reducing the number of models will greatly help to solve these issues. In most articles, the returns-to-scale estimates for decision-making units (DMU) are performed with at least two LP models. Furthermore have problems in some cases and may not be able to investigate all DMUs. In this paper, a method has been proposed which determines returns-to-scale and identifies boundary units by solving only one model. In fact, unlike some articles which explore the boundary DMUs, the assessment area in this paper is all the DMUs observed in PPS (Production Possible Set). Therefore, based on this approach, identifying boundary units and estimating their returns-to-scale will be possible by solving only one LP model. Further, according to the proposed dual model, a different perspective has been proposed to identify and evaluate boundary units and to estimate the returns-to-scale by solving a single LP model. Additionally and according to numerous observations, the dual model is able to estimate the returns-to-scale of the image point of the inefficient units in the BCC model in the input orientation. The correctness of all content has been shown theoretically as well as intuitively, with due substantiation.

Keywords: Boundary unit, Data envelopment analysis, Panel data, Returns-to-scale
2010 MSC: 90C08

1 Introduction

Estimating returns-to-scale is one of the most important issues in Data Envelopment Analysis (DEA). the estimating returns to scale (RTS) helps managers to have more accurate predictions about the size variations of a DMU

*Corresponding author
Email addresses: mar_jam_na@yahoo.com (Maryam Jamshidy Naeiny), Mohsen_rostamy@yahoo.com (Mohsen Rostamy Malkhalifeh), hosseinzadeh_lotfi@yahoo.com (Farhad Hosseinzadeh Lotfi), raziehmehrjooiau@gmail.com (Razieh Mehrjoo)

Received: May 2019 *Accepted:* May 2020

. There are different methods to do this. [2] have paid particular attention to the BCC linear model with a unique optimal solution. A unique reliant hyper plane has been estimated using the DEA method for Production Possible Set(PPS).Their approach does not guarantee that RTS will be unique in the presence of multiple optimal responses. This limitation caused considerable concern in many applications where multiple optimal solutions are available. Thus, some methods have been developed to overcome this problem. [4] presented a method based on solving two LP models by examining all the optimal solutions of BCC and CCR models. Also, [9, 14, 7, 11], and [2, 3] have introduced alternative methods to estimate RTS by solving at least two LP models in DEA.

[8] introduced a method for determining the left and right return toscale. They used two LP models to estimate RTS for each DMUin this method: however, this method will fail when at least one of these models is not possible. [10] introduced an advanced method for solving two LP models for estimating RTS in suspected cases where Golany and Yu's method is not applicable. [1] presented a method based on solving two LP models: one of these was in the input orientation and the other in the output orientation, taking into account the concept of left and right RTS from the point of view of [8] method. The models presented by [6] and by [1] are always feasible and do not have the infeasibility of the [8] model. [12] examine this assessment in a more general sense, the whole of the PPS boundary. By solving two DEA models, the boundary units are identified, and then the returns-to-scale of the boundary units are determined.

In statistics and econometrics, panel data or longitudinal data are multi-dimensional data involving measurements over time. Panel data contain observations of multiple inputs and outputs obtained over multiple time periods for the same DMUs. [13] illustrated that access to a large panel data set allows us to deal with Demsetz effects and measurement errors. Contrary to theoretical studies and attitudinal surveys, the RTS-based econometric estimates of the data panel are derived from the time changes observed in inputs and outputs. Therefore, their description of returns to scale is closer to those that are related to changes in demand and changes in the policy of managers during the sample period.It is well known that novice DMUs tend to be small and to have relatively high failure rates. also that among these DMUs, the less efficient units fail more frequently. Therefore, if DMUs that have not been observed over all years (have not been active for all years of a sample period), their data for all sample years will be eliminated from the analysis and used cross-sectional data for analysis, the ratio of estimated change in input per unit to the change in output may be biased. For example, if a novice efficient DMU needs to have a relatively high changes in inputs for outputs changes, then eliminating information during the period and using cross-sectional data will probably result in estimate of increasing returns to scale that diverges from The views of managers. A corollary of this finding is that most estimating RTS based on cross-sectional data will result in exaggerated estimating returns to scale that may not provide The opinion of managers.

Using this explanation, it's clear that the use of the data panel, in addition to estimating returns to scale closer to the view of the managers, leads to a significant increase in the number of inputs and outputs for each DMU.Since we know that the DEA models have computational complexity for the great practical issues, use particularly when there are large numbers of outputs and inputs reducing the number of models will greatly help to solve these issues.

The our proposed estimation approach is of practical use particularly when there are large numbers of outputs and inputs, a situation that often results using standard econometric methods and aggregate annual data even when firm or establishment data are used.

Therefore, based on this approach, identifying boundary units and estimating their returns-to-scale will be possible by solving only one LP model. Therefore, a lot of computational complexity is reduced.

In this paper, our approach is to estimate the returns-to-scale for a DMU. The present paper aims to achieve the goal of identifying boundary units and determining RTS by solving a single LP model. Our approach is a one-step method that, in addition to identifying boundary units, determines the returns-to-scale of a DMU in the case of a boundary. In fact, the field of evaluation in this method is not necessarily boundary DMUs, and there is no need to solve an independent LP model for identifying boundary units and then estimating returns-to-scale. The proposed model performs both actions simultaneously in a LP model. The model's duals also provide an intuitive and theoretical view for identifying boundary units and estimating their returns-to-scale from duality perspective:it leads to interesting and remarkable results in this regard. The results of this assessment are consistent with the results of [3].

2 Basic Concepts

Suppose that $\{DMU_j | j = 1, \cdots, n\}$ is the set of n observation generating m inputs $x_{ij} \geq 0$ $(i = 1, \cdots, m)$ and s outputs $y_{rj} \geq 0$ $(r = 1, \cdots, s)$ for each $j = 1, \cdots, n$. The PPS with variable return-to-scale will be in the following form:

$$T_v = \Big\{(x,y)|x \geq \sum_{j=1}^{n}\lambda_j x_j,\ y \leq \sum_{j=1}^{n}\lambda_j y_j,\ \sum_{j=1}^{n}\lambda_j = 1, \lambda_j \geq 0,\ \ j = 1,\cdots n\Big\}$$

$DMU_0 \in T_v$. If the efficiency value of a specific DMU_0 is evaluated with the BCC multiplier model in DEA (in the input orientation):

$$\begin{aligned} \max \quad & uy_0 + u_0 \\ s.t \quad & uy_j - vx_j + u_0 \leq 0 \\ & vx_0 = 1 \\ & (u,v) \geq 0 \end{aligned} \tag{2.1}$$

Definition 2.1. If (u^*, v^*, u_0^*) is the optimal solution for the model (2.1), we denote the DMU_0 in the boundary $T_v(DMU_0 \in \partial T_v)$ whenever $u^* y_0 + u_0^* = 1$.
If $(DMU_0 \notin \partial T_v)$, then is an internal unit in T_v and it can be written that $(DMU_0 \in int\ T_v)$ and DMU_0 is known as the inefficient unit.

3 RTS Estimation with Banker and Thrall (1992)

[4] have shown that the return-to-scale for each boundary unit is estimated as follows:

Theorem 3.1. If $(DMU_0 \in T_v)$ is strongly efficient, by solving the two models (3.1) and (3.2) we have:

i) If $u_0^+ < 0$, then DMU_0 has decreasing returns-to-scale (DRS).

ii) If $u_0^- > 0$, then DMU_0 has increasing returns-to-scale (IRS).

iii) If $u^+ \geq 0, u_0^- \leq 0$ then DMU_0 has constant returns-to-scale (CRS).

The correctness of this issue has been shown in Theorem 7 of Seiford and Zhu [13] (1999).

Result 3.2. Clearly, if $u_0^+ \geq 0$ and $u_0^- \leq 0$, then there is a feasible solution of (u', v', u_0') for the model (3.1) and (3.2) that $u_0' = 0$.

The correctness of this is apparent because the feasible regions of the models (3.1) and (3.2) are convex and equal. Result 3.2 has been used in this paper.

$$\begin{aligned} u_0^+ &= \max u_0 \\ s.t \quad & u^t y_j - v^t x_j + u_0 \leq 0, \quad j = 1,\cdots,n \\ & u^t y_0 + u_0 = 1 \\ & v^t x_0 = 1, \qquad u \geq 0, \quad v \geq 0 \end{aligned} \tag{3.1}$$

$$\begin{aligned} u_0^- &= \min u_0 \\ s.t \quad & u^t y_j - v^t x_j + u_0 \leq 0, \quad j = 1,\cdots,n \\ & u^t y_0 + u_0 = 1 \\ & v^t x_0 = 1, \qquad u \geq 0, \quad v \geq 0 \end{aligned} \tag{3.2}$$

4 Proposed method for estimating returns-to-scale with a model

4.1 Single-step primal model for estimating returns-to-scale

Researchers can use panel data for cases where issues can not be explored merely as cross-sectional data. One of these cases is the estimation of the returns to scale and management decisions that result from it. The integration of panel data with cross-sectional data can provide useful information for estimating econometric models. Also, policy-making inferences can be considered based on the results obtained. Panel data contains more information and broader variation, and therefor more reliable estimates can be made. Over the past few years, panel data (panels or compilations) have become increasingly important in econometrics, especially in practical applications. These data are most widely used in economic subjects. But the real issue is not gathering a lot of data; that's how these data are analyzed. The hopeful view is that units can collect data, analyze data and ultimately make smarter decisions. Panel data means a collection of information that has a high volume and They are produced at high speed and require innovative processing approaches at logical computational complexity that can be used to make accurate estimates and Improve decision making and insight.

In most models presented for estimating RTS, the efficient assumption of power is considered for DMU_0. In this model, the field of evaluation and assessment has been considered by all the observed DMUs for the Production Possible Set (PPS). By solving only one model, the boundary units are first identified and then the RTS of these units will be estimated. We introduce the model (4.1) as a single-stage model for determining the type of returns-to-scale of a specific DMU_0 (in the case of a boundary) as the following:

$$\begin{aligned} &\min |u_0| \\ &s.t \quad u^t y_j - v^t x_j + u_0 \leq 0, \quad j = 1, \cdots, n \\ &\qquad v^t x_0 = 1 \\ &\qquad u^t y_0 + u_0 = 1, \qquad u \geq 0, \quad v \geq 0 \end{aligned} \tag{4.1}$$

u_0 is a sign free variable, with variable transformation of $u_0 = u_0^+ - u_0^-$ where $u_0^+ \geq 0$ and $u_0^- \geq 0$ and the condition $u_0^+.u_0^- : |u_0| = u_0^+ + u_0^-$. Therefore, we can write the model (4.1) as follows:

$$\begin{aligned} &\min u_0^+ + u_0^- \\ &s.t \quad u^t y_j - v^t x_j + u_0^+ - u_0^- \leq 0, \quad j = 1, \cdots, n \\ &\qquad v^t x_0 = 1 \\ &\qquad u^t y_0 + u_0^+ - u_0^- = 1 \\ &\qquad u \geq 0, \quad v \geq 0, \quad u_0^+ \geq 0, \quad u_0^- \geq 0 \end{aligned} \tag{4.2}$$

Theorem 4.1. If $(u^*, v^*, u_0^{+^*}, u_0^{-^*})$ is an optimal solution for the model (4.3), then $u_0^{+^*}.u_0^{-^*} = 0$. Therefore, $u_0^+.u_0^- = 0$ in model (4.2) is an added constraint for an optimal solution, and the optimal solution set of models (4.2) and (4.3) are equal.

Proof . Suppose $(u^*, v^*, u_0^{+^*}, u_0^{-^*})$ is the optimal solution for model (4.3). According to proof by contradiction, let us assume $u_0^{+^*}.u_0^{-^*} \neq 0$, so $u_0^{+^*} > 0, u_0^{-^*} > 0$. If we consider $\delta = \min\{u_0^{+^*}, u_0^{-^*}\}$, clearly $\delta > 0$. Now, we define $u_0^{+^{'}} = u_0^{+^*} - \delta \geq 0$ and $u_0^{-^{'}} = u_0^{-^*} - \delta \geq 0$. Obviously $u_0^{+^{'}}.u_0^{-^{'}} = 0$ and $(u^*, v^*, u_0^{+^{'}}, u_0^{-^{'}})$ is a feasible solution for model (4.3). Further, since $\delta > 0$, we have:

$$u_0^{+^{'}} + u_0^{-^{'}} = u_0^{+^*} - \delta + u_0^{-^*} - \delta = u_0^{-^*} + u_0^{-^*} - 2\delta < u_0^{+^*} + u_0^{-^*}$$

Given that model (4.3) is minimized, therefore the above expression is contradictory with the optimality $(u^*, v^*, u_0^{+^*}, u_0^{-^*})$ and so the contradicted assumption is null and void. Hence $u_0^{+^*}.u_0^{-^*} = 0$. □

$$\begin{aligned} &\min u_0^+ + u_0^- \\ &s.t \quad u^t y_j - v^t x_j + u_0 \leq 0, \quad j = 1, \cdots, n \\ &\qquad v^t x_0 = 1 \\ &\qquad u^t y_0 + u_0^+ - u_0^- = 1 \\ &\qquad u \geq 0, \quad v \geq 0, \quad u_0^+ \geq 0, \quad u_0^- \geq 0 \end{aligned} \tag{4.3}$$

Using Theorem 3.1, we were able to write non-linear models (4.1) and (4.2) onto the linear model form (4.3).Therefore, the optimal solutions for the two models (4.2) and (4.3) are equal.

Theorem 4.2. In model (4.3):

i) If it is an impractical model, then DMU_0 is inefficient.

If the model is feasible and $(u^*, v^*, u_0^{+^*}, u_0^{-^*})$ is the optimal solution, then:

ii) If $u_0^{+^*} = u_0^{-^*} = 0$, then there is constant returns-to-scale (CRS) of DMU_0.

iii) If $u_0^{+^*} > 0$ then there is increasing returns-to-scale of $DMU_0(IRS)$ is observed.

iv) If $u_0^{-^*} > 0$ then there is decreasing returns-to-scale of $DMU_0(DRS)$ is observed.

Proof . i) It is clear from the model constraints and Definition 2.1.

ii) Assume $(u^*, v^*, u_0^{+^*}, u_0^{-^*})$ that is the optimal solution for model (4.3) so that $u_0^{+^*}.u_0^{-^*} = 0$. If $u_0^* = u_0^{+^*} - u_0^{-^*}$ is a feasible solution for models (3.1) and (3.2) such that $u_0^* = 0$, then DMU_0 has a constant returns-to-scale (CRS) with respect to Theorem 3.1 and the result of 3.2.

iii) If $u_0^{+^*} > 0$, then according to Theorem 4.1, $u_0^{-^*} = 0$ and therefore $(u^*, v^*, u_0^{+^*})$ is a feasible solution for the model (3.2). It can also be easily demonstrated that considering $u_0^{+^*} > 0$ for each feasible solution of model (3.2) $(u^{'}, v^{'}, u_0^{'})$, the condition $u_0^{'} > 0$ must apply. (Otherwise, the model (3.2) will have a feasible solution $(\bar{u}, \bar{v}, \bar{u}_0)$, where $\bar{u}_0 = 0$. If $\bar{u}_0^+ = 0$ and $\bar{u}_0^- = 0$, then $(\bar{u}, \bar{v}, \bar{u}_0^+, \bar{u}_0^-)$ is a feasible solution for the model (4.3). This is in contradiction with the optimality $(\bar{u}, \bar{v}, u_0^{+^*}, u_0^{-^*})$). Therefore, with respect to Theorem 3.1, the increasing returns-to-scale (IRS) of DMU_0 will be proven.

iv) If $u_0^{-^*} > 0$, then according to Theorem 4.1, $u_0^{+^*} = 0$ and $(u^*, v^*, -u_0^{-^*})$ is a feasible solution for model (3.1). This is similar to proof (iii) above, and it can be concluded that each feasible solution of model (3.1) $(u^{'}, v^{'}, u_0^{'})$ must be $u_0^{'} < 0$. Therefore, according to Theorem 3.1, there will be decrease in the returns-to-scale $DMU_0(DRS)$. □

With the help of Theorem 4.1, we have been able to provide an approach to estimate the returns-to-scale by solving only one model (4.1) or, equivalently, by solving a single linear model (4.3) in addition to the identification of the boundary unit: we thus showed its correctness according to [4].

4.2 Single-stage dual model for estimating returns-to-scale

Now we want to provide an intuitive vision and a different theory for identifying boundary units and estimating their returns -to-scale by solving a single model by examining model (4.3).The dual model (4.3) can be written as follows:

$$
\begin{aligned}
&\max \varphi - \theta \\
&s.t \quad \sum_{j=1}^{n} \lambda_j x_j \leq \theta x_0 \\
&\qquad \sum_{j=1}^{n} \lambda_j y_j \geq \varphi y_0 \\
&\qquad \sum_{j=1}^{n} \lambda_j \geq \varphi - 1 \qquad (i) \\
&\qquad \sum_{j=1}^{n} \lambda_j \leq \varphi + 1 \qquad (ii) \\
&\qquad \lambda_j \geq 0, \quad \varphi \ and\ \theta\ is\ free\ sign
\end{aligned}
\tag{4.4}
$$

If we write the two constraints (i) and (ii) in model (4.4) by slack variables, then model (4.4) will be:

$$\begin{aligned} &\max \varphi - \theta \\ s.t \quad &\sum_{j=1}^{n} \lambda_j x_j \le \theta x_0 \\ &\sum_{j=1}^{n} \lambda_j y_j \ge \varphi y_0 \\ &\sum_{j=1}^{n} \lambda_j - S^- = \varphi - 1 \\ &\sum_{j=1}^{n} \lambda_j - S^+ = \varphi + 1 \\ &\lambda_j \ge 0 \quad S^- \ge 0, \quad S^+ \ge 0, \quad \varphi \; and \; \theta \; is \; free \; sign \end{aligned} \tag{4.5}$$

Theorem 4.3. Model (4.4) is always feasible.

Proof . It is sufficient to consider $\theta = 1, \varphi = 1, \lambda_0 = 1, \lambda_j = 0, \forall j = 1, \cdots, n, j \neq 0$. $(\theta, \varphi, \lambda)$ by this definition is a feasiblesolution for the model (4.4), so model (4.4) is always feasible. □

Since model (4.4) is dual of model (4.3), each optimal solution of the model (4.1) (u^*, v^*, u_0^*) corresponds to an optimal solution of the model (4.3) with the optimum $|u_0^*| \ge 0$ and if it is feasible. Therefore, for each optimal solution of the model (4.4), $(\theta^*, \varphi^*, \lambda^*) : |U_0^*| = \varphi^* - \theta^* \ge 0$.

Theorem 4.4. If $(\theta^*, \varphi^*, \lambda^*, S^{+^*}, S^{-^*})$ is the optimal solution for model (4.5):

i) If $\varphi^* - \theta^* = 0$, then the returns-to-scale of DMU_0 is constant (CRS).

ii) If $0 < \varphi^* - \theta^* < +\infty$ and $\sum_{j=1}^{n} \lambda_j^* = \varphi^* - 1$, then there is increasing returns-to-scale of DMU_0 is observed (IRS).

iii) If $0 < \varphi^* - \theta^* < +\infty$ and $\sum_{j=1}^{n} \lambda_j^* = \varphi^* + 1$, then there is decreasing returns-to-scale of DMU_0 is observed (DRS).

iv) If there is infinite optimum, then DMU_0 is inefficient.

Proof . If model (4.5) has a finite optimal value, then model (4.3) is also feasible and has a finite optimum. Therefore, assume that $(u^*, v^*, u_0^{+^*}, u_0^{-^*})$ is an optimal solution to the model (4.3):

i) If $\varphi^* - \theta^* = 0$, then according to the duality Theorem $u_0^{+^*} + u_0^{-^*} = \varphi^* - \theta^* = 0(u_0^{+^*} \ge 0, u_0^{-^*} \ge 0)$. So, $u_0^{+^*} = 0, u_0^{-^*} = 0$: thus, $u_0^* = u_0^{+^*} - u_0^{-^*} = 0$ and with respect to Theorem 4.2 DMU_0 is a unit with constant returns-to-scale (CRS).

ii) If $0 < \varphi^* - \theta^* < +\infty$ and $\sum_{j=1}^{n} \lambda_j^* = \varphi^* - 1$ then $\sum_{j=1}^{n} \lambda_j^* < \varphi^* + 1$. Therefore, $S^{+^*} > 0$, with respect to the Complementary Slackness Theorem ([5] Theorem 6.2) for each optimal solution of the model (4.3),such as for $(u^*, v^*, u_0^{+^*}, u_0^{-^*})$ we have $u_0^{-^*} = 0$. Since $u_0^{+^*} + u_0^{-^*} = \varphi^* - \theta^* > 0$, so $u_0^{+^*} > 0$, and with respect to Theorem 4.2 DMU_0 is a unit with increasing returns-to-scale (IRS).

iii) If $0 < \varphi^* - \theta^* < +\infty$ and $\sum_{j=1}^{n} \lambda_j^* = \varphi^* + 1$ then $\sum_{j=1}^{n} \lambda_j^* > \varphi^* - 1$. Thus, $S^{-^*} > 0$ with respect to the Complementary Slackness Theorem ([5] Theorem 6.2). For any optimal solution of the model (4.3), such as for $(u^*, v^*, u_0^{+^*}, u_0^{-^*}) : u_0^{+^*} = 0$ and since $u_0^{+^*} + u_0^{-^*} = \varphi^* - \theta^* > 0$, then $u_0^{-^*} > 0$, and With respect to Theorem 4.2, DMU_0 is a unit with decreasing returns-to-scale (DRS).

iv) Model (4.5) has an infinite optimal value: since model (4.5) is the dual of model (4.3), then model (4.3) is either infeasible or has an infinite optimal value (Fundamental Theorem of Duality).Since it is a minimization problem and the feasible solution set of model (4.3) is bounded,theinfinite optimal value will not be possible and, in fact, the model (4.3) will not have a divergent and improving direction. Thus, model (4.3) is not feasible,model (4.1) is impossible, and DMU_0 is an inefficient unit.

□

Theorem 4.5. If model (4.5) has an optimal solution such that $(\theta^{'}, \varphi^{'}, \lambda^{'}, S^{+^{'}}, S^{-^{'}})$ so that $\varphi^{'} - 1 < \sum_{j=1}^{n} \lambda_j^{'} < 1 + \varphi^{'}$, then return-to-scale is constant and,hence, we have $\varphi^{'} - \theta^{'} = 0$.

Proof . If $(u^{'}, v^{'}, u_0^{+^{'}}, u_0^{-^{'}})$ is an optimal solution for model (4.3), since $\sum_{j=1}^{n} \lambda_j^{'} < \varphi^{'} + 1$, then $S^{+^{'}} > 0$ according to the weak complementary Slackness Theorem $u_0^{-^{'}} = 0$. Since $\sum_{j=1}^{n} \lambda_j^{'} > \varphi^{'} - 1$, then $S^{-^{'}} > 0$ and, with respect to the weak complementary Slackness Theorem, $u_0^{+^{'}} = 0$. Consequently, $u_0^{'} = u_0^{+^{'}} + u_0^{-^{'}} = 0$: therefore, with respect to the duality theorem, $\varphi^{'} - \theta^{'} = u_0^{'} = 0$, and the returns-to-scale of DMU_0 is constant. □

Theorems 4.4 and 4.5 lead to an examination of all possible modes of the optimal solution of model (4.5). Therefore, it provides an approach and a method with a different view from the perspective of the duality to identify the boundary units and estimate their returns-to-scale.In addition to the theoretical view, Theorem 4.4 can provide a specific geometric and intuitive view of the estimation of the return-to-scale of the boundary units as well as the return-to-scale of the bench mark point of the inefficient units in the BCC model of the input orientation.

Theorem 4.6. The optimal solution set (4.4) is boundless.

Proof . Model (4.4) is always possible with the assumption of $(\theta^*, \varphi^*, \lambda^*)$ as the optimal solution for model (4.4), thus for $\alpha > 0, (\theta^* + \alpha, \varphi^* + \alpha, \lambda_0^* + \alpha, \lambda_j^*, j \neq 0)$ is also a feasible solution for model (4.4), where $(\varphi^* + \alpha) - (\theta^* + \alpha) = \varphi^* - \theta^*$ then this is an optimal solution. Therefore, the optimal solution set is boundless. □

Therefore, the optimal solution set of model (4.4) has at least one divergent direction such as $0 \neq d = (d_\theta, d_\varphi, d_0, d_j, j \neq 0) = (1, 1, 1, 0)$. This is perpendicular to the vector of the objective function of the model (4.4) $c = (-1, 1, 0, 0), (c.d = 0)$.

5 Numerical examples

Example 5.1. A simple example reviewed in [4] was investigated by adding two inefficient DMUs and a DMU on the weak efficient boundary with the one-step model presented in this article (Model (4.3)).Its dual (model (4.3)) and the correctness of these two models are shown by comparing their results with the results of [4].

m=s=1 and n=10										
y	1	3.5	6	7	8	9	10	10	2	3
x	1	1.5	2	2.5	3	4	5	8	4	6

Table 1 shows the results of model (4.3) for the above data. The column u_{0m} shows the value $u_0^{-^*}$ and u_{0p} for the value $u_0^{+^*}$ in the evaluation of each DMU. In fact, the column u_0 represents $|u_0^*| = u_0^{+^*} + u_0^{-^*}$. The optimal value of the model (4.3) was also evaluated for each DMU. As shown in Table 1, model (4.3) estimates the return-to-scale and identifies the boundary units. These results are perfectly consistent with those of [4] for the boundary units. However, the accuracy of the results of Table 1 can also be adapted with diagram 1:

Table 1: Data of output of model (4.3)

uo	uop	uom	RTS
0.80	0.80	0.00	irs
0.53	0.53	0.00	irs
0.00	0.00	0.00	crs
0.40	0.00	0.40	drs
0.33	0.00	0.33	drs
1.25	0.00	1.25	drs
1.00	0.00	1.00	drs
0.38	0.00	4.00	drs
0.70	0.20	0.00	infeasible
0.77	0.13	0.00	infeasible

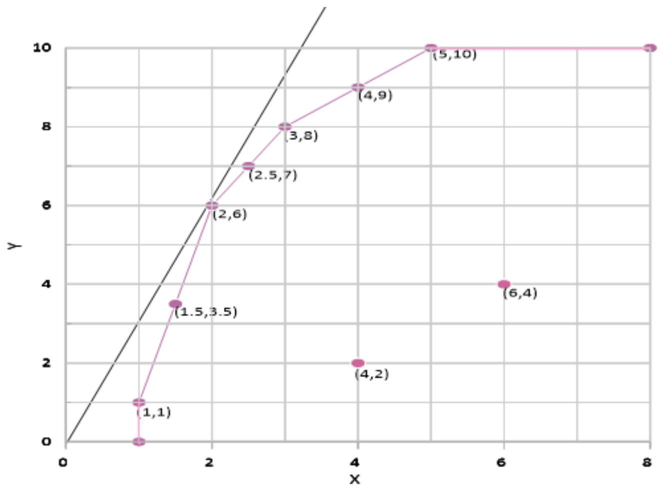

Figure 1: A data domain with $s = m = 1, n = 10$

Table 2 shows the values of model (4.5) (dual of model (4.3)). The phi-beta column evaluates the optimal value of $\varphi^* - \theta^*$ for model (4.5) for each DMU. As mentioned, with respect to the uo column in Table 1 and the phi-beta column in Table 2, if models (4.3) and (4.5) are both possible and finite, then $|u_0^*| = \varphi^* - \theta^* \geq 0$ and the results of estimating returns-to-scale for boundary units of the two models (4.3) and (4.5) are consistent with each other. Attending Table 2, it can be concluded that model (4.5), in addition to identifying inefficiencies, estimates the returns-to-scale of the image point ofthe inefficient unit in the BCC model in the input orientation.Therefore, model (4.5) in addition to identifying boundary units and determining the type of their returns-to-scale identifies inefficient units and estimates the returns-to-scale of their benchmark point in the BCC model in the input orientation.

Table 2: Data of output of model (4.5)

phi-beta	RTS
0.80	irs
0.53	irs
0.00	crs
0.40	drs
0.33	drs
1.25	drs
1.00	drs
0.38	drs
?	irs
?	irs

Example 5.2. This example is provided to assess the determination trend of the returns-to-scale of model (4.5) from an intuitive view.Assuming $DMU_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is under consideration, we have the set of observable below:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 2 & \frac{5}{2} & 6 & 12 \\ 1 & 3 & 5 & 12 & 18 \end{bmatrix}$$

Table 3 shows the review process of model (4.5) to find the optimal value of $\varphi - \theta$ and, ultimately, their returns-to-scale estimation.

Table 3: The process of finding the optimal value by model (4.5)

	step 1	*step* 2	*step* 3	*step* 4	*step* 5	*step* 6	*step* 7
φ	1	$\frac{5}{3}$	$\frac{20}{9}$	4	5	6	7
$[\varphi - 1, \varphi + 1]$	$[0, 2]$	$[\frac{2}{3}, \frac{8}{3}]$	$[\frac{11}{9}, \frac{29}{4}]$	$[3, 5]$	$[4, 6]$	$[5, 7]$	$[6, 8]$
θ	$\frac{3}{4}$	$\frac{5}{4}$	$\frac{10}{6}$	$\frac{27}{8}$	$\frac{35}{8}$	$\frac{43}{8}$	$\frac{51}{8}$
$\varphi - \theta$	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{5}{9}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$
BM_φ	$(\frac{3}{2}, 3)$	$(\frac{5}{2}, 5)$	$(\frac{10}{3}, \frac{20}{3})$	$(\frac{27}{4}, 12)$	$(\frac{35}{4}, 15)$	$(\frac{43}{4}, 18)$	$(\frac{51}{4}, 21)$
$\sum_{j=1}^{n} \lambda_j$	< 1	1	1	3	4	5	6

In Table 3, φ is an arbitrary constant value at each step: we obtain the rest of the values according to the constraints of model (4.5) and the maximization process accurately. With respect to the values of $\varphi - \theta$ in Table 3, it is evident that from step (4) we reach a constant value of $\frac{5}{8}$: this, in fact, is the optimal value of the model (4.3) in the evaluation. Considering the quantity $\sum_{j=1}^{n} \lambda_j$ for step (4), we have: $\sum_{j=1}^{n} \lambda_j = \varphi - 1$. Since $\varphi - \theta = \frac{5}{8}$ can be deduced according to Theorem 4.4, it can be concluded that the return-to-scale of DMU_0 is increasing (IRS). Of course, given the optimal solution of the model (4.3) for the above example and the DMU_0 evaluation we have: $(u^*, v^*, u_0^{+^*}, u_0^{-^*}) = (\frac{1}{8}, \frac{1}{2}, \frac{5}{8}, 0)$. So $u_0^{+^*} > 0$ and this is another confirmation for the increasing feature of the returns-to-scale of DMU_0. Also,

$$\varphi^* - \theta^* = \frac{5}{8} = u_0^{+^*} + u_0^{-^*} = |u_0^*|.$$

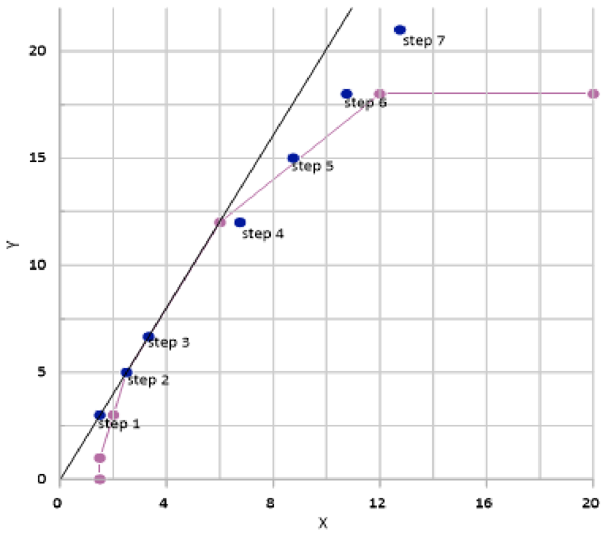

Figure 2: model (4.5) behavior in finding the optimal answer

Diagram 2 shows the trend of table 3 from an intuitive perspective. The point BM_φ is also the image point of DMU_0 obtained from model (4.5) for a constant φ. In fact, the direction of motion of the BM_φ point leads to the identification and estimation of the returns-to-scale of DMU_0. The BM_φ movement trend corresponding to Step 1 through Step 7 has been shown by diagram 2. This example shows a different intuitive view of how to determine the returns-to-scale for boundary units, and provides a good view of how the model (4.5) works.

6 Conclusion

In this paper, we were able to provide a single-stage alternative model for identifying boundary units and estimating the returns-to-scale of a particular DMU. We consider the possibility that the boundary of production is not a separate set and thus includes a general approach, including all units in the PPS and is able to separate the points on the production boundary. Since we developed this idea within the framework of the PPS used in DEA, these concepts in each PPS can be generalizable. It should be emphasized that this proposed model accurately identifies boundary units and their RTS type without having to judge multiple optimal responses. The proposed dual model also offers a different perspective on how to identify boundary units and estimate the returns-to-scale by solving only one LP model. Corresponding to the reciprocating super-plate obtained by model (4.1) and given the fact that for a constant φ the image point model is the corresponding super-plate, the dual model of movement on the parallel super-plate sand the parallel reciprocal plate is obtained from model (4.1). The type of returns-to-scale was identified accordingly: this means that the properties of BM_φ, the type of returns-to-scale, can be identified.

This model has a finite optimal value for boundary units and has an infinite value for inefficient units.

As shown in Examples 5.1, we can establish the correctness and accuracy of the performance of both the primal and the dual models by comparing the numerical example of [4] with the results of the two proposed methods in the paper. Also, to illustrate the PPS boundary resolution by these two models, [4] added two inefficient DMUs and a DMU on a weak efficient boundary. The results in the tables here indicate the identification of the boundary units by model (4.3) as well as model (4.5). Table 3 also examines the intuitive and geometric trend of model (4.5) for estimating returns-to-scale in Example 5.2. We showed the relationship between the optimal value of model (4.5) and model (4.3) with a numerical example and then a plot of its results: this could lead to interesting results in determining the type of returns-to-scale with a LP model. For inefficient units, model (4.5) can also determine the type of returns-to-scale of the image in the BCC model of the input-oriented. Of course, this has been observed in the face of numerous examples that require further investigation into the future. Therefore, in addition to identifying efficient units and estimating returns-to-scale for the image of inefficient units in the BCC model of input-oriented, the dual model also determines the returns-to-scale. Obviously, if the returns to scale of a particular DMU_0 is detected by model (4.1) or model (4.5) is increasing (descending) , both left and right returns to scale are increasing (descending), but in case of the DMU_0 has constant returns to scale ,a method for determining the left and right returns to scale by a single model It can be the subject of future research.Also Future research can lead to the development of a single-step proposed model for estimating returns-to-scale for units with negative data, as well as reviewing the dual model and interpreting it for such data.

References

[1] M. Allahyar and M. Rostamy-Malkhalifeh, *An improved approach for estimating returns to scale in DEA*, Bull. Malay. Math. Sci. Soc. **37** (2014), no. 4, 1185–1194.

[2] R.D. Banker, A. Charnes and W.W. Cooper, *Some models for estimating technical and scale inefficiencies in data envelopment analysis*, Manag. Sci. **30** (1984), 1078–1092.

[3] R.D. Banker, W.W. Cooper, R.M. Thrall, L.M. Seiford and J. Zhu, *Returns to scale in different DEA modeles*, Eur. J. Oper. Res. **2** (2004), 345–362.

[4] R.D. Banker and R.M. Thrall, *Estimation of returns to scale using data envelopment analysis*, Eur. J. Oper. Res. **62** (1992), 74–84.

[5] S, Bazaraa Mokhtar, J. Jarvis John and D. Sherali Hanif, *Linear programming and network flows*, A. John Wiley & Sons, Inc, Publication, 1952.

[6] R. Eslami and M. Khoveyni, *Right and left returns to scales in data envelopment analysis: Determining type and measuring value*, Comput. Ind. Engin. **65** (2013), 500–508.

[7] R. Fare and S. Grosskopf, *Estimating of returns to scale using data envelopment analysis, a comment*, Eur. J. Oper. Res. **79** (1995), no. 3, 379–382.

[8] B. Golany and G. Yu, *Estimating returns to scale in DEA*, Eur. J. Oper. Res. **103** (1997), no. 1, 28–37.

[9] G.R. Jahanshahloo and M. Soleimani-Damaneh, *Estimating returns to scale in data envelopment analysis: A new procedure*, Appl. Math. Comput. **150** (2004), no.1, 89–98.

[10] G.R. Jahanshahloo, M. Soleimani-Damaneh and M. Rostamy-Malkhalifeh, *An enhanced procedure for estimating returns-to-scale in DEA*, Appl. Math. Comput. **171** (2005), no. 2, 1226–1238.

[11] K. Kerstens and P. Vanden Eeckaut, *Estimating returns-to-scale using non-parametric deterministic technologies: a new method based on goodness-of-fit*, Eur. J. Oper. Res. **113** (1999), no. 1, 206–214.

[12] M. Omidi, M. Rostamy-Malkhalifeh, A. Payan and F. Hoseinzadehlotfi, *Estimation of overall returns to scale (RTS) of a frontier unit using the left and right RTS*, Springer Science Business Media, 2017.

[13] R.J. Tybout and M.D. Westbrook, *Estimating returns to scale with large imperfect panels (English)*, World Bank Econ. Rev. **7** (1993), 85–112.

[14] M. Zarepisheh, M. Soleimani-damaneh and L. Pourkarimi, *Determination of returns to scale by CCR formulation without chasing down alternative optimal solutions*, Appl. Math. Lett. **19** (2006), no. 9, 964–967.