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Pairwise connectedness in Čech fuzzy soft bi-closure spaces

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Abstract

The concept of Čech fuzzy soft bi-closure space (Čfs bi-csp) $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is initiated and studied by the authors in [6]. The notion of pairwise fuzzy soft separated sets in Čfs bi-csp is defined in this study, and various features of this notion are proved. Then, we introduce and investigate the concept of connectedness in both Čfs bi-csps and its associated fuzzy soft bitopological spaces utilizing the concept of pairwise fuzzy soft separated sets. Furthermore, the concept of pairwise feebly connected is introduced, and the relationship between pairwise connected and pairwise feebly connected is discussed. Finally, we provide various instances to further explain our findings.

Keywords: Fuzzy soft set, pairwise fuzzy soft separated, pairwise connected Čfs bi-csp, and pairwise feebly connected Čfs bi-csp
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1 Introduction

Zadeh [23] introduces the concept of a Fuzzy set. Moldtsov [17] discusses the fundamental concept of soft set theory and offers the theory's initial results. Maji et al. [15] combine the concepts of fuzzy set and soft set to create a new idea called fuzzy soft set. Tanay and Kandemir [20] proposed the idea of a fuzzy soft set-based topological structure.

The concept of $\check{C}$ech closure spaces were first developed by $\check{C}$ech [1]. When Mashhour and Ghanim [16] replaced sets with fuzzy sets in the description of $\check{C}$ech closure space, they established a new concept of $\check{C}$ech fuzzy soft closure space. The concept of biclosure space $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2)$. was introduced by Rao and Gowri [2]. $\mathcal{L}_1$ and $\mathcal{L}_2$ are two arbitrary $\check{C}$ech closure operators available in such a space. Later, Tapi and Navalakhe [21] proposed the idea of fuzzy biclosure spaces. After the concept of soft theory appeared by Moldtsov [17], Gowri and Jegadeesan [5], and Krishnaveni and Sekar [9] used the principle of soft sets to introduce the concept of soft $\check{C}$ech closure spaces In the same year, Gowri and Jegadeesan [4] introduced the concept of soft bi Čech closure spaces.

Majeed [12] recently defined $\check{C}$ech fuzzy soft closure spaces, which were inspired by Chang's fuzzy soft set and fuzzy soft topology notions [3]. Majeed and Maibed investigated the architecture of $\check{C}$ech fuzzy soft closure spaces, as well as separation axioms and connectedness [10, 12, 13, 14]. We recently developed and examined the notion of $\check{C}$ech fuzzy soft bi-closure spaces (Cfs bi-csps) [6] as a generalization of Čech fuzzy soft closure space [11] and some further aspects of Cfs bi-csps have been studied in [7, 8].

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In the current work, we extend the concept of connectedness in Čfs bi-csps, which is a generalization of the connectedness in Čech fuzzy soft closure spaces [13] in border sense. We define pairwise fuzzy soft separated sets in Čfs bi-csps and discuss some of their basic features in Section 3. The concept of disconnected is then introduced in Čfs bi-csps as well as its associated fuzzy soft bitopological spaces based on pairwise fuzzy soft separated sets. The concept of feebly disconnected Čfs bi-csps is introduced in Section 4. We demonstrate that the concepts of pairwise connected and pairwise feebly disconnected are mutually exclusive (see Examples 4.9 and 4.10).

2 Preliminaries

Throughout this paper, $\mathcal{U}$ will refer to the initial universe, $\mathrm{I} = [0, 1], \mathrm{I}_0 = (0, 1]$, and S the set of parameters for $\mathcal{U}$ and $\mathcal{A}$ be an empty subset of $S.\beta_{\mathcal{A}}$ is called a fuzzy set of $\mathcal{U}$ [23] if $\beta_{\mathcal{A}}$ is a mapping $\mathcal{U}$ from into I. The family of all fuzzy sets of $\mathcal{U}$ is denoted by I^u.

Definition 2.1. [19] A fuzzy soft set (fss, briefly) $\beta_{\mathcal{A}}$ on the universe set, $\mathcal{U}$ is a mapping from the parameters set S to I^u, i. e., $\beta_{\mathcal{A}} : S \to I^u$, where $\beta_{\mathcal{A}}(s) \neq \overline{0}$ if $s \in \mathcal{A} \subseteq S$ and $\beta_{\mathcal{A}}(s) = \overline{0}$ if $s \notin \mathcal{A}$, where $\overline{0}$ is the empty fuzzy set on $\mathcal{U}$. The family of all $fss's$ over $\mathcal{U}$ denoted by $\mathcal{FS}(\mathcal{U}, S)$.

Definition 2.2. [22] Let $\beta_{\mathcal{A}}, \mathcal{S}_{\mathcal{B}} \in \mathcal{FS}(\mathcal{U}, S)$. Then, some basic set operations of $fss's$ are defined as follows:

1. (Inclusion): $\beta_{\mathcal{A}} \sqsubseteq \mathcal{S}_{\mathcal{B}}$ iff $\beta_{\mathcal{A}}(s) \leq \mathcal{S}_{\mathcal{B}}(s)$, for all $s \in S$.
2. (Equality): $\beta_{\mathcal{A}} = \mathcal{S}_{\mathcal{B}}$ iff $\beta_{\mathcal{A}} \sqsubseteq \mathcal{S}_{\mathcal{B}}$ and $\delta_{\mathcal{B}} \sqsubseteq \beta_{\mathcal{A}}$.
3. (Union): $\rho_{\mathcal{A}\cup\mathcal{B}} = \beta_{\mathcal{A}} \sqcup \mathcal{S}_{\mathcal{B}}$ iff $\rho_{\mathcal{A}\cup\mathcal{B}}(s) = \beta_{\mathcal{A}}(s) \vee \mathcal{S}_{\mathcal{B}}(s)$, for all $s \in S$.
4. (Intersection): $\rho_{\mathcal{A}\cap\mathcal{B}} = \beta_{\mathcal{A}} \sqcap \mathcal{S}_{\mathcal{B}}$ iff $\rho_{\mathcal{A}\cap\mathcal{B}}(s) = \beta_{\mathcal{A}}(s) \wedge \mathcal{S}_{\mathcal{B}}(s)$, for all $s \in S$.
5. (Complement): $\mathcal{S}_{\mathcal{B}} = \beta^c_{\mathcal{A}}$ iff $\beta^c_{\mathcal{A}}(s) = \overline{1} - \beta_{\mathcal{A}}(s)$, for all $s \in S$, where $\overline{1}(x) = 1 \forall x \in \mathcal{U}$.
6. (Null fss) : β_s is called null fss, denoted $\widetilde{0}_s$, if $\beta_s(s) = \overline{0}$, for all $s \in S$.
7. (Universal fss)β_s is called Universal fss, denoted $\widetilde{1}_s$, if $\beta_s(s) = \overline{1}$, for all $s \in S$.

Proposition 2.3. [11] Let $(\mathcal{U}, \mathcal{L}, S)$ be a Čfs csp and $\beta_{\mathcal{A}}, \mathcal{S}_{\mathcal{B}} \in \mathcal{FS}(\mathcal{U}, S)$ such that $\beta_{\mathcal{A}} \sqsubseteq \mathcal{S}_{\mathcal{B}}$, then $\mathcal{L}(\beta_{\mathcal{A}}) \sqsubseteq \mathcal{L}(\mathcal{S}_{\mathcal{B}})$.

Definition 2.4. [20] A triple $(\mathcal{U}, \tau, S)$ is called a fuzzy soft topological space where $\mathcal{U}$ is a following properties.

1. $\tilde{0}_S, \widetilde{1}_S \in \tau$,
2. If $\beta_{\mathcal{A}}, \mathcal{S}_{\mathcal{B}} \in \tau$, then $\beta_{\mathcal{A}} \sqcap \mathcal{S}_{\mathcal{B}} \in \tau$,
3. If $(\beta_{\mathcal{A}})_i \in \tau \forall i$, then $\sqcup_{i \in J} (\beta_{\mathcal{A}})_i \in \tau$.

τ is called a topology of $fss's$ on $\mathcal{U}$. Each member of τ is called an open $fss.\delta_{\mathcal{B}}$ is called a closed f_{SS} in $(\mathcal{U}, \tau, S)$ if $\mathcal{S}^c_{\mathcal{B}} \in \tau$.

Definition 2.5. [18] A quadruple $(\mathcal{U}, \tau_1, \tau_2, S)$ is called fuzzy soft bi-topological space where τ_1, τ_2 are arbitrary fuzzy soft topologies on $\mathcal{U}$ whit a fixed set of parameters S.

The following recall the concept of Čfs bi-csp and its fundamental properties. For $i, j = 1, 2$ where $i \neq j$.

Definition 2.6. [6] A Čfs bi-csp is a quadruple $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ where $\mathcal{U}$ is a nonempty set, and $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{FS}(\mathcal{U}, \mathrm{S}) \to \mathcal{FS}(\mathcal{U}, \mathrm{S})$ are two fuzzy soft closure operators on $\mathcal{U}$ which satisfy the following axioms:

(A_1) $\mathcal{L}_i\left(\widetilde{0}_S\right) = \widetilde{0}_S$,

(A_2) $\beta_{\mathcal{A}} \sqsubseteq \mathcal{L}_i(\beta_{\mathcal{A}})$ for all $\beta_{\mathcal{A}} \in \mathcal{FS}(\mathcal{U}, S)$,

(A_3) $\mathcal{L}_i(\beta_{\mathcal{A}} \sqcup \mathcal{S}_{\mathcal{B}}) = \mathcal{L}_i(\beta_{\mathcal{A}}) \sqcup \mathcal{L}_i(\mathcal{S}_{\mathcal{B}})$ for all $\beta_{\mathcal{A}}, \mathcal{S}_{\mathcal{B}} \in \mathcal{FS}(\mathcal{U}, S)$.

Definition 2.7. [6] A $fss\beta_{\mathcal{A}}$ of a Čfs bi-csp $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is said to be $\mathcal{L}_i$-closed ($\mathcal{L}_i$-open, respectively) f_{SS} if $\mathcal{L}_i(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$ (respectively, $\mathcal{L}_i(\beta^c_{\mathcal{A}}) = \beta^c_{\mathcal{A}}$) And, it is called a closed fss if and only if $\mathcal{L}_i(\mathcal{L}_j(\beta_{\mathcal{A}})) = \beta_{\mathcal{A}}$. For $i, j = 1$ or 2 where $i \neq j$. The complement of a closed fss is called an open fss.

Proposition 2.8. [6] Let $\beta_{\mathcal{A}}$ be a fss of a Čfs bi-csp $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$. Then,

1. $\beta_{\mathcal{A}}$ is a closed fss in $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ if and only if $\beta_{\mathcal{A}}$ is $\mathcal{L}_j$-closed fss.
2. If $\beta_{\mathcal{A}}$ is an open fss in $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$, then $\mathcal{L}_i(\mathcal{L}_j(\beta^c_{\mathcal{A}})) = \mathcal{L}_j(\mathcal{L}_i(\beta^c_{\mathcal{A}}))$. For $i, j = 1$ or 2 where $i \neq j$.

Definition 2.9. [7] Let $(U, \mathcal{L}_1, \mathcal{L}_2, S)$ be a Čfs bi-csp, the induced fuzzy soft bitopological space (induced fs-bits, for short) of $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$, denoted by $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ where $\tau_{\mathcal{L}_i} = \{\beta^c_{\mathcal{A}} : \mathcal{L}_i(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}\}$.

Definition 2.10. [7] Let $(U, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ be the induced fs-bits of the Čfs bi-csp $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ and $\beta_{\mathcal{A}} \in \mathcal{FS}(\mathcal{U}, S), \beta_{\mathcal{A}}$ is called an $\tau_{\mathcal{L}_i}$-open fss in $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$, if $\beta_{\mathcal{A}} \in \tau_{\mathcal{L}_i}$. The complement of an $\tau_{\mathcal{L}_i}$-open $f_{SS}\beta_{\mathcal{A}}$ is a $\tau_{\mathcal{L}_i}$-closed f_{SS}, and $\beta_{\mathcal{A}}$ is called an open f_{SS} in $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$, if $\beta_{\mathcal{A}}$ is an $\tau_{\mathcal{L}_i}$-open, for $i = 1, 2$.

Proposition 2.11. [8] Let $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ be Čfs bi-csp and $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ be the induced fsbits of $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$. Then for any $\beta_{\mathcal{A}} \in \mathcal{FS}(\mathcal{U}, S)$.

$$\tau_{\mathcal{L}_i}\text{- int}(\beta_{\mathcal{A}}) \sqsubseteq \text{Int}_i(\beta_{\mathcal{A}}) \sqsubseteq \beta_{\mathcal{A}} \sqsubseteq \mathcal{L}_i(\beta_{\mathcal{A}}) \sqsubseteq \tau_{\mathcal{L}_i}\text{- cl}(\beta_{\mathcal{A}}).$$

Definition 2.12. [6] Let $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ be a Čfs bi-csp and $\mathcal{V} \subseteq \mathcal{U}$. The quadruple $(\mathcal{V}, \mathcal{L}_{1_v}, \mathcal{L}_{2_v}, S)$ is called a Čech fuzzy soft bi-closure subspace (Čfs bi-csubsp, for short) of $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$, where $\mathcal{L}_{i_v} : \mathcal{FS}(\mathcal{V}, S) \to \mathcal{FS}(\mathcal{V}, S)$ defind by $\mathcal{L}_{i_v}(\beta_{\mathcal{A}}) = \tilde{\mathcal{V}}_S \sqcap \mathcal{L}_i(\beta_{\mathcal{A}})$ for all $\beta_{\mathcal{A}} \in \mathcal{FS}(\mathcal{V}, S)$. The Čfs bi-csubsp $(\mathcal{V}, \mathcal{L}_{1_v}, \mathcal{L}_{2_v}, S)$ is said to be a closed (resp. open) subspace if $\tilde{\mathcal{V}}_S$ is a closed (resp. open) fss over $\mathcal{U}$.

3 Pairwise Connectedness Čech Fuzzy Soft bi-Closure Spaces

In this section we introduce and study pairwise fuzzy soft separated sets in Čfs bi-csps, then we use it to introduce the notion of pairwise connectedness in Čfs bi-csp's.

Definition 3.1. Let $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ be a Čfs bi-csp, and let $\beta_{\mathcal{A}}, \mathcal{S}_{\mathcal{B}} \in \mathcal{FS}(\mathcal{U}, S)$. The two fuzzy soft sets. $\beta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ are said to be pairwise fuzzy soft separated sets (P-fuzzy soft separated, for short) if and only if $\beta_{\mathcal{A}} \sqcap \mathcal{L}_i(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\mathcal{L}_j(\beta_{\mathcal{A}}) \sqcap \mathcal{S}_{\mathcal{B}} = \tilde{0}_S$.

In other words, two non-empty $fss's\beta_{\mathcal{A}}, \mathcal{S}_{\mathcal{B}}$ of Čfs bi-csp $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2,$ S) are said to be P-fuzzy soft separated sets if and only if $(\beta_{\mathcal{A}} \sqcap \mathcal{L}_i(\mathcal{S}_{\mathcal{B}})) \sqcup (\mathcal{L}_j(\beta_{\mathcal{A}}) \sqcap \mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$.

Remark 3.2. It is clear that if $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are P-fuzzy soft separated sets in $(U, \mathcal{L}_1, \mathcal{L}_2,$ S), then $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are disjoint fuzzy soft sets. The following example shows that the converse is not true.

Example 3.3. Let $\mathcal{U} = \{x, y, z\}, S = \{s_1, s_2\}$ and let $\rho_C = \{(s_1, y_{0.5}), (s_2, y_{0.5})\}$. Define fuzzy soft closure operators $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{FS}(\mathcal{U}, S) \to \mathcal{FS}(\mathcal{U}, S)$ as follows:

$$\mathcal{L}_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_S & \text{if } \beta_{\mathcal{A}} = \tilde{0}_S, \\ \{(s_1, x_{0.5} \vee y_{0.5}), \quad (s_2, x_{0.5} \vee y_{0.5})\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq \rho_C, \\ \tilde{1}_S & \text{otherwise.} \end{cases}$$

And

$$\mathcal{L}_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_S & \text{if } \beta_{\mathcal{A}} = \tilde{0}_S, \\ \{(s_1, x_{0.8} \vee y_{0.8}), \quad (s_2, x_{0.8} \vee y_{0.8})\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq \rho_C, \\ \tilde{1}_S & \text{otherwise.} \end{cases}$$

Then $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is Čfs bi-csp on $\mathcal{U}$. Here we have $\beta_{\mathcal{A}} = \{(s_1, y_{0.5})\}$ and $\mathcal{S}_{\mathcal{B}} = \{(s_1, x_{0.5}), (s_2, z_{0.5})\}$ are non-empty disjoint fuzzy soft sets but $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are not P-fuzzy soft separated sets.

Theorem 3.4. Let $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ be a Čfs bi-csp. Then all fuzzy soft subsets of P-fuzzy soft separated sets are also P-fuzzy soft separated sets.

Proof . Let $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are P-fuzzy soft separated sets in $(U, \mathcal{L}_1, \mathcal{L}_2, S)$, and let $\rho_C \sqsubseteq \beta_{\mathcal{A}}$ and $\eta_D \sqsubseteq \mathcal{S}_{\mathcal{B}}$. Since $\rho_C \sqsubseteq \beta_{\mathcal{A}}$ and $\eta_D \sqsubseteq \mathcal{S}_{\mathcal{B}}$, then by Proposition 2.3, we have $\mathcal{L}_i(\rho_C) \sqsubseteq \mathcal{L}_i(\beta_{\mathcal{A}})$ and $\mathcal{L}_i(\eta_D) \sqsubseteq \mathcal{L}_i(\mathcal{S}_{\mathcal{B}})$ for $i = 1, 2$. This implies $\mathcal{L}_j(\rho_C) \sqcap \eta_D \sqsubseteq \mathcal{L}_j(\beta_{\mathcal{A}}) \sqcap \mathcal{S}_{\mathcal{B}}$ for $j = 1, 2$ and $\mathcal{L}_i(\eta_D) \sqcap \rho_C \sqsubseteq \mathcal{L}_i(\mathcal{S}_{\mathcal{B}}) \sqcap \beta_{\mathcal{A}}$ for $i = 1, 2$. But $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are P-fuzzy soft separated sets, it follows $\mathcal{L}_j(\rho_C) \sqcap \eta_D \sqsubseteq \mathcal{L}_j(\beta_{\mathcal{A}}) \sqcap \mathcal{S}_{\mathcal{B}} = \tilde{0}_S$ and $\mathcal{L}_i(\eta_D) \sqcap \rho_C \sqsubseteq \mathcal{L}_i(\mathcal{S}_{\mathcal{B}}) \sqcap \beta_{\mathcal{A}} = \tilde{0}_S$. Hence $\mathcal{L}_j(\rho_C) \sqcap \eta_D = \tilde{0}_S$ and $\mathcal{L}_i(\eta_D) \sqcap \rho_C = \tilde{0}_S$. Thus ρ_C and η_D are P-fuzzy soft separated sets. □

Theorem 3.5. Let $(\mathcal{V}, \mathcal{L}_{1_v}, \mathcal{L}_{2_v}, S)$ be a Čfs bi-csubsp of $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ and let $\beta_{\mathcal{A}}, \mathcal{S}_{\mathcal{B}} \in \mathcal{FS}(\mathcal{V}, S)$, then $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are P-fuzzy soft separated sets in $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ if and only if $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are P-fuzzy soft separated sets in $(\mathcal{V}, \mathcal{L}_{1_v}, \mathcal{L}_{2_v}, S)$.

Proof . Let $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ be a Čfs bi-csp and $(\mathcal{V}, \mathcal{L}_{1_v}, \mathcal{L}_{2v}, S)$ be a Čfs bi-csubsp of $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$. Assume that $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are P-fuzzy soft separated sets in $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$, this implies that $\beta_{\mathcal{A}} \sqcap \mathcal{L}_i(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\mathcal{L}_j(\beta_{\mathcal{A}}) \sqcap \mathcal{S}_{\mathcal{B}} = \tilde{0}_S$. Which means $(\beta_{\mathcal{A}} \sqcap \mathcal{L}_i(\mathcal{S}_{\mathcal{B}})) \sqcup (\mathcal{L}_j(\beta_{\mathcal{A}}) \sqcap \mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$.

Now,

$$\begin{aligned}
(\beta_{\mathcal{A}} \sqcap \mathcal{L}_{i_v}(\mathcal{S}_{\mathcal{B}})) \sqcup (\mathcal{L}_{j_\mathcal{V}}(\beta_{\mathcal{A}}) \sqcap \mathcal{S}_{\mathcal{B}}) &= \left(\beta_{\mathcal{A}} \sqcap \left(\tilde{V}_S \sqcap \mathcal{L}_i(\mathcal{S}_{\mathcal{B}})\right)\right) \sqcup \left(\left(\tilde{V}_S \sqcap \mathcal{L}_j(\beta_{\mathcal{A}})\right) \sqcap \mathcal{S}_{\mathcal{B}}\right) \\
&= \left(\left(\beta_{\mathcal{A}} \sqcap \tilde{\mathcal{V}}_S\right) \sqcap \mathcal{L}_i(\mathcal{S}_{\mathcal{B}})\right) \sqcup \left(\left(\tilde{\mathcal{V}}_S \sqcap \mathcal{S}_{\mathcal{B}}\right) \sqcap \mathcal{L}_j(\beta_{\mathcal{A}})\right) \\
&= (\beta_{\mathcal{A}} \sqcap \mathcal{L}_i(\mathcal{S}_{\mathcal{B}})) \sqcup (\mathcal{S}_{\mathcal{B}} \sqcap \mathcal{L}_j(\beta_{\mathcal{A}})) \\
&= \tilde{0}_S.
\end{aligned}$$

Therefore, $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are P-fuzzy soft separated sets in $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ if and only if $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are P-fuzzy soft separated sets in $(\mathcal{V}, \mathcal{L}_{1_v}, \mathcal{L}_{2_v}, S)$. □

Definition 3.6. A Čfs bi-csp $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2,$ S) is said to be pairwise disconnected (P-disconnected, for short) Čs bi-csp if there exist p-fuzzy soft separated $\beta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ such that $\mathcal{L}_i(\beta_{\mathcal{A}}) \sqcap \mathcal{L}_j(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\mathcal{L}_i(\beta_{\mathcal{A}}) \sqcup \mathcal{L}_j(\mathcal{S}_{\mathcal{B}}) = \tilde{1}_S$.

Definition 3.7. A Čfs bi-csp $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2,$ S) is P-connected $\check{C}fs$ bi-csp if it is not P-disconnected Čfs bi-csp.

Now we give two examples one is P-disconnected Čfs bi-csp and the other is Pconnected Čfs bi-csp.

Example 3.8. Let $\mathcal{U} = \{x, y\}, S = \{s_1, s_2\}$. Define fuzzy soft closure operators $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{FS}(\mathcal{U}, S) \to \mathcal{FS}(\mathcal{U}, S)$ as follows:

$$\mathcal{L}_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_S & \text{if } \beta_{\mathcal{A}} = \tilde{0}_S \\ \{(s_1, x_1 \vee y_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq \{(s_1, x_1)\} \\ \{(s_2, x_1 \vee y_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq \{(s_2, x_1)\} \\ \tilde{1}_S & \text{other wise} \end{cases}$$

And

$$\mathcal{L}_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_s & \text{if } \beta_{\mathcal{A}} = \tilde{0}_S, \\ \{(s_1, x_1 \vee y_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq \{(s_1, x_1 \vee y_1)\}, \\ \{(s_2, x_1 \vee y_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq \{(s_2, x_1 \vee y_1)\}, \\ \tilde{1}_S & \text{other wise.} \end{cases}$$

Then $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-disconnected Čfs bi-csp. To explain that taking $\beta_{\mathcal{A}} = \{(s_1, x_{0.5})\}$ and $\mathcal{S}_{\mathcal{B}} = \{(s_2, x_{0.2})\}$. It is clear that $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are P-fuzzy soft separated sets such that $\mathcal{L}_1(\beta_{\mathcal{A}}) \sqcap \mathcal{L}_2(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\mathcal{L}_1(\beta_{\mathcal{A}}) \sqcup \mathcal{L}_2(\mathcal{S}_{\mathcal{B}}) = \tilde{1}_S$. And $\mathcal{L}_2(\beta_{\mathcal{A}}) \sqcap \mathcal{L}_1(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\mathcal{L}_2(\beta_{\mathcal{A}}) \sqcup \mathcal{L}_1(\mathcal{S}_{\mathcal{B}}) = \tilde{1}_S$

Example 3.9. Let $\mathcal{U} = \{x, y\}, S = \{s_1, s_2\}$. Define fuzzy soft closure operators $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{FS}(\mathcal{U}, S) \to \mathcal{FS}(\mathcal{U}, S)$ as follows:

$$\mathcal{L}_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_S & \text{if } \beta_{\mathcal{A}} = \tilde{0}_S, \\ \{(s_1, x_1 \vee y_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq \{(s_1, x_1 \vee y_1)\}, \\ \tilde{1}_S & \text{otherwise.} \end{cases}$$

And

$$\mathcal{L}_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_S & \text{if } \beta_{\mathcal{A}} = \tilde{0}_S, \\ \{(s_1, x_1 \vee y_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq \{(s_1, x_1)\}, \\ \tilde{1}_S & \text{otherwise.} \end{cases}$$

Then $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2,\ S)$ is P-connected Čfs bi-csp.

Remark 3.10. P-connectedness in Čfs bi-csp is not hereditary property. The following example explain that.

Example 3.11. Let $\mathcal{U} = \{x, y, z\}$, $S = \{s_1, s_2\}$ and let $(\beta_{\mathcal{A}})_1, (\beta_{\mathcal{A}})_2 \in \mathcal{FS}(\mathcal{U}, S)$ such that $(\beta_{\mathcal{A}})_1 = \{(s_1, x_1 \vee y_1 \vee z_{0.5})\}$ and $(\beta_{\mathcal{A}})_2 = \{(s_2, x_1 \vee y_1 \vee z_{0.7})\}$. Define fuzzy soft closure operators $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{FS}(\mathcal{U}, S) \to \mathcal{FS}(\mathcal{U}, S)$ as follows:

$$\mathcal{L}_1(\beta_{\mathcal{A}}) = \mathcal{L}_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_S & \text{if } \beta_{\mathcal{A}} = \tilde{0}_S \\ \{(s_1, x_1 \vee y_1 \vee z_{0.5})\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq (\beta_{\mathcal{A}})_1, \\ \{(s_2, x_1 \vee y_1 \vee z_{0.7})\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq (\beta_{\mathcal{A}})_2, \\ \mathcal{L}_i((\beta_{\mathcal{A}})_1) \sqcup \mathcal{L}_i((\beta_{\mathcal{A}})_2) & \text{if } \beta_{\mathcal{A}} \sqsubseteq (\beta_{\mathcal{A}})_1 \sqcup (\beta_{\mathcal{A}})_2, i = 1, 2 \\ \tilde{1}_S & \text{otherwise.} \end{cases}$$

Then $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2,\ S)$ is P-connected Čfs bi-csp. Let $\mathcal{V} = \{x, y\}$, then $\mathcal{L}_{1_v}, \mathcal{L}_{2_v} : \mathcal{FS}(\mathcal{V}, S) \to \mathcal{FS}(\mathcal{V}, S)$ defined as

$$\mathcal{L}_{1_v}(\beta_{\mathcal{A}}) = \mathcal{L}_{2_v}(\beta_{\mathcal{A}}) = \begin{cases} \tilde{S}_S & \text{if } \beta_{\mathcal{A}} = \tilde{0}_S, \\ \{(s_1, x_1 \vee y_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq \{(h_1, x_1 \vee y_1)\}, \\ \{(s_2, x_1 \vee y_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq \{(h_2, x_1 \vee y_1)\}, \\ \tilde{V}_S & \text{otherwise.} \end{cases}$$

Then $(\mathcal{V}, \mathcal{L}_{1_v}, \mathcal{L}_{2v}, S)$ is P-disconnected Čfs bi-csubsp of $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$. Since there exist $\beta_{\mathcal{A}} = \{(s_1, x_1 \vee y_1)\}$ and $\mathcal{S}_{\mathcal{B}} = \{(s_2, x_1 \vee y_1)\}$ are p-fuzzy soft separated sets such that $\mathcal{L}_{1_v}(\beta_{\mathcal{A}}) \sqcap \mathcal{L}_{2_v}(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\mathcal{L}_{1_v}(\beta_{\mathcal{A}}) \sqcup \mathcal{L}_{2_v}(\mathcal{S}_{\mathcal{B}}) = \tilde{\mathcal{V}}_S$. And $\mathcal{L}_{2_v}(\beta_{\mathcal{A}}) \sqcap \mathcal{L}_{1_v}(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\mathcal{L}_{2_v}(\beta_{\mathcal{A}}) \sqcup \mathcal{L}_{1_v}(\mathcal{S}_{\mathcal{B}}) = \tilde{\mathcal{V}}_S$

Now, we introduce the concept of P-fuzzy soft separated sets in the induced fs-bits of Čfs bi-csp.

Definition 3.12. Two non-empty $fss's\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are said to be p-fuzzy soft separated sets in the induced fs-bits $(U, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2},\ S)$, if $\beta_{\mathcal{A}} \sqcap \tau_{\mathcal{L}_i} - cl(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\tau_{\mathcal{L}_j} - cl(\beta_{\mathcal{A}}) \sqcap \mathcal{S}_{\mathcal{B}} = \tilde{0}_S$.

Theorem 3.13. If $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are P-fuzzy soft separated sets in the induced fs-bits $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2},\ S)$, then $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are also P-fuzzy soft separated sets in $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2,\ S)$.

Proof . Let $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are P-fuzzy soft separated sets in $(U, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2},\ S)$. Then $\beta_{\mathcal{A}} \sqcap \tau_{\mathcal{L}_i}$ - $\mathrm{cl}(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\tau_{\mathcal{L}_j}$-$cl(\beta_{\mathcal{A}}) \sqcap \mathcal{S}_{\mathcal{B}} = \tilde{0}_S$. By proposition 2.11, we get, $\beta_{\mathcal{A}} \sqcap \mathcal{L}_i(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\mathcal{L}_j(\beta_{\mathcal{A}}) \sqcap \mathcal{S}_{\mathcal{B}} = \tilde{0}_S$. This implies $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ are P-fuzzy soft separated sets in $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2,\ S)$. □

Definition 3.14. The induced fs-bits $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ of a Čfs bi-csp $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is said to be P-disconnected fs-bits, if there exist P-fuzzy soft separated $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ in $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ such that $\tau_{\mathcal{L}_i}$-$cl(\beta_{\mathcal{A}}) \sqcap \tau_{\mathcal{L}_j}$-$cl(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\tau_{\mathcal{L}_i}$-$cl(\beta_{\mathcal{A}}) \sqcup \tau_{\mathcal{L}_j}$-$cl(\mathcal{S}_{\mathcal{B}}) = \tilde{1}_S$.

Definition 3.15. The induced fs-bits $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ of a Čfs bi-csp $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is said to be P-connected fs-bits, if it is not P-disconnected fs-bits.

Theorem 3.16. If $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ is a P-disconnected fs-bits, then $(\mathcal{U}_1, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-disconnected Čfs bi-csp.

Proof . Let $(U, \tau_{L_1}, \tau_{\mathcal{L}_2}, S)$ be P-disconnected fs-bits, then there exist two P-fuzzy soft separated sets $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ in $(U, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2},\ S)$ such that $\tau_{\mathcal{L}_i}$- $\mathrm{cl}(\beta_{\mathcal{A}}) \sqcap \tau_{\mathcal{L}_j}$-$cl(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\tau_{\mathcal{L}_i}$- $\mathrm{cl}(\beta_{\mathcal{A}}) \sqcup \tau_{\mathcal{L}_i}$-$cl(\mathcal{S}_{\mathcal{B}}) = \tilde{1}_S$. Since $\tau_{\mathcal{L}_i}$-$cl(\beta_{\mathcal{A}})$ and $\tau_{\mathcal{L}_i}$-$cl(\mathcal{S}_{\mathcal{B}})$ are closed $fss's$, then $\mathcal{L}_i(\tau_{\mathcal{L}_i}$- $\mathrm{cl}(\beta_{\mathcal{A}})) = \tau_{\mathcal{L}_i}$-$cl(\beta_{\mathcal{A}})$ and $\mathcal{L}_j\left(\tau_{\mathcal{L}_j}\text{-}cl(\mathcal{S}_{\mathcal{B}})\right) = \tau_{\mathcal{L}_j}$-$cl(\mathcal{S}_{\mathcal{B}})$. Let $\rho_C = \tau_{\mathcal{L}_i}$-$cl(\beta_{\mathcal{A}})$ and $\eta_D = \tau_{\mathcal{L}_j}$- $\mathrm{cl}(\mathcal{S}_{\mathcal{B}})$. Then we have ρ_C and η_D are P-fuzzy soft separated sets in $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2},\ S)$ such that $\mathcal{L}_i(\rho_C) \sqcap \mathcal{L}_j(\eta_D) = \rho_C \sqcap \eta_D = \tilde{0}_S$ and $\mathcal{L}_i(\rho_C) \sqcup \mathcal{L}_j(\eta_D) = \rho_C \sqcup \eta_D = \tilde{1}_S$. Hence, $(U, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-disconnected Čfs bi-csp. □

Corollary 3.17. If $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-connected Čs bi-csp, then $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ is a Pconnected fs-bits.

Proof . The proof follows by suppose $(U, \tau_{L_1}, \tau_{L_2}, S)$ is P-disconnected fs-bits. From Theorem 3.16, we get $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-disconnected Čfs bi-csp which is a contradiction with hypothesis. Hence, the result. □

Remark 3.18. The converse of Theorem 3.16 and its corollary is not true in general. That is, if $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-disconnected Čfs bi-csp, then $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ need not to P-disconnected fs-bits. The following example shows that.

Example 3.19. In Example 3.8, $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-disconnected Čfs bi-csp. But it's The induced fs-bits $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ is P-connected fs-bits, because $\tau_{\mathcal{L}_i} = \{\tilde{0}_S, \tilde{1}_S\} \cdot i = 1, 2$.

4 Pairwise feebly Connected Čech Fuzzy Soft bi-Closure Spaces

Definition 4.1. A Čfs bi-csp $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is said to be P-feebly disconnected Čs bi-csp if there exist two non-empty disjoint fuzzy soft sets $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ such that $\beta_{\mathcal{A}} \sqcup \mathcal{L}_i(\mathcal{S}_{\mathcal{B}}) = \tilde{1}_S$ and $\mathcal{L}_j(\beta_{\mathcal{A}}) \sqcup \mathcal{S}_{\mathcal{B}} = \tilde{1}_S$.

Definition 4.2. A Čfs bi-csp $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is said to be P-feebly connected Čfs bi-csp if it is not P-feebly disconnected Čfs bi-csp.

Remark 4.3. P-feebly disconnected in Čfs bi-csp is not hereditary property. The following example explain that.

Example 4.4. Let $\mathcal{U} = \{x, y\}, S = \{s_1, s_2\}$ and let $(\beta_{\mathcal{A}})_1, (\beta_{\mathcal{A}})_2 \in \mathcal{FS}(\mathcal{U}, S)$ such that $(\beta_{\mathcal{A}})_1 = \{(s_1, x_1)\}$ and $(\beta_{\mathcal{A}})_2 = \{(s_1, y_1), (s_2, x_1 \vee y_1)\}$. Define fuzzy soft closure operators $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{FS}(\mathcal{U}, S) \to \mathcal{FS}(\mathcal{U}, S)$ as follows:

$$\mathcal{L}_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_S & \text{if } \beta_{\mathcal{A}} = \tilde{0}_S \\ \{(s_1, y_1), (s_2, x_1 \vee y_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq (\beta_{\mathcal{A}})_2 \\ \{(s_1, x_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq (\beta_{\mathcal{A}})_1 \\ \tilde{1}_S & \text{otherwise} \end{cases}$$

And

$$\mathcal{L}_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_S & \text{if } \beta_{\mathcal{A}} = \tilde{0}_S, \\ \{(s_1, y_1), (s_2, x_1 \vee y_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq (\beta_{\mathcal{A}})_2, \\ \{(s_1, x_{0.5})\} & \text{if } \beta_{\mathcal{A}} \in \{(s_1, x_{k_1}), 0 < k_1 < 0.5\}, \\ \{(s_1, x_1)\} & \text{if } \beta_{\mathcal{A}} \in \{(s_1, x_{k_1}), 0.5 \leq k_1 \leq 1\}, \\ \tilde{1}_S & \text{otherwise.} \end{cases}$$

Then $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-feebly disconnected Čfs bi-csp. Since there exist $\beta_{\mathcal{A}} = \{(s_1, x_1)\}$ and $\mathcal{S}_{\mathcal{B}} = \{(s_1, y_1), (s_2, x_1 \vee y_1)\}$ are disjoint fuzzy soft sets such that $\mathcal{L}_1(\mathcal{S}_{\mathcal{B}}) \sqcup \beta_{\mathcal{A}} = \tilde{1}_S$ and $\mathcal{S}_{\mathcal{B}} \sqcup \mathcal{L}_2(\beta_{\mathcal{A}}) = \tilde{1}_S$, and $\mathcal{L}_2(\mathcal{S}_{\mathcal{B}}) \sqcup \beta_{\mathcal{A}} = \tilde{1}_S$ and $\mathcal{S}_{\mathcal{B}} \sqcup \mathcal{L}_1(\beta_{\mathcal{A}}) = \tilde{1}_S$
Let $\mathcal{V} = \{y\}$, then $\mathcal{L}_{1_v}, \mathcal{L}_{2_v} : \mathcal{FS}(\mathcal{V}, S) \to \mathcal{FS}(\mathcal{V}, S)$ defined as:

$$\mathcal{L}_{1_v}(\beta_{\mathcal{A}}) = \mathcal{L}_{2_v}(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_S & \text{if } \beta_{\mathcal{A}} = \tilde{0}_S, \\ \tilde{V}_S & \text{otherwise.} \end{cases}$$

Then $(\mathcal{V}, \mathcal{L}_{1_v}, \mathcal{L}_{2_v}, S)$ is P-feebly connected Čfs bi-csubsp of $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$.

Definition 4.5. The induced fs-bits $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ of a Čfs bi-csp $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is said to be P-feebly disconnected fs-bits, if there exist two non-empty disjoint fuzzy soft sets $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ in $(U, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ such that $\beta_{\mathcal{A}} \sqcup \tau_{\mathcal{L}_i}\text{-}cl(\mathcal{S}_{\mathcal{B}}) = \tilde{1}_S$ and $\tau_{\mathcal{L}_j}\text{-}cl(\beta_{\mathcal{A}}) \sqcup \mathcal{S}_{\mathcal{B}} = \tilde{1}_S$, for $i, j = 1,2$ and $i \neq j$.

Theorem 4.6. $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-feebly disconnected Čfs bi-csp, then $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ is P- feebly disconnected fs-bits.

Proof . The proof follows from Definition 4.1 and Proposition 2.11 . □

Corollary 4.7. $(U, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ is P-feebly connected fs-bits, then $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-feebly connected Čfs bi-csp.

Proof . The proof follows by suppose $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-feebly disconnected Čfs bi-csp. From Theorem 4.6, we get $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ is P-feebly disconnected fs-bits which is a contradiction with hypothesis. Hence, the result. □

Next we discuss the relationship between P-disconnectedness and P-feebly disconnectedness Čfs bi-csp's.

Remark 4.8. The concept of P-disconnected Čfs bi-csp and P-feebly disconnected-Čfs bi-csp are independent. The next two examples explain our claim.

The following example shows that if $(U, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-disconnected Čfs bi-csp, then $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2,$ S) need not to be P-feebly disconnected Čfs bi-csp.

Example 4.9. Let $\mathcal{U} = \{x, y\}, S = \{s\}$. Define fuzzy soft closure operators $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{FS}(\mathcal{U}, S) \to \mathcal{FS}(\mathcal{U}, S)$ as follows:

$$\mathcal{L}_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_s & \text{if } \beta_{\mathcal{A}} = \tilde{0}_s, \\ \{(s, x_1)\} & \text{if } \beta_{\mathcal{A}} = \{(s, x_t); 0 < t < 1\}, \\ \{(s, y_1)\} & \text{if } \beta_{\mathcal{A}} = \{(s, y_s); 0 < s < 1\}, \\ \tilde{1}_s & \text{other wise.} \end{cases}$$

And

$$\mathcal{L}_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_s & \text{if } \beta_{\mathcal{A}} = \tilde{0}_s, \\ \{(s, x_t)\} & \text{if } \beta_{\mathcal{A}} = \{(s, x_t); 0 < t < 0.2\}, \\ \{(s, x_1)\} & \text{if } \beta_{\mathcal{A}} = \{(s, x_t); 0.2 \le t < 1\}, \\ \{(s, y_s)\} & \text{if } \beta_{\mathcal{A}} = \{(s, y_s); 0 < s < 0.2\}, \\ \{(s, y_1)\} & \text{if } \beta_{\mathcal{A}} = \{(s, y_s); 0.2 \le s < 1\}, \\ \tilde{1}_s & \text{other wise.} \end{cases}$$

Then $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-disconnected Čfs bi-csp, since there exist $\beta_{\mathcal{A}} = \{(s, x_{0.5})\}$ and $\mathcal{S}_{\mathcal{B}} = \{(s, y_{0.3})\}$ are P-fuzzy soft separated sets such that $\mathcal{L}_1(\beta_{\mathcal{A}}) \sqcap \mathcal{L}_2(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\mathcal{L}_1(\beta_{\mathcal{A}}) \sqcup \mathcal{L}_2(\mathcal{S}_{\mathcal{B}}) = \tilde{1}_S$, and $\mathcal{L}_2(\beta_{\mathcal{A}}) \sqcap \mathcal{L}_1(\mathcal{S}_{\mathcal{B}}) = \tilde{0}_S$ and $\mathcal{L}_2(\beta_{\mathcal{A}}) \sqcup \mathcal{L}_1(\mathcal{S}_{\mathcal{B}}) = \tilde{1}_S$. However, $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is not P-feebly disconnected Čfs bi-csp. Since for any non-empty disjoint $fSs's$ $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$, we have $\beta_{\mathcal{A}} \sqcup \mathcal{L}_1(\mathcal{S}_{\mathcal{B}}) \neq \tilde{1}_S$.

The next example shows that if $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-feebly disconnected Čs bi-csp, then $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2,$ S) need not to be P-disconnected Čfs bi-csp.

Example 4.10. Let $\mathcal{U} = \{x, y\}, S = \{s_1, s_2\}$. Define fuzzy soft closure operators $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{FS}(\mathcal{U}, S) \to \mathcal{FS}(\mathcal{U}, S)$ as follows:

$$\mathcal{L}_1(\beta_{\mathcal{A}}) = \mathcal{L}_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_S & \text{if } \beta_{\mathcal{A}} = \tilde{0}_S, \\ \{(s_1, x_1 \vee y_1), (s_2, y_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq \{(s_1, y_1)\}, \\ \{(s_1, x_1), (s_2, x_1 \vee y_1)\} & \text{if } \beta_{\mathcal{A}} \sqsubseteq \{(s_2, x_1)\}, \\ \tilde{1}_S & \text{other wise.} \end{cases}$$

Then $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2, S)$ is P-feebly disconnected Čs bi-csp. Since there are non-empty disjoint fuzzy soft sets $\beta_{\mathcal{A}} = \{(s_1, y_1)\}$ and $\mathcal{S}_{\mathcal{B}} = \{(s_2, x_1)\}$ such that $\mathcal{L}_2(\beta_{\mathcal{A}}) \sqcup \mathcal{S}_{\mathcal{B}} = \tilde{1}_S$ and $\beta_{\mathcal{A}} \sqcup \mathcal{L}_1(\mathcal{S}_{\mathcal{B}}) = \tilde{1}_S$, and $\mathcal{L}_1(\beta_{\mathcal{A}}) \sqcup \mathcal{S}_{\mathcal{B}} = \tilde{1}_S$ and $\beta_{\mathcal{A}} \sqcup \mathcal{L}_2(\mathcal{S}_{\mathcal{B}}) = \tilde{1}_S$. But $(\mathcal{U}, \mathcal{L}_1, \mathcal{L}_2,$ S) is p-connected Čfs bi-csp. Since for any P-fuzzy soft separated sets $\beta_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$, we have $\mathcal{L}_1(\beta_{\mathcal{A}}) \sqcup \mathcal{L}_2(\mathcal{S}_{\mathcal{B}}) = \tilde{1}_S$ but $\mathcal{L}_1(\beta_{\mathcal{A}}) \sqcap \mathcal{L}_2(\mathcal{S}_{\mathcal{B}}) \neq \tilde{0}_S$.

Remark 4.11. It is worth noting that the definitions of P-disconnected Čfs bi-csp and P-feebly disconnected Čfs bi-csp (see Definitions 4.6 and 4.1, respectively) turn to be every Pdisconnected Čfs bi-csp is P-feebly disconnected Čfs bi-csp, if the P-fuzzy soft separated sets which are satisfying the conditions of P-disconnected Čfs bi-csp are closed $fss's$.

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