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On automorphisms of strong semilattice of π-groups

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Abstract

In this paper, we make a start by considering the automorphisms of strong semilattice of π-groups, relating them to the automorphisms of underlying π-groups. We also provide a condition under which an automorphism of strong semilattice of π-groups can be constructed.

Keywords: Automorphisms, Linking homomorphisms, π-groups, π-regular
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1 Introduction

Let Λ be a semilattice and for each $\alpha \in \Lambda$, let S_α be a semigroup and suppose $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$. For every $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$, there exists a homomorphism $f_{\alpha,\beta} : S_\alpha \to S_\beta$ satisfying the following conditions:

(i) $f_{\alpha,\alpha} = \mathrm{Id}_{S_\alpha}$ for any $\alpha \in \Lambda$.
(ii) For any $\alpha, \beta, \gamma \in \Lambda$ with $\alpha \geq \beta \geq \gamma$, $f_{\beta,\gamma}\ f_{\alpha,\beta}\ = f_{\alpha,\gamma}$.

The semigroup operation on $S = \cup_{\alpha\in\Lambda} S_\alpha$ is defined in terms of the multiplication in the components S_α and the homomorphism $f_{\alpha,\beta}$ (called linking homomorphism) by $st = f_{\alpha,\gamma}(s) f_{\beta,\gamma}(t)$ for $s \in S_\alpha$ and $t \in S_\beta$, where $\gamma = \alpha \wedge \beta$. Then S with multiplication defined above is a strong semilattice Λ of semigroup S_α, and is denoted by $S = (\Lambda, \{S_\alpha\}_{\alpha\in\Lambda}, \{f_{\alpha,\beta}\}_{\alpha\geq\beta})$.

A semigroup S is said to be a π-group if there exists a subgroup G^S of S which is an ideal, and for any $s \in S$, there exists a natural number $n \in \mathbb{N}$ such that $s^n \in G^S$. An element $s \in S$ is said to be regular if there exists an element $a \in S$ such that $sas = s$ and S is said to be regular if every element of S is regular. An element s of S is said to be π-regular if there exists a positive integer $n \in \mathbb{N}$ such that $s^n \in s^n S s^n$ and S is said to be π-regular if every element of S is π-regular. Infact, π-regular semigroups is one of the important classes of non-regular semigroups. Let R^S denote the set of all regular elements of S. We write, $S = R^S \cup N^S$, where $N^S = S \setminus R^S$ is the set of non-regular elements of S.

The set of idempotents in S will be denoted by E_S. Thus $E_S = \{e_\alpha; \alpha \in \Lambda\}$. If S is a π-group and $s \in R^S$, then $s = se$ for the (unique) idempotent e, and so $s \in G^S$. Since obviously $G^S \subseteq R^S$, so we have $G^S = R^S$ in a π-group. In this paper, we are looking for automorphisms of strong semilattice of π-groups.

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2 Automorphisms

In this section, first, we fix some notations without further mention. Let S be a strong semilattice of π-groups. We write $S_\alpha = R_\alpha \cup N_\alpha$, where $N_\alpha = S_\alpha \setminus R_\alpha$ is the set of non-regular elements of S_α and it is the partial semigroup by definition of π-group.

Lemma 2.1. Let S be a strong semilattice of π-groups. Let $\phi \in \mathrm{Aut}(S)$, then the following hold:

(i) $\phi|_{E_S}$ is an automorphism of semilattices.
(ii) If $G \subseteq S$ is a group, then there exists $\alpha \in \Lambda$ such that $G \subseteq S_\alpha$.
(iii) For each $\alpha \in \Lambda$, $\phi|_{s_\alpha}$ is an isomorphism of π-groups from S_α to S_τ, where $\phi(e_\alpha) = e_\tau$.

Proof . Let $\phi \in \mathrm{Aut}(S)$.

(i). Suppose $e_\alpha \in E_S$, we have $\phi(e_\alpha) = \phi(e_\alpha e_\alpha) = \phi(e_\alpha)\phi(e_\alpha)$, that is, $\phi(e_\alpha)$ is idempotent, hence $\phi(E_S) \subseteq E_S$.

Now for any $e_\gamma \in E_S$, since ϕ is onto, therefore there exists some $s \in S$ such that $\phi(s) = e_\gamma$. Now we show that $s \in E_S$. For this we have

$$\begin{aligned}\phi(s) &= e_\gamma \\ &= e_\gamma e_\gamma \quad (\text{as } e_\gamma \in E_S) \\ &= \phi(s)\phi(s) \\ &= \phi(s^2) \quad (\text{as } \phi \text{ is homomorphism}).\end{aligned}$$

That is, $\phi(s) = \phi(s^2)$. Since ϕ is injective, therefore we have $s = s^2$, implies, s is idempotent. Hence we have $e_\gamma = \phi(s) \in \phi(E_S)$, that is, $E_S \subseteq \phi(E_S)$. Thus we have $\phi(E_S) = E_S$. Since each S_α contains a unique idempotent e_α, and $\phi \in \mathrm{Aut}(S)$ permutes the idempotents, ϕ induces a bijection on Λ. Since $e_\alpha e_\beta = e_\beta$ if and only if $\alpha \geq \beta$, then ϕ preserves the order on Λ.

(ii). Suppose G is a subgroup of S. Let e be the identity element of G. Then $e = e_\alpha \in S_\alpha$ for some $\alpha \in \Lambda$. We show that G is a subgroup of S_α. Let $g \in G$, then $g \in S_\beta$ for some $\beta \in \Lambda$. Since e is the identity element of G, so $ge = g$. Also, $g = ge = f_{\beta,\alpha\beta}(g)f_{\alpha,\alpha\beta}(e) \in S_{\alpha\beta}$. So $\beta = \alpha\beta$, as $S_\beta \cap S_{\alpha\beta} = \emptyset$, this implies, $\beta \leq \alpha$.

Let g^{-1} be the inverse of g, then $g^{-1} \in S_\eta$ for some $\eta \in \Lambda$. Thus $gg^{-1} = e \in S_\alpha$. Also, $e = gg^{-1} = f_{\beta,\beta\eta}(g)f_{\eta,\beta\eta}(g^{-1}) \in S_{\beta\eta}$. So $e \in S_{\beta\eta}$, implies, $\alpha = \beta\eta$ and so $\alpha \leq \beta$. Hence $\alpha = \beta$ and $g \in S_\alpha$, that is, $G \subseteq S_\alpha$.

(iii). Let $g \in S_\alpha$, since S_α is a π-group, so there exists a subgroup G^{S_α} of S_α which is an ideal of S_α and there exists $n \in \mathbb{N}$ such that $g^n \in G^{S_\alpha}$. Since G^{S_α} is a group, it implies the inverse of g^n exists in G^{S_α}. That is, $g^{-n} \in G^{S_\alpha} \subseteq S_\alpha$ such that $g^n g^{-n} = e_\alpha \in S_\alpha$. Let $\phi(g^n) \in S_\gamma$ for some $\gamma \in \Lambda$. Also, by part (i), $\phi(e_\alpha) = e_\tau \in S_\tau$ for some $\tau \in \Lambda$ and $\phi(g^{-n}) \in S_\gamma$.

Now we have

$$\begin{aligned}\phi(e_\alpha) &= \phi(g^n g^{-n}) \\ &= \phi(g^n)\phi(g^{-n}) \in S_\gamma.\end{aligned}$$

That is, $S_\tau = S_\gamma$. Hence $\phi(S_\alpha) \subseteq S_\tau$. Since ϕ is an automorphism and so ϕ^{-1} exists and will do same and hence $\phi^{-1}(S_\tau) \subseteq S_\alpha$, that is, $S_\tau \subseteq \phi(S_\alpha)$ and from part (i), we are done. □

By the above lemma we know that every automorphism of S induces an automorphism of Λ. We will denote this automorphism of semilattices by ϕ_Λ. Hence, we can write $\phi_\Lambda(\alpha) = \tau$, where $\phi(e_\alpha) = e_\tau$. Let $\phi \in \mathrm{Aut}(S)$. Then we write ϕ_α for $\phi|_{S_\alpha}$, where $\alpha \in \Lambda$. By Lemma 2.1, we know ϕ_α is an isomorphism. So given an automorphism ϕ of S, we obtain family $\{\phi_\alpha : \alpha \in \Lambda\}$ of π-group isomorphisms and a semilattice automorphism denoted by ϕ_Λ. Thus we have ϕ_Λ and $\{\phi_\alpha : \alpha \in \Lambda\}$ determines ϕ completely.

Following lemma is due to Lallement; for the proof, one can see [2].

Lemma 2.2. Let $\varphi : S \to T$ be a homomorphism from a regular semigroup S into a semigroup T. Then $\mathrm{im}(\varphi)$ is regular. □

Let $S = (\Lambda, \{S_\alpha\}_{\alpha\in\Lambda}, \{f_{\alpha,\beta}\}_{\alpha\geq\beta})$ be strong semilattices of π-groups. Note that $R^S = \bigsqcup_{\alpha\in\Lambda} R^{S_\alpha}$. Now, for any $s \in S$, we define a mapping $\psi : S \to R^S$ by

$$\psi(s) = e_\alpha s \ \text{ if } s \in S_\alpha.$$

Where e_α is the unique idempotent of the π-group S_α. Since we know R^S is an ideal of S. Thus the map ψ is well defined. The following lemma shows that the map ψ commutes with the linking homomorphisms.

Lemma 2.3. Let $S = (\Lambda, \{S_\alpha\}_{\alpha\in\Lambda}, \{f_{\alpha,\beta}\}_{\alpha\geq\beta})$ be strong semilattices of π-groups. Then for any $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$ and for any $s \in S_\alpha$, we have

$$\psi \ f_{\alpha,\beta} = f_{\alpha,\beta} \ \psi.$$

Proof . Let $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$. Then for any $s \in S_\alpha$ we have

$$\begin{aligned} f_{\alpha,\beta} \ \psi(s) &= f_{\alpha,\beta}(e_\alpha s) \\ &= f_{\alpha,\beta}(e_\alpha) f_{\alpha,\beta}(s) \\ &= e_\beta(f_{\alpha,\beta}(s)) \\ &= \psi \ f_{\alpha,\beta}(s). \end{aligned}$$

Thus we have

$$\psi \ f_{\alpha,\beta} = f_{\alpha,\beta} \ \psi.$$

□ Next, we start from semilattices automorphism and a family of π-group isomorphisms satisfying a condition under which an automorphism of strong semilattices of π-groups can be constructed.

Theorem 2.4. Let $S = (\Lambda, \{S_\alpha\}_{\alpha\in\Lambda}, \{f_{\alpha,\beta}\}_{\alpha\geq\beta})$ be strong semilattices of π-groups. Let $\phi_\Lambda \in \mathrm{Aut}(\Lambda)$ and for each $\alpha \in \Lambda$, $\phi_\alpha : S_\alpha \to S_{\phi_\Lambda(\alpha)}$ be an isomorphism of π-groups. Also, assume that the following conditions are satisfied.

(1) $\phi|N^S$ is a partial automorphism of N^S, and for any $s, s' \in N^S$, if $ss' \notin N^S$, then $\phi(s)\phi(s') \notin N^S$.
(2) $\psi \ \phi_\beta \ f_{\alpha,\beta} = \psi \ f_{\phi_\Lambda(\alpha),\phi_\Lambda(\beta)} \ \phi_\alpha$.

Define a mapping ϕ on S by $\phi(s) = \phi_\alpha(s)$ if $s \in S_\alpha$. Then ϕ is an automorphism of S. Conversely, every automorphism of strong semilattices of π-groups satisfies the conditions.

Proof . Suppose there exists a semilattice automorphism $\phi_\Lambda : \Lambda \to \Lambda$ and a family of π-group isomorphisms $\{\phi_\alpha : \alpha \in \Lambda\}$ where $\phi_\alpha : S_\alpha \to S_{\phi_\Lambda(\alpha)}$ satisfying the above two conditions. Let $\phi : S \to S$ be a map defined by $\phi(s) = \phi_\alpha(s)$ if $s \in S_\alpha$. We show that $\phi \in \mathrm{Aut}(S)$. Let $s_1, s_2 \in S$. If $s_1 = s_2$, then there exists $\alpha \in \Lambda$ such that $s_1, s_2 \in S_\alpha$. Since $\phi_\alpha : S_\alpha \to S_{\phi_\Lambda(\alpha)}$ is an isomorphism, therefore we have

$$\begin{aligned} & s_1 = s_2 \\ \Leftrightarrow \ & \phi_\alpha(s_1) = \phi_\alpha(s_2) \\ \Leftrightarrow \ & \phi(s_1) = \phi(s_2). \end{aligned}$$

That is, ϕ is well defined and injective. Now for any $t \in S$, there exists some $\alpha \in \Lambda$ with $\phi_\Lambda(\alpha) = \delta \in \Lambda$ such that $t \in S_\delta$. As $\phi_\alpha : S_\alpha \to S_{\phi_\Lambda(\alpha)}$ is an isomorphism. Therefore there exists some $s \in S_\alpha$ such that $t = \phi_\alpha(s) = \phi(s)$, and so ϕ is surjective. Hence ϕ is bijective.

Now we need to show ϕ is a homomorphism. For this, let $s_\alpha \in S_\alpha$ and $s_\beta \in S_\beta$. Then we have the following cases.

Case 1: If $s_\alpha s_\beta \in N_{\alpha\beta}$, then $s_\alpha \in N_\alpha$ and $s_\beta \in N_\beta$, hence by condition (1), we have $\phi(s_\alpha s_\beta) = \phi(s_\alpha)\phi(s_\beta)$.

Case 2: If $s_\alpha s_\beta \notin N_{\alpha\beta}$, then we have $\phi(s_\alpha s_\beta) \notin N^S$, therefore we have

$$\begin{aligned} \phi(s_\alpha s_\beta) &= \phi_{\alpha\beta}(s_\alpha s_\beta) \\ &= \phi_{\alpha\beta}(f_{\alpha,\alpha\beta}(s_\alpha) f_{\beta,\alpha\beta}(s_\beta)) \\ &= \phi_{\alpha\beta}(f_{\alpha,\alpha\beta}(s_\alpha))\phi_{\alpha\beta}(f_{\beta,\alpha\beta}(s_\beta)) \\ &= e_{\phi_\Lambda(\alpha\beta)}\phi_{\alpha\beta}(f_{\alpha,\alpha\beta}(s_\alpha))\phi_{\alpha\beta}(f_{\beta,\alpha\beta}(s_\beta)) \\ &= (e_{\phi_\Lambda(\alpha\beta)}\phi_{\alpha\beta}(f_{\alpha,\alpha\beta}(s_\alpha)))(e_{\phi_\Lambda(\alpha\beta)}\phi_{\alpha\beta}(f_{\beta,\alpha\beta}(s_\beta))) \\ &= (\psi \ \phi_{\alpha\beta} \ f_{\alpha,\alpha\beta}(s_\alpha))(\psi \ \phi_{\alpha\beta} \ f_{\beta,\alpha\beta}(s_\beta)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\phi(s_\alpha)\phi(s_\beta) &= \phi_\alpha(s_\alpha)\phi_\beta(s_\beta)\\ &= f_{\phi_\Lambda(\alpha),\phi_\Lambda(\alpha\beta)}(\phi_\alpha(s_\alpha))f_{\phi_\Lambda(\beta),\phi_\Lambda(\alpha\beta)}(\phi_\beta(s_\beta))\\ &= (e_{\phi_\Lambda(\alpha\beta)}f_{\phi_\Lambda(\alpha),\phi_\Lambda(\alpha\beta)}(\phi_\alpha(s_\alpha)))\ (e_{\phi_\Lambda(\alpha\beta)}f_{\phi_\Lambda(\beta),\phi_\Lambda(\alpha\beta)}(\phi_\beta(s_\beta)))\ \ \text{(by condition (1))}\\ &= (\psi\ f_{\phi_\Lambda(\alpha),\phi_\Lambda(\alpha\beta)}\ \phi_\alpha(s_\alpha))\ (\psi\ f_{\phi_\Lambda(\beta),\phi_\Lambda(\alpha\beta)}\ \phi_\beta(s_\beta))\\ &= (\psi\ \phi_{\alpha\beta}\ f_{\alpha,\alpha\beta}(s_\alpha))(\psi\ \phi_{\alpha\beta}\ f_{\beta,\alpha\beta}(s_\beta))\ \ \text{(by condition (2))}.\end{aligned}$$

Hence we have $\phi(s_\alpha s_\beta) = \phi(s_\alpha)\phi(s_\beta)$. Thus ϕ is an automorphism of S.

Conversely, suppose ϕ is an automorphism of S. By Lemma 2.1, we have the existence of semilattice automorphism ϕ_Λ and a family $\{\phi_\alpha : S_\alpha \to S_{\phi_\Lambda(\alpha)}\}$ of π-group isomorphisms. Since $\phi \in \mathrm{Aut}(S)$, then image of N^S is N^S, by Lemma 2.2. Therefore, condition (1) holds clearly.

Now for any $\alpha \geq \beta$, then $\alpha\beta = \beta$. For $s \in S_\alpha$, we have $e_\beta s = f_{\beta,\beta}(e_\beta)f_{\alpha,\beta}(s) = e_\beta f_{\alpha,\beta}(s) = \psi f_{\alpha,\beta}(s)$. Thus we have,

$$\begin{aligned}\phi(e_\beta s) &= \phi(\psi f_{\alpha,\beta}(s))\\ &= \phi_\beta(\psi\ f_{\alpha,\beta}(s))\\ &= \phi_\beta\ \psi\ f_{\alpha,\beta}(s).\end{aligned}$$

Also, we have

$$\begin{aligned}\phi(e_\beta)\phi(s) &= e_{\phi_\Lambda(\beta)}\phi_\alpha(s)\\ &= (e_{\phi_\Lambda(\beta)})(e_{\phi_\Lambda(\alpha)})(\phi_\alpha(s))\\ &= e_{\phi_\Lambda(\beta)}\ \psi\ \phi_\alpha(s)\\ &= f_{\phi_\Lambda(\alpha),\phi_\Lambda(\beta)}\ \psi\ \phi_\alpha(s).\end{aligned}$$

Thus we have

$$\phi_\beta\ \psi\ f_{\alpha,\beta} = f_{\phi_\Lambda(\alpha),\phi_\Lambda(\beta)}\ \psi\ \phi_\alpha. \tag{1}$$

Now for any $\alpha \in \Lambda$ and $s \in S_\alpha$ we have

$$\begin{aligned}\phi_\alpha\psi(s) &= \phi_\alpha(e_\alpha s)\\ &= \phi_\alpha(e_\alpha)\phi_\alpha(s)\\ &= e_{\phi_\Lambda(\alpha)}\phi_\alpha(s)\\ &= \psi\ \phi_\alpha(s).\end{aligned}$$

Therefore, we have

$$\phi_\alpha\ \psi = \psi\ \phi_\alpha. \tag{2}$$

Hence we have

$$\begin{aligned}\psi\ \phi_\beta\ f_{\alpha,\beta} &= \phi_\beta\ \psi\ f_{\alpha,\beta}\ \ \text{(by equation (2))}\\ &= f_{\phi_\Lambda(\alpha),\phi_\Lambda(\beta)}\ \psi\ \phi_\alpha\ \ \text{(by equation (1))}\\ &= \psi\ f_{\phi_\Lambda(\alpha),\phi_\Lambda(\beta)}\ \phi_\alpha\ \ \text{(by Lemma (2.3))}.\end{aligned}$$

That is,

$$\psi\ \phi_\beta\ f_{\alpha,\beta} = \psi\ f_{\phi_\Lambda(\alpha),\phi_\Lambda(\beta)}\ \phi_\alpha.$$

Thus the proof is completed. □ In the following theorem, we provide a construction for the automorphisms of S from the automorphisms of underlying π-groups S_α.

Theorem 2.5. Suppose all the linking homomorphisms are bijective and $\Lambda = \{\alpha, \beta\}_{\alpha\leq\beta}$. Consider $S = S_\alpha \cup S_\beta$, then every automorphism of S_α or S_β gives rise to an automorphism of S.

Proof . Suppose all the linking homomorphisms be bijective. Then $S_\alpha \cong S_\beta \cong \mathbb{G}$. Let $\theta \in \mathrm{Aut}(\mathbb{G})$ be the arbitrary automorphism of $\mathbb{G}$. Since $S_\alpha \cong S_\beta \cong \mathbb{G}$, therefore we have the isomorphisms $\phi_\alpha : \mathbb{G} \to S_\alpha$ and $\phi_\beta : \mathbb{G} \to S_\beta$.

Let $\theta^\phi : S \to S$ be the map defined by

$$\theta^\phi(s) = \begin{cases} \phi_\alpha\theta\phi_\alpha^{-1}(s) & \text{if } s \in S_\alpha \\ \phi_\beta\theta\phi_\beta^{-1}(s) & \text{if } s \in S_\beta. \end{cases}$$

We show that $\theta^\phi \in \mathrm{Aut}(S)$. For this, we first show that for all $s \in S_\beta$

$$\theta^\phi f_{\beta,\alpha}(s) = f_{\beta,\alpha}\theta^\phi(s). \tag{3}$$

Since $\phi_\beta : \mathbb{G} \to S_\beta$ and $f_{\beta,\alpha} : S_\beta \to S_\alpha$ are isomorphisms, we can define $\phi_\alpha = f_{\beta,\alpha}\phi_\beta$. Therefore, we have

$$\begin{aligned} & \phi_\alpha^{-1} = (f_{\beta,\alpha}\phi_\beta)^{-1} \\ \Rightarrow\quad & \phi_\alpha^{-1} = \phi_\beta^{-1} f_{\beta,\alpha}^{-1} \\ \Rightarrow\quad & \phi_\alpha\theta\phi_\alpha^{-1} = \phi_\alpha\psi\phi_\beta^{-1} f_{\beta,\alpha}^{-1} \\ \Rightarrow\quad & \phi_\alpha\theta\phi_\alpha^{-1} = f_{\beta,\alpha}\phi_\beta\theta\phi_\beta^{-1} f_{\beta,\alpha}^{-1} \\ \Rightarrow\quad & \phi_\alpha\theta\phi_\alpha^{-1} f_{\beta,\alpha} = f_{\beta,\alpha}\phi_\beta\theta\phi_\beta^{-1}. \end{aligned}$$

Now for any $s \in S_\beta$, we have

$$\begin{aligned} & \phi_\alpha\theta\phi_\alpha^{-1} f_{\beta,\alpha}(s) = f_{\beta,\alpha}\phi_\beta\theta\phi_\beta^{-1}(s) \\ \Rightarrow\quad & \phi_\alpha\theta\phi_\alpha^{-1}(f_{\beta,\alpha}(s)) = f_{\beta,\alpha}\theta^\phi(s) \\ \Rightarrow\quad & \theta^\phi(f_{\beta,\alpha}(s)) = f_{\beta,\alpha}\theta^\phi(s). \end{aligned}$$

Hence for all $s \in S_\beta$ we have $\theta^\phi f_{\beta,\alpha}(s) = f_{\beta,\alpha}\theta^\phi(s)$.
It is clear that θ^ϕ is bijective. Now we show that θ^ϕ is a homomorphism. For this, we have the following cases.
Case(i). If $s, t \in S_\alpha$ or S_β, then we have

$$\begin{aligned} \psi^\phi(st) &= \phi_\alpha\theta\phi_\alpha^{-1}(st) \\ &= \phi_\alpha\theta\phi_\alpha^{-1}(s)\phi_\alpha\theta\phi_\alpha^{-1}(t) \\ &= \theta^\phi(s)\theta^\phi(t). \end{aligned}$$

Case(ii). If $s \in S_\alpha$ and $t \in S_\beta$, then we have

$$\begin{aligned} \theta^\phi(st) &= \theta^\phi(f_{\alpha,\alpha}(s)f_{\beta,\alpha}(t)) \\ &= \theta^\phi(sf_{\beta,\alpha}(t)) \\ &= \theta^\phi(ss_\alpha) \quad (\text{where } s_\alpha = f_{\beta,\alpha}(t)) \\ &= \phi_\alpha\theta\phi_\alpha^{-1}(ss_\alpha) \\ &= \phi_\alpha\theta\phi_\alpha^{-1}(s)\phi_\alpha\theta\phi_\alpha^{-1}(s_\alpha) \\ &= \theta^\phi(s)\theta^\phi(f_{\beta,\alpha}(t)) \\ &= \theta^\phi(s)f_{\beta,\alpha}\theta^\phi(t) \quad (\text{by equation (3)}) \\ &= \theta^\phi(s)\theta^\phi(t). \end{aligned}$$

Hence we have $\theta^\phi \in \mathrm{Aut}(S)$ and every automorphism of S can be constructed in this way. □

The next lemma helps us to prove the above theorem for arbitrary semilattices.

Lemma 2.6. Let $S = (\Lambda, \{S_\alpha\}_{\alpha\in\Lambda}, \{f_{\alpha,\beta}\}_{\alpha\geq\beta})$ be strong semilattices of π-groups with all the linking homomorphisms bijective, then for any $\lambda \in \Lambda$, we have $S \cong \Lambda \times S_\lambda$ ($\cong \Lambda \times \mathbb{G}$).

Proof . Fix $\lambda \in \Lambda$, then for each $\alpha \in \Lambda$ we have an isomorphism

$$\sigma_\alpha = f_{\lambda,\lambda\alpha}^{-1}\, f_{\alpha,\lambda\alpha} : S_\alpha \to S_\lambda.$$

Now define a map $\chi : S \to \Lambda \times S_\lambda$ by $\chi(s) = (\alpha, \sigma_\alpha(s))$ if $s \in S_\alpha$. We show χ is an isomorphism. Let $s_1, s_2 \in S$. If $s_1 = s_2$, then there exists $\alpha \in \Lambda$ such that $s_1, s_2 \in S_\alpha$. Since $\sigma_\alpha : S_\alpha \to S_\lambda$ is an isomorphism, therefore we have

$$\begin{aligned} & s_1 = s_2 \\ \Leftrightarrow\ & \sigma_\alpha(s_1) = \sigma_\alpha(s_2) \\ \Leftrightarrow\ & (\alpha, \sigma_\alpha(s_1) = (\alpha, \sigma_\alpha(s_2)). \end{aligned}$$

That is, χ is well defined and injective. Now for any $(\alpha, t) \in \Lambda \times S_\lambda$, there exists some $s' \in S_\alpha$ such that $t = \sigma_\alpha(s')$ as σ_α is surjective, therefore we have $(\alpha, t) = (\alpha, \sigma_\alpha(s')) = \chi(s')$, that is, χ is surjective.

Now for any $s, t \in S$ then $s \in S_\alpha$ and $t \in S_\beta$ for some $\alpha, \beta \in \Lambda$. If $\alpha = \beta$, then there is nothing to prove. Now suppose $\alpha \neq \beta$, we have

$$\begin{aligned} \chi(st) &= (\alpha\beta, \sigma_{\alpha\beta}(st)) \\ &= (\alpha\beta, \sigma_\alpha(s)\sigma_\beta(t)) \quad \text{(as all the linking homomorphisms are bijective)} \\ &= (\alpha, \sigma_\alpha(s))(\beta, \sigma_\beta(t)) \\ &= \chi(s)\chi(t). \end{aligned}$$

Therefore χ is an isomorphism. □

Corollary 2.7. Let $S = (\Lambda, \{S_\alpha\}_{\alpha\in\Lambda}, \{f_{\alpha,\beta}\}_{\alpha\geq\beta})$ be strong semilattices of π-groups with all the linking homomorphisms bijective, then every automorphism of S_α for some $\alpha \in \Lambda$ gives rise to an automorphism of S.

Proof . *The proof follows from Lemma 2.6 and Theorem 2.5.* □

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