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On fixed point approximation method for finite family of k-strictly pseudo-contractive mappings and pseudomonotone equilibrium problem in Hadamard space

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Abstract

In this paper, we first introduce the Halpern iteration process for approximating the solution of the fixed point problem of a finite family of k-strictly pseudo-contractive mappings in Hadamard spaces. We also propose an extra gradient Halpern iterative algorithm for approximating a common solution of a finite family of k_j-strictly pseudocontractive mappings and a pseudomonotone equilibrium problem in Hadamard space. We prove a strong convergence result without imposing any strict (compactness) conditions for approximating the solutions to the aforementioned problems. We state some consequences of our results and display some numerical examples to show the performance of our results. Our results improve and generalize many recent results in the literature.

Keywords: Strictly pseudo-contractive mapping, Hadamard spaces, pseudomonotone equilibrium, extragradient algorithm; fixed point problem
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1 Introduction

Let C be a nonempty, closed and convex subset of a CAT(0) space (see Section 2 for details) X and $T : C \to C$ be a nonlinear mapping. A point $p \in C$ is called a fixed point of T if $Tp = p$. We denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is said to be

(i) nonexpansive, if

$$d(Tx, Ty) \leq d(x, y) \quad \forall \ \ x, y \in C,$$

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(ii) k-strictly pseudo-contractive, if $p \in F(T)$ and $k \in [0, 1)$, then

$$d^2(Tx, p) \le d^2(x, p) + kd^2(x, Tx), \ \forall \ x \in C.$$

Note that if $k = 0$ in (ii) above, then we obtain (i), which implies that the class of k-strictly pseudo-contractive mappings contains the class of nonexpansive mappings. The class of strictly pseudo-contractive mappings have been considered by many authors in Hilbert and Banach spaces as well as Hadamard spaces (see [19] and other references contained in).

Fixed point theory has been the center of attractions in the area of convex and nonlinear analysis. The study of fixed point theory in generalized metric spaces (particularly CAT(0) spaces) began with the work of Kirk [31]. After that, numerous authors have continued to obtain interesting results on fixed point theory in metric spaces and its several generalizations (see [4, 18, 21] and other references therein.)

On the other hand, the equilibrium problem (EP) is one of the most important optimization problems that unifies other optimization problems and other problems of interest in many applications. The EP has been extensively studied in the framework of Hilbert and Banach spaces (see [5] and other references contained in). Recently, Khatibzadeh and Mohebbi [28] extended the study of monotone and pseudo-monotone EP to Hadamard spaces.

Let X be a Hadamard space, $K \subset X$ and $f : K \times K \to \mathbb{R}$ be a bifunction. f is said to be monotone, if

$$f(x, y) + f(y, x) \le 0, \ \forall \ x, y \in X,$$

and pseudo monotone, if

$$f(x, y) \ge 0 \implies f(y, x) \le 0, \ \forall \ x, y \in X.$$

An EP for f and K consists of finding $x^* \in K$ such that

$$f(x^*, y) \ge 0, \ \forall \ y \in C. \tag{1.1}$$

The point x^* is called an equilibrium point. We denote the set of all equilibrium points for EP by $S(f, K)$. To study the EP, it is required that the the bifuntion f satisfy the following conditions:

(B1) $f(x, .)$ is convex and lower semicontinuous for all $x \in X$,
(B2) $f(., y)$ is Δ-upper semicontinuous for all $y \in X$,
(B3) f is Lipschitz-type continuous, i.e. there exist two positive constants c_1 and c_2 such that

$$f(x, y) + f(y, z) \ge f(x, z) - c_1 d^2(x, y) - c_2 d^2(y, z), \ \forall \ x, y, z \in X. \tag{1.2}$$

(B4) f is pseudo-monotone.

Khatibzadeh and Mohebbi [28] studied the existence of solutions of (1.1) associated with pseudomonotone bifunctions. They employed the proximal point algorithm to approximate the equilibrium point of f and established that the sequence generated by the algorithm converges. In 2019, Khatibzadeh and Mohebbi [29] introduced the extragradient method to approximate the equilibrium point of pseudo-monotone function of (1.1) in Hadamard spaces. Precisely, they proposed the following extragraident method for EP in Hadamard spaces as follows:

$$y_n \in Arg \min_{y \in C} \big\{ f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \big\}, \tag{1.3}$$

$$x_{n+1} \in Arg \min_{y \in C} \big\{ f(y_n, y) + \frac{1}{2\lambda_n} d^2(y_n, y) \big\}. \tag{1.4}$$

It was also established that the sequence generated by (1.3) Δ-converges to a solution of (1.1). In the same vein, Moharami and Eskandani [37] proposed a hybrid extragradient method to approximate a common element of the set of solutions of an EP and a common zero of a finite family of monotone operators in Hadamard spaces. They established the convergence theorems of their hybrid extagradient methods under suitable assumptions.

Motivated by the works of the aforementioned authors in literature, we propose a Halpern extragradient method for approximating the solution of fixed point problem of a finite family of k-strictly pseudo-contractive mappings and pseudo-monotone EP in Hadamard space. Our method strongly converges to a common element in the intersection of the set of fixed points of pseudo-contractive mappings and the set of equilibrium points of the pseudo-monotone function f. We display a numerical example to show the the behaviour of our proposed method. Our result improve and generalize the results of [1, 2, 13, 19, 39, 40] and many recent results in the literature.

2 Preliminaries

Let X be a metric space and $x, y \in X$. A geodesic from x to y is a map (or a curve) c from the closed interval $[0, d(x, y)] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(d(x, y)) = y$ and $d(c(t), c(t')) = |t - t'|$, for all $t, t' \in [0, d(x, y)]$. The image of c is called a geodesic segment joining from x to y. The geodesic segment is denoted by $[x, y]$ when it is unique. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset D of a geodesic space X is said to be convex, if for any two points $x, y \in D$, the geodesic joining x and y is contained in D, that is, if $c : [0, d(x, y)] \to X$ is a geodesic such that $x = c(0)$ and $y = c(d(x, y))$, then $c(t) \in D$, for all $t \in [0, d(x, y)]$. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three vertices (points in X) with unparameterized geodesic segments between each pair of vertices. For any geodesic triangle there is comparison (Alexandrov) triangle $\bar{\Delta} \subset \mathbb{R}^2$, such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$, for $i, j \in \{1, 2, 3\}$. A geodesic space X is a CAT(0) space if the distance between an arbitrary pair of points on a geodesic triangle Δ does not exceed the distance between its corresponding pair of points on its comparison triangle $\bar{\Delta}$. If Δ and $\bar{\Delta}$ are geodesic and comparison triangles in X respectively, then Δ is said to satisfy the CAT(0) inequality for all points x, y of Δ and $\bar{x}, \bar{y}$ of $\bar{\Delta}$ if

$$d(x, y) = d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{2.1}$$

Let x, y, z be points in X and y_0 be the midpoint of the segment $[y, z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \le \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d(y, z). \tag{2.2}$$

Berg and Nikolaev [9] introduced the notion of quasi-linearization in a CAT(0) space as follows: Let a pair $(a, b) \in X \times X$ which is denoted by $\overrightarrow{ab}$ be a vector. Then, the quasilinearization map $\langle ., . \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)), \quad \text{for all } a, b, c, d \in X. \tag{2.3}$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, for all $a, b, c, d, e \in X$. Furthermore, a geodesic space X is said to satisfy the Cauchy-Schwartz inequality, if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \le d(a, b)d(c, d),$$

for all $a, b, c, d \in X$. It is well known that a geodesically connected space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality [18]. Also, it is known that complete CAT(0) spaces are called Hadamard spaces.

In 2010, Kakavandi and Amini [26] introduced the dual space of a Hadamard space X as follows: Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ which is define by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \quad (t \in \mathbb{R},\ a, b, x \in X),$$

where $C(X, \mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwartz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b)$ $(t \in \mathbb{R}, a, b \in X)$, where $L(\phi) = \sup\{(\phi(x) - \phi(y))/d(x, y) : x, y \in X,\ x \ne y\}$ is the Lipschitz semi-norm for any function $\phi : X \to \mathbb{R}$. A pseudometric $\mathcal{D}$ on $\mathbb{R} \times X \times X$ is defined by

$$\mathcal{D}((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)), \quad (t, s \in \mathbb{R}, a, b, c, d \in X).$$

In a Hadamard space X, the psuedometric space $(\mathbb{R} \times X \times X, \mathcal{D})$ can be considered as a subset of the pseudometric space of all real-valued Lipschitz functions $(Lip(X, \mathbb{R}), L)$. It is well known from [26] that $\mathcal{D}((t, a, b), (s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$, for all $x, y \in X$. Thus, $\mathcal{D}$ induces an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t, a, b) is defined as

$$[t\overrightarrow{ab}] := \{s\overrightarrow{cd} : \mathcal{D}((t, a, b), (s, c, d)) = 0\}.$$

The set $X^* = \{[t\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X\}$ is a metric space with the metric $\mathcal{D}([t\overrightarrow{ab}], [s\overrightarrow{cd}) := \mathcal{D}((t, a, b), (s, c, d))$. The pair $(X^*, \mathcal{D})$ is called the dual space of the metric space (X, d). It is shown in [26] that the dual of a closed and convex subset of a Hilbert space H with nonempty interior is H and $t(b-a) \equiv [t\overrightarrow{ab}]$ for all $t \in \mathbb{R}$ and $a, b \in H$. We also note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle, \quad (x^* = [t\overrightarrow{ab}] \in X^*, \quad x, y \in X).$$

Let $\{x_n\}$ be a bounded sequence in X and $r(., \{x_n\}) : X \to [0, \infty)$ be a continuous mapping defined by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}) : \inf\{r(x, x_n) : x \in X\},$$

while the asymptotic center of $\{x_n\}$ is the set

$$A(\{x_n\}) = x \in X : r(x, \{x_n\}) = r(\{x_n\}).$$

It is well known from [17, 32] that in a complete CAT(0) space X, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim\limits_{n\to\infty} x_n = x$.

Definition 2.1. Let C be a nonempty, closed convex subset of a CAT(0) space X. The metric projection $P_C : X \to C$ which assigns to each $x \in X$ the unique point $P_C x$ is defined by

$$d(x, P_C x) = \inf\{d(x, y) : y \in C\}.$$

Lemma 2.2. [14, 18] Let X be a CAT(0) space. Then for all $w, x, y, z \in X$ and all $t \in [0, 1]$, we have

1. $d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z)$,
2. $d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$,
3. $d^2(z, tx \oplus (1-t)y) \leq t^2 d^2(z, x) + (1-t)^2 d^2(z, y) + 2t(1-t)\langle \overrightarrow{zx}, \overrightarrow{zy} \rangle$.

Lemma 2.3. [18] Every bounded sequence in a complete CAT(0) space has a $\triangle$-convergence subsequence.

Definition 2.4. Let C be a nonempty, closed and convex subset of a Hadamard space X. A mapping $T : C \to C$ is said to be Δ-demiclosed, if for any bounded sequence $\{x_n\}$ in X such that Δ-$\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$, then $x = Tx$.

Lemma 2.5. [19] Let C be a nonempty, closed convex subset of an Hadamard space X and $T : X \to X$ be a k-strictly pseudo-contractive mapping, then $I - T$ is demiclosed at the origin.

Lemma 2.6. [15] Let C be a nonempty, closed convex subset of a CAT(0) space X, $x \in X$ and $u \in C$. Then $u = P_C x$ if and only if $\langle \overrightarrow{xu}, \overrightarrow{uy} \rangle \leq 0$ for all $y \in C$.

Lemma 2.7. [11] Let X be a CAT(0) space, $\{u_1, u_2, \cdots, u_N\} \subset X$ and $\{\beta_1, \beta_2, \cdots, \beta_N\} \subset (0, 1)$ with $\sum\limits_{i=0}^{N} \beta_i = 1$. Then,

$$d^2\left(\sum_{i=1}^{N} \oplus \beta_i u_i, x\right) \leq \sum_{i=1}^{N} \beta_i d^2(u_i, x) - \sum_{i,j=1, i\neq j}^{N} \beta_i \beta_j d^2(u_i, u_j).$$

Lemma 2.8. [6, 30] Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\gamma_n\}$ be a sequence of real numbers in $(0, 1)$ with conditions $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Assume that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n d_n, \quad n \geq 1.$$

If $\limsup\limits_{k\to\infty} d_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition:
$\limsup\limits_{k\to\infty}(a_{n_k} - a_{n_k+1}) \leq 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

3 Main results

Lemma 3.1. Let X be a Hadamard space and $T_j : X \to X, j = 1, 2, .., N$ be a finite family of k_j- strictly pseudo-contractive mappings for some $0 < k_j < 1$ such that $\Gamma := \cap_{j=1}^{N} F(T_j) \neq \emptyset$. For arbitrary $x_1, u \in X$, let the sequence $\{x_n\}$ be generated iteratively by

$$\begin{cases} u_n = \gamma_{n,0} x_n \oplus \sum_{j=1}^{N} \oplus \gamma_{n,j} T_j x_n; \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n) u_n; \end{cases} \tag{3.1}$$

where $\gamma_{n,0} \subset [a, b]$ for some $a, b \in (k_j, 1)$ and $\alpha_n \in (0, 1)$ satisfying $\sum_{i=0}^{N} \gamma_{n,j} = 1$. Then, $\{x_n\}$ is bounded.

Proof . Let $z \in \Gamma$, then we obtain from (3.1) and Lemma 2.7 that

$$\begin{aligned} d^2(u_n, z) &= d^2(\gamma_{n,0} x_n \oplus \sum_{j=1}^{N} \oplus \gamma_{n,j} T_j x_n, z) \\ &\leq \gamma_{n,0} d^2(x_n, z) + \sum_{j=1}^{N} \gamma_{n,j} d^2(T_j x_n, z) - \gamma_{n,0} \sum_{j=1}^{N} \gamma_{n,j} d^2(x_n, T_j x_n) \\ &\leq \gamma_{n,0} d^2(x_n, z) + \sum_{j=1}^{N} \gamma_{n,j} (d^2(x_n, z) + k_j d^2(x_n, T_j x_n)) - \gamma_{n,0} \sum_{j=1}^{N} \gamma_{n,j} d^2(x_n, T_j x_n) \\ &= d^2(x_n, z) - (\gamma_{n,0} - k_j) \sum_{j=1}^{N} \gamma_{n,j} d^2(x_n, T_j x_n). \end{aligned} \tag{3.2}$$

This implies that

$$d(u_n, z) \leq d(x_n, z). \tag{3.3}$$

Using (3.1) and (3.3), we have that

$$\begin{aligned} d(x_{n+1}, z) &= d(\alpha_n u \oplus (1-\alpha_n) u_n, z) \\ &\leq \alpha_n d(u, z) + (1-\alpha_n) d(u_n, z) \\ &\leq \alpha_n d(u, z) + (1-\alpha_n) d(x_n, z) \\ &\leq \max\{d(u, z), d(x_n, z)\}. \end{aligned} \tag{3.4}$$

By induction, we obtain that $\{x_n\}$ is bounded. Consequently, $\{u_n\}$ is also bounded. □

Theorem 3.2. Let X be a Hadamard space and $T_j : X \to X, j = 1, 2, .., N$ be a finite family of k_j- strictly pseudo-contractive mappings for some $0 < k_j < 1$ such that $\Gamma := \cap_{j=1}^{N} F(T_j) \neq \emptyset$. For arbitrary $x_1, u \in X$, let the sequence $\{x_n\}$ be generated iteratively by (3.1), where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

1. $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;
2. $\sum\limits_{j=0}^{N} \gamma_{n,j} = 1$ and $\gamma_{n,0} \subset [a, b]$ for some $a, b \in (k_j, 1)$.

Then, $\{x_n\}$ converges strongly to $x^* = P_\Gamma u$, where P_Γ is the metric projection of X onto Γ.

Proof . Let $z \in \Gamma$, then we have from (3.1), (3.2) and Lemma 2.2 (3) that

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2(\alpha_n u \oplus (1-\alpha_n) u_n, z) \\ &\leq \alpha_n^2 d^2(u, z) + (1-\alpha_n)^2 d^2(u_n, z) + 2\alpha_n (1-\alpha_n) \langle \overrightarrow{uz}, \overrightarrow{u_n z} \rangle \\ &\leq \alpha_n^2 d^2(u, z) + (1-\alpha_n) d^2(x_n, z) - (1-\alpha_n)(\gamma_{n,0} - k_j) \sum_{j=1}^{N} \gamma_{n,j} d^2(x_n, T_j x_n) \\ &+ 2\alpha_n (1-\alpha_n) \langle \overrightarrow{uz}, \overrightarrow{u_n z} \rangle \qquad (3.5) \\ &\leq \alpha_n^2 d^2(u, z) + (1-\alpha_n) d^2(x_n, z) + 2\alpha_n (1-\alpha_n) \langle \overrightarrow{uz}, \overrightarrow{u_n z} \rangle \\ &= (1-\alpha_n) d^2(x_n, z) + \alpha_n b_n, \qquad (3.6) \end{aligned}$$

where $b_n = \alpha_n d^2(u,z) + 2(1-\alpha_n)\langle\overrightarrow{uz}, \overrightarrow{u_n z}\rangle$. From Lemma 2.8, it suffices that

$$\limsup_{k\to\infty}(d(x_{n_k}, z) - d(x_{n_{k+1}}, z)) \leq 0. \tag{3.7}$$

To prove (3.7), suppose $\{d(x_{n_k}, z)\}$ is a subsequence of $\{d(x_n, z)\}$, then

$$\limsup_{k\to\infty}\Big(d^2(x_{n_k}, z) - d^2(x_{n_{k+1}}, z)\Big) = \limsup_{k\to\infty}\Big((d(x_{n_k}, z) - d(x_{n_{k+1}}, z))(d(x_{n_k}, z) + d(x_{n_{k+1}}, z))\Big) \leq 0. \tag{3.8}$$

Now from (3.5), (3.7) and condition (i) of (3.1), we have that

$$\begin{aligned}\limsup_{k\to\infty}\Big((1-\alpha_{n_k})(\gamma_{n_k,0} - k_j)\sum_{j=1}^{N}\gamma_{n_k,j}d^2(x_{n_k}, T_j x_{n_k})\Big) &\leq \limsup_{k\to\infty}\Big(\alpha_{n_k}^2 d^2(u,z) + (1-\alpha_{n_k})d^2(x_{n_k}, z)\\ &\quad - d^2(x_{n_k+1}, z)\\ &\quad + \limsup_{k\to\infty}(2\alpha_{n_k}(1-\alpha_{n_k})\langle\overrightarrow{uz}, \overrightarrow{u_{n_k}z}\rangle\Big)\\ &= \limsup_{k\to\infty}\Big(d^2(x_{n_k}, z) - d^2(x_{n_{k+1}}, z)\Big)\\ &\leq 0.\end{aligned} \tag{3.9}$$

Hence, we obtain that

$$\lim_{k\to\infty} d(x_{n_k}, T_j x_{n_k}) = 0. \tag{3.10}$$

Using (3.1) and (3.10), we have that

$$d(u_{n_k}, x_{n_k}) \leq \sum_{j=1}^{N}\gamma_{n_k,j}d(x_{n_k}, T_j x_{n_k}) \to 0 \text{ as } k\to\infty. \tag{3.11}$$

Also, using (3.1) and (3.11), we have that

$$\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) = 0. \tag{3.12}$$

Since $\{x_{n_k}\}$ is bounded on an Hadamard space X, it follows from Lemma 2.3 that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\triangle - \lim_{j\to\infty}\{x_{n_{k_j}}\} = x^*$. Also, using (3.11) and Lemma 2.3, there exists a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ such that $\triangle - \lim_{j\to\infty}\{u_{n_{k_j}}\} = x^*$. Now, applying (3.10) and Lemma 2.5, we obtain that $x^* \in \cap_{j=1}^N F(T_j)$, which also implies that $x^* \in \Gamma$. Next, we prove that $\{x_n\}$ converges strongly to x^*. To prove this, we first show that $\lim_{k\to\infty}\langle\overrightarrow{uz}, \overrightarrow{u_{n_k}z}\rangle \leq 0$. Let $x^* = P_\Gamma u$. Since $\{u_{n_k}\}$ is bounded, we choose a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ which Δ-converges to x^* such that

$$\begin{aligned}\lim_{k\to\infty}\langle\overrightarrow{uz}, \overrightarrow{u_{n_k}z}\rangle &= \lim_{j\to\infty}\langle\overrightarrow{uz}, \overrightarrow{u_{n_{k_j}}z}\rangle\\ &\leq \langle\overrightarrow{uz}, \overrightarrow{x^*z}\rangle.\end{aligned} \tag{3.13}$$

Applying Lemma (2.6) and (3.11), we obtain that

$$\begin{aligned}\lim_{k\to\infty}\langle\overrightarrow{uz}, \overrightarrow{u_{n_k}z}\rangle &\leq \langle\overrightarrow{uz}, \overrightarrow{x^*z}\rangle\\ &\leq 0.\end{aligned} \tag{3.14}$$

From Lemma 2.8 and (3.5), we obtain

$$d^2(x_{n_{k+1}}, x^*) \leq (1-\alpha_{n_k})d^2(x_{n_k}, x^*) + \alpha_{n_k}\Big(\alpha_{n_k}d^2(u, x^*) + 2(1-\alpha_{n_k})\langle\overrightarrow{ux^*}, \overrightarrow{u_{n_k}x^*}\rangle\Big). \tag{3.15}$$

It suffices from condition (i) of (3.1) and (3.14) that $\lim_{k\to\infty} b_{n_k} \leq 0$. Applying Lemma 2.8 to (3.15), we obtain that $d(x_n, x^*) \to 0$ as $n\to\infty$. Therefore, we conclude that $\{x_n\}$ converges strongly to $x^* = P_\Gamma u$. $\square$ We now consider a Halpern extragradient method for approximating a common solution to the fixed point problem of k-strictly pseudo-contractive mappings and EP with a pseudo-monotone function f.

Theorem 3.3. Let K be a nonempty, closed and convex subset of a Hadamard space X and assume that the bifunction f satisfies B_1, B_2, B_3 and B_4. Let $T_j : X \to X$ be a finite family of k_j-strictly pseudocontractive mappings for some $0 < k_j < 1$ such that $\Gamma := \{\cap_{j=1}^N F(T_j) \cap S(f, K)\} \neq \emptyset$. Let the sequence $\{x_n\}$ be generated iteratively by

Step 1: solve the minimization problem and let w_n be defined as

$$w_n \in Argmin_{y \in K}\{f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y)\}. \tag{3.16}$$

Step 2: solve the following minimization problem and let z_n be defined as

$$z_n \in Argmin_{y \in K}\{f(w_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y)\}. \tag{3.17}$$

Step 3: determine the next approximation u_n as

$$u_n = \gamma_{n,0} z_n \oplus \sum_{j=1}^{N} \gamma_{n,j} T_j z_n.$$

Step 4: Compute the next iterate x_{n+1} as

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) u_n.$$

Step 5: $n := n + 1$ and go back to step 1, where $0 < \rho \leq \lambda_n \leq \mu < \{\frac{1}{2c_1}, \frac{1}{2c_2}\}$ for $n = 0, 1, 2, ..$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

1. $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;
2. $\sum_{j=0}^{N} \gamma_{n,j} = 1$ and $\gamma_{n,0} \subset [a, b]$ for some $a, b \in (k_j, 1)$. Then, $\{x_n\}$ converges strongly to $x^* = P_\Gamma u$, where P_Γ is the metric projection of X onto Γ.

In order to prove Theorem 3.3, we need to establish the following result.

Lemma 3.4. Assume that $\{x_n\}, \{w_n\}$ and $\{z_n\}$ are generated by (3.36) and $z \in S(f, K)$, then we have

$$d^2(z_n, z) \leq d^2(x_n, z) - (1 - 2c_1\lambda_n) d^2(x_n, w_n) - (1 - 2c_2\lambda_n) d^2(w_n, z_n). \tag{3.18}$$

Proof . Let $z \in S(f, K)$, since z_n is the solution of the minimization problem in (3.17), by letting $y = v z_n \oplus (1 - v) z$ such that $v \in [0, 1)$, we have

$$\begin{aligned} f(w_n, z_n) + \frac{1}{2\lambda_n} d^2 2(x_n, z_n) &\leq f(w_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \\ &= f(w_n, v z_n \oplus (1 - v) z) + \frac{1}{2\lambda_n} d^2(x_n, v z_n \oplus (1 - v) z) \\ &\leq v f(w_n, z_n) + (1 - v) f(w_n, z) + \frac{1}{2\lambda_n}\{v d^2(x_n, z_n) + (1 - v) d^2(x_n, z) \\ &\quad - v(1 - v) d^2(z_n, z)\}. \end{aligned} \tag{3.19}$$

Since $f(z, w_n) \geq 0$, then pseudo-monotonicity property of f implies that $f(w_n, z) \leq 0$. Hence, we can re-write (3.19) as

$$f(w_n, z_n) \leq \frac{1}{2\lambda_n}\{d^2(x_n, z) - d^2(x_n, w_n) - v d^2(z_n, z)\}.$$

Now, if $v \to 1^-$, we have

$$f(w_n, z_n) \leq \frac{1}{2\lambda_n}\{d^2(x_n, z) - d^2(x_n, z_n) - d^2(z_n, z)\}. \tag{3.20}$$

Also, since $\{w_n\}$ solves the minimization problem in (3.16), by letting $y = vw_n \oplus (1-v)z_n$ such that $v \in [0,1)$, we have

$$\begin{aligned} f(x_n, w_n) + \frac{1}{2\lambda_n} d^2(x_n, w_n) &\leq f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \\ &= f(x_n, vw_n \oplus (1-v)z_n) + \frac{1}{2\lambda_n} d^2(x_n, vy_n \oplus (1-v)z_n) \\ &\leq vf(x_n, w_n) + (1-v)f(x_n, z_n) + \frac{1}{2\lambda_n}\{vd^2(x_n, w_n) + (1-v)d^2(x_n, z_n) \\ &- v(1-v)d^2(w_n, z_n)\}, \end{aligned} \tag{3.21}$$

which implies that

$$f(x_n, w_n) - f(x_n, z_n) \leq \frac{1}{2\lambda_n}\{d^2(x_n, z_n) - d^2(x_n, w_n) - vd^2(w_n, z_n)\}.$$

Now, if $v \to 1^-$, we get

$$f(x_n, w_n) - f(x_n, z_n) \leq \frac{1}{2\lambda_n}\{d^2(x_n, z_n) - d^2(x_n, w_n) - d^2(w_n, z_n)\}. \tag{3.22}$$

Using the fact that f is Lipschitz-continuous with constants c_1 and c_2, and by applying B_3, we have that

$$-c_1 d^2(x_n, w_n) - c_2 d^2(w_n, z_n) + f(x_n, z_n) - f(x_n, w_n) \leq f(w_n, z_n). \tag{3.23}$$

From (3.22) and (3.23), we get

$$(\frac{1}{2\lambda_n} - c_1)d^2(x_n, w_n) + (\frac{1}{2\lambda_n} - c_2)d^2(w_n, z_n) - \frac{1}{2\lambda_n} d^2(x_n, z_n) \leq f(w_n, z_n). \tag{3.24}$$

We therefore conclude from (3.20) and (3.24) that

$$d^2(z_n, z) \leq d^2(x_n, z) - (1 - 2c_1\lambda_n)d^2(x_n, w_n) - (1 - 2c_2\lambda_n)d^2(w_n, z_n).$$

□

Proof . Let $z \in \Gamma$. Using Lemma 3.4, Lemma 2.7 and step 3 of (3.36), we have that

$$\begin{aligned} d^2(u_n, z) &= d^2(\gamma_{n,0} z_n \oplus \sum_{j=1}^{N} \oplus \gamma_{n,j} T_j z_n, z) \\ &\leq \gamma_{n,0} d^2(z_n, z) + \sum_{j=1}^{N} \gamma_{n,j} d^2(T_j z_n, z) - \gamma_{n,0} \sum_{j=1}^{N} \gamma_{n,j} d^2(z_n, T_j z_n) \\ &\leq \gamma_{n,0} d^2(z_n, z) + \sum_{j=1}^{N} \gamma_{n,j}(d^2(z_n, z) + k_j d^2(z_n, T_j z_n)) - \gamma_{n,0} \sum_{j=1}^{N} \gamma_{n,j} d^2(z_n, T_j z_n) \\ &= d^2(x_n, z) - (\gamma_{n,0} - k_j) \sum_{j=1}^{N} \gamma_{n,j} d^2(z_n, T_j z_n) \\ &\leq d^2(x_n, z) - (1 - 2c_1\lambda_n)d^2(x_n, w_n) - (1 - 2c_2\lambda_n)d^2(w_n, z_n) \\ &- (\gamma_{n,0} - k_j) \sum_{j=1}^{N} \gamma_{n,j} d^2(z_n, T_j z_n). \end{aligned} \tag{3.25}$$

This implies that

$$d(u_n, z) \leq d(x_n, z). \tag{3.26}$$

Following the same process as in (3.4), we have that $\{x_n\}$ is bounded. Consequently, $\{u_n\}$, $\{w_n\}$ and $\{z_n\}$ are all bounded. From (3.36), Lemma 2.2 (3) and (3.25), we obtain

$$\begin{aligned}
d^2(x_{n+1},z) &= d^2(\alpha_n u \oplus (1-\alpha_n)u_n, z)\\
&\le \alpha_n^2 d^2(u,z) + (1-\alpha_n)^2 d^2(u_n,z) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{uz}, \overrightarrow{u_n z}\rangle\\
&\le \alpha_n^2 d^2(u,z) + (1-\alpha_n) d^2(x_n,z) - (1-\alpha_n)(1-2c_1\lambda_n)d^2(x_n,w_n)\\
&- (1-\alpha_n)(1-2c_2\lambda_n)d^2(w_n,z_n) - (1-\alpha_n)(\gamma_{n,0}-k_j)\sum_{j=1}^{N}\gamma_{n,j}d^2(z_n,T_j z_n)\\
&+ 2\alpha_n(1-\alpha_n)\langle \overrightarrow{uz}, \overrightarrow{u_n z}\rangle\\
&\le \alpha_n^2 d^2(u,z) + (1-\alpha_n) d^2(x_n,z) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{uz}, \overrightarrow{u_n z}\rangle. \qquad (3.27)
\end{aligned}$$

Using (3.9), (3.27) and condition (i) of (3.1), we have that

$$\begin{aligned}
\limsup_{k\to\infty}\Big((1-\alpha_{n_k})(1-2c_2\lambda_{n_k})d^2(w_{n_k},z_{n_k})\Big) &\le \limsup_{k\to\infty}\Big(\alpha_{n_k}^2 d^2(u,z) + (1-\alpha_{n_k})d^2(x_{n_k},z) - d^2(x_{n_k+1},z)\\
&+ \limsup_{k\to\infty}(2\alpha_{n_k}(1-\alpha_{n_k})\langle \overrightarrow{uz}, \overrightarrow{u_{n_k} z}\rangle\Big)\\
&= \limsup_{k\to\infty}\Big(d^2(x_{n_k},z) - d^2(x_{n_{k+1}},z)\Big)\\
&\le 0. \qquad (3.28)
\end{aligned}$$

Hence, we obtain from (3.28) and the fact that $(1-2c_2\lambda_{n_k}) > 0$ that

$$\lim_{k\to\infty} d(w_{n_k}, z_{n_k}). \qquad (3.29)$$

Following the same approach in (3.28), we have that

$$\lim_{k\to\infty} d(w_{n_k}, x_{n_k}) = 0. \qquad (3.30)$$

Also, applying (3.27) in (3.28), we get

$$\lim_{k\to\infty} d(z_{n_k}, T_j z_{n_k}) = 0. \qquad (3.31)$$

Using (3.29) and (3.30), we get

$$\lim_{k\to\infty} d(z_{n_k}, x_{n_k}) = 0. \qquad (3.32)$$

On replacing n by n_k in (3.20) and (3.24), and taking the limit , we obtain that

$$\lim_{k\to\infty} f(w_{n_k}, z_{n_k}) = 0. \qquad (3.33)$$

Since $\{z_{n_k}\}$ is bounded, using Lemma 2.3, there exists a subsequence $\{z_{n_{k_j}}\}$ of $\{z_{n_k}\}$ such that $\triangle-\lim_{j\to\infty}\{z_{n_{k_j}}\} = x^*$. Now, since $\{z_n\}$ solves the minimization problem in (3.17) and by letting $z = vz_n \oplus (1-v)y$ such that $v \in [0,1)$ and $y \in K$, we have

$$\begin{aligned}
f(w_n,z_n) + \frac{1}{2\lambda_n}d^2(x_n,z_n) &\le f(w_n,z) + \frac{1}{2\lambda_n}d^2(x_n,z)\\
&= f(w_n, vz_n \oplus (1-v)y) + \frac{1}{2\lambda_n}d^2(x_n, vz_n\oplus(1-v)y)\\
&\le vf(w_n,z_n) + (1-v)f(w_n,y) + \frac{1}{2\lambda_n}\{vd^2(x_n,z_n)\\
&+ (1-v)d^2(x_n,y) - v(1-v)d^2(z_n,y)\}.
\end{aligned}$$

This implies that

$$f(w_n, z_n) - f(w_n, y) \leq \frac{1}{2\lambda_n}\{d^2(x_n, y) - d^2(x_n, z_n) - vd^2(z_n, y)\}.$$

Now, if $t \to 1^{-1}$, we obtain

$$\frac{1}{2\lambda_n}\{d^2(x_n, z_n) + d^2(z_n, y) - d^2(x_n, y)\} \leq f(w_n, y) - f(w_n, z_n), \tag{3.34}$$

which implies that

$$\frac{-1}{2\lambda_n} d(x_n, z_n)\{d(z_n, y) + d(x_n, y)\} \leq f(w_n, y) - f(w_n, z_n). \tag{3.35}$$

Since $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ such that $\triangle - \lim_{j\to\infty}\{y_{n_{k_j}}\} = x^*$. Using (3.32), (3.33), replacing n with n_k in (3.35) and taking *limsup* as $k \to \infty$, we get

$$0 \leq \limsup_{j\to\infty} f(y_{n_{k_j}}, y), \ \forall \ y \in K.$$

Now, since $f(., y)$ is $\triangle$-upper semicontinuous, we get

$$f(x^*, y) \geq 0, \forall \ y \in K.$$

Hence, $x^* \in S(f, K)$. Therefore, we conclude that $x^* \in \Gamma$. Also, following the same argument as in (3.14) and (3.15), we conclude that $\{x_n\}$ converges strongly to $x^* = P_\Gamma u$. $\square$

Corollary 3.5. Let X be a Hadamard space and $T_j : X \to X, j = 1, 2, .., N$ be a finite family of nonexpansive mappings such that $\Gamma := \cap_{j=1}^N F(T_j) \neq \emptyset$. For arbitrary $x_1, u \in X$, let the sequence $\{x_n\}$ be generated iteratively by (3.1), where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

1. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;
2. $\sum_{j=0}^N \gamma_{n,j} = 1$ and $\gamma_{n,0} \subset (0, 1)$.

Then, $\{x_n\}$ converges strongly to $x^* = P_\Gamma u$, where P_Γ is the metric projection of X onto Γ.

Corollary 3.6. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T_j : C \to C,\ j = 1, 2, .., N$ be a finite family of k_j- strictly pseudo-contractive mappings for some $0 < k_j < 1$ such that $\Gamma := \cap_{j=1}^N F(T_j) \neq \emptyset$. For arbitrary $x_1, u \in X$, let the sequence $\{x_n\}$ be generated iteratively by

$$\begin{cases} u_n = \gamma_{n,0}x_n + \sum_{j=1}^N \gamma_{n,j}T_jx_n; \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)u_n; \end{cases} \tag{3.36}$$

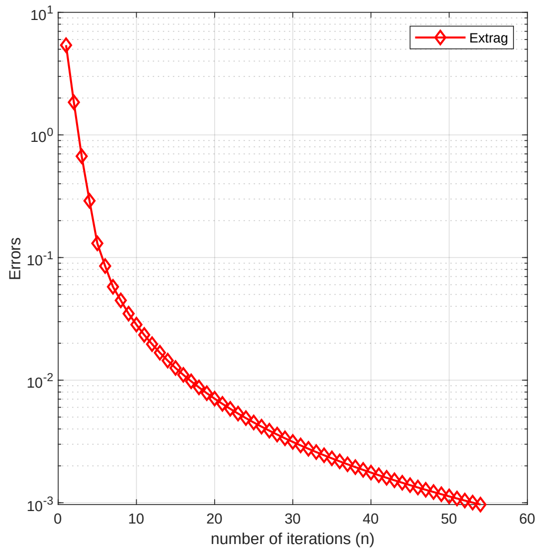
where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

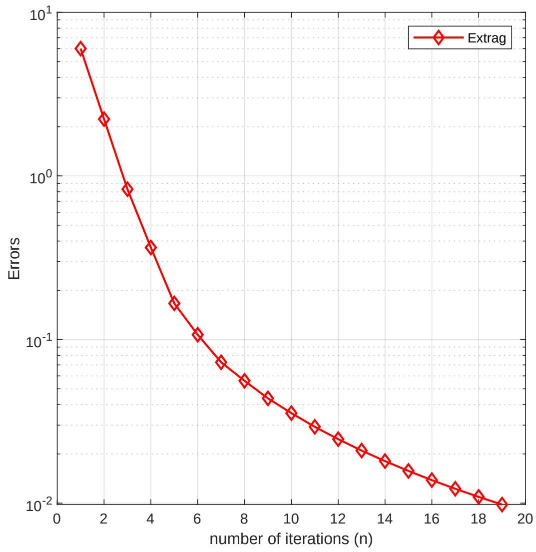
1. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;
2. $\sum_{j=0}^N \gamma_{n,j} = 1$ and $\gamma_{n,0} \subset [a, b]$ for some $a, b \in (k_j, 1)$.

Then, $\{x_n\}$ converges strongly to $x^* = P_\Gamma u$, where P_Γ is the metric projection of H onto Γ.

We highlight our contributions as follows:

1. We considered the pseudomonotone EP in a more general Hadamard space, where [20, 42] considered the Hilbert spaces.
2. The class of mappings considered in this article generalizes the ones considered in [3, 13].
3. Our method of proof is simple and unique.





Example 3.7. Let $X = \mathbb{R}$, $T_1(x) = -2x$ be $\frac{1}{3}$-strict pseudocontractive mapping and $T_2(x) = -5x$ be $\frac{2}{3}-$ strict pseudocontractive mapping with $\{0\} = F(T_1) = F(T_2)$. Suppose that $f(x,y) = 2xy(y-x) + xy|y-x|$ with $\lambda_n = 0.5$ and $\gamma_{n,0} = \frac{3n}{2n^2+3n+7}$, $\gamma_{n,1} = \frac{2n^2}{2n^2+3n+7}$, $\gamma_{n,2} = \frac{7}{2n^2+3n+7}$ and $\mu_n = 0.56$ and $\alpha_n = \frac{1}{2n+1}$. It can be observed that all the conditions of (3.36) have been satisfied. A simple computation shows that (3.36) takes the following form.

Step 1: solve the minimization problem and let w_n be defined as

$$w_n \in Argmin_{y\in K}\{f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y)\}, \tag{3.37}$$

Step 2: solve the following minimization problem and let z_n be defined as

$$z_n \in Argmin_{y\in K}\{f(w_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y)\}, \tag{3.38}$$

Step 3: determine the next approximation u_n as

$$u_n = \gamma_{n,0} z_n \oplus \sum_{j=1}^{2} \gamma_{n,j} T_j z_n,$$

Step 4: Compute the next iterate x_{n+1} as

$$x_{n+1} = \frac{1}{2n+1} u \oplus \frac{2n}{2n+1} u_n,$$

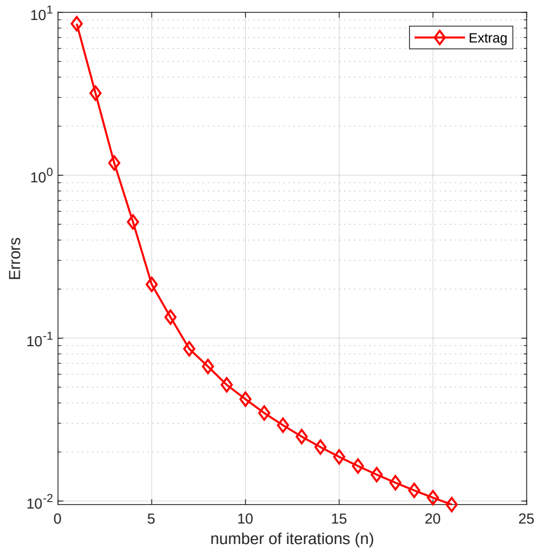
Step 5: $n := n+1$ and go back to step 1,

Remark

To check the efficiency of our main result, we run a numerical simulation with operators and sequences as defined in our Algorithm. The convergence is shown in the figures presented above with a stopping criterion $||x_{n+1} - x_n|| \leq \epsilon$, where $\epsilon = 10^{-4}$.

4 Conclusion

In this paper, we proposed and analyzed the strong convergence theorem of an extragradient Halpern iterative method for solving the fixed point problem of a finite family of k-strictly pseudo-contractive mappings and a pseudomonotone equilibrium problem in Hadamard space. Strong convergence result of these two iterative algorithms were proved. Furthermore, the method of proof employed in this paper is quite different from the usual two cases approach (see [1, 2, 3, 7, 8, 11, 13, 22, 24]). Consequences and numerical examples were displayed to show the applicability of our result.



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