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Existence, uniqueness and continuous dependence of solution to random delay differential equation of fractional order

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Abstract

In this paper, we aim to prove the existence, uniqueness of the solution to the random delay differential equation of fractional order involving the successive approximation method. Moreover, using the Gronwall inequality, we study the continuous dependence of solution in the mean square sense of the problem. Finally, the fractional ϵ-approximate solution in the mean square sense is also considered.

Keywords: Mean square calculus; Random differential equation; Fractional Stochastic Calculus; Random fractional differential equation.
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1 Introduction and Preliminaries

Many dynamic systems can be characterized in detail through fractional differential and integral equations (FDIEs). A several models can be found in Physics [11] , Chemistry [20], Biology [13], Engineering [23], and Economics [2]. Hence the study of these equations has a widespread interest.

Researchers in these last two decades have vigorously studied the theory of the FDIEs. We can refer to the monographs [14] and the papers [1, 14, 15, 18, 19]. Unfortunately, the FDIEs may be limited because the uncertainties inherent in dynamic systems may not be mentioned. As a result, fractional random differential and integral equations (R-FDIEs) have been used more and more over the years (see [4, 6, 16, 17, 25, 12]). Therefore, it makes sense to develop a fractional calculus that considers the "randomness" of this situation.

In 2001, Hafiz et al. [9] established the theory mean square fractional calculus, which transfers from the deterministic fractional calculus to a mean square setting. The mean square fractional integration and differentiation for mean square continuous second-order stochastic processes in the sense of Caputo are introduced by themselves in [9]. Next, Hafiz [8] studied the mean square fractional integration in the sense of Riemann—Liouville for mean square integrable stochastic processes. The properties of the mean square fractional derivative in the sense of Caputo and Riemann—Liouville are also discussed.

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El-Sayed et al. [5] considered the following the delay stochastic differential equation of fractional order:

$$\begin{cases} \Upsilon'(t) = \mathcal{F}\big(t, \Upsilon(\phi_1(t)), \mathbf{D}^\beta \Upsilon(\phi_2(t))\big), \\ \Upsilon(0) = \Upsilon_0. \end{cases}$$

where $\phi_1, \phi_2 : [0,T] \to [0,T]$ are continuous functions satisfy $\phi_1(t), \phi_2(t) \le t$ and $\mathcal{F}$ satisfies Lipschitz condition. The authors proved the existence local of the mean square solution to this problem. Besides, the continuous dependence on initial data also established. Afterward, Vu and Hoa [24] showed that the technique used in [5] could be applied to yield existence results in the mean square sense for the problem as follows:

$$\begin{cases} \Upsilon'(t) = \mathcal{F}\big(t, \Upsilon(t), \mathbf{D}^\beta \Upsilon(t)\big), \quad t \in [0, t_1] \cup (0, t_2] \cup \ldots \cup (0, T], \\ \Delta \Upsilon(t_k) = I_k\big(\Upsilon(t_k)\big), \quad k = 1, 2, \ldots, \\ \Upsilon(0) = \Upsilon_0. \end{cases}$$

Using mean square random calculus, Bouros et al. [3] constructed the fractional forward Euler-like method to solve the following random fractional differential equation

$$\begin{cases} \mathbf{D}^\alpha \Upsilon(t) = \mathcal{F}(t, \Upsilon(t)), \quad t \in [0, T], \\ \Upsilon(a) = \Upsilon_0, \end{cases}$$

where $\mathcal{F}$ is mean square continuous function and satisfies Lipschitz condition. Besides, the mean square convergence of this method also proved. Base on the Maximum Entropy Principle, the authors discussed an approach to approximation the first probability density function of the mean square solution of the above problem.

Yfrah et al. [7] proved the existence and uniqueness of solution of the high order random fractional equation with nonlocal conditions in the Banach space as follows:

$$\begin{cases} \mathbf{D}^\beta \Upsilon(t) = B(t)\mathcal{F}(\Upsilon(t)) + C(t)\mathcal{G}\big(\mathbf{D}^{\beta-1}\Upsilon(t), \ldots, \mathbf{D}^{\beta-n+1}\Upsilon(t)\big) \\ \Upsilon_0 = \Upsilon(0) + \sum_{k=1}^n a_k \Upsilon(\tau_k), \\ \Upsilon_j = \Upsilon^j(0), \quad j = 1, 2, \ldots, n-1, \end{cases}$$

where $\beta \in (n-1, n]$, $n = [\beta]+1$ and $n = 0, 1, 2, \ldots$, a_k are non-negative constant, $B, C : [0,T] \to \mathbb{R}$ and $\mathcal{F} : L_2(\Omega) \to \mathbb{R}$, $\mathcal{G} : \big(L_2(\Omega)\big)^{n-1} \to \mathbb{R}$ satisfy some suitable conditions. In the proofs, the Banach's fixed point theorem is used. The continuous dependence on the initial condition of solution and high order fractional derivative dependence also discussed. Before, Slimane el al. [21] also considered this problem in cases $n = 0$.

To the best of our knowledge, the existence and unique solution of the random delay differential equation of fractional order in mean square sense is not still considered. From the above discussions, in this paper, we will consider this problem. The outcomes of our work include the following new features:

- Using the successive approximation methods, we prove the existence and uniqueness of the mean square solution of the random delay differential equation of fractional order.
- The dependence of solution of the random delay differential equation of fractional order in mean square sense is considered.
- We establish the $\epsilon-$solution of the random delay differential equation of fractional order in mean square sense.

Next, we present some important theorems, definitions, and notations related to the mean square calculus of the stochastic process, which will be used throughout this paper.

The triplet $(\Omega, F, \mathbb{P})$ will denote a complete probability space. A random variable $\Upsilon(t) = \big\{\Upsilon(t, \omega) \,|\, t \in [0, a], \omega \in \Omega\big\}$ is called a second order random variable, if

$$\mathbf{E}[\Upsilon^2(t)] := \int_\Omega \Upsilon^2 d\mathbb{P} < \infty,$$

where $\mathbf{E}[\cdot]$ is the expectation operator. If $\Upsilon(t)$ is a second order random variable, then $\Upsilon(t)$ is termed a second order stochastic process.

The set $L_2(\Omega) = \{\Upsilon : \Omega \to \mathbb{R} : E(\Upsilon^2(t)) < \infty\}$ of all the second order random variables endowed with the norm

$$\|\Upsilon(t)\|_2 = \sqrt{\mathbf{E}[\Upsilon^2(t)]}.$$

It is easy to see that $L_2(\Omega)$ is a Banach space.

Let $\{\Upsilon_m\}_{m\geq 0}$ be a sequence in $L_2(\Omega)$. We say that $\{\Upsilon_m\}_{m\geq 0}$ is converges in the mean square to $\Upsilon \in L_2(\Omega)$, if

$$\lim_{m\to\infty} \left\|\Upsilon_m - \Upsilon\right\|_2 = 0.$$

Let $C := C([0,a], L_2(\Omega))$ denotes the space of all second order stochastic processes, which are mean square Riemann integrable on $[0,a]$, i.e.

$$\int_{[0,a]} \mathbf{E}[\Upsilon^2(t)]dt < \infty.$$

Denote $C([0,a], L_2(\Omega))$ as the Banach space of all mean square continuous functions from $[0,a]\times\Omega$ into $\mathbb{R}$ with the norm

$$\|\Upsilon\|_C = \max_{t\in[0,a]} \|\Upsilon(t)\|_2.$$

Definition 1.1. ([22]) Let $\Upsilon(t)$ be a second order stochastic process. We say that $\Upsilon(t)$ has a mean square derivative at t, denoted by $\Upsilon'(t)$, if

$$\lim_{h\to 0} \left\| \frac{\Upsilon(t+h)-\Upsilon(t)}{h} - \Upsilon'(t) \right\|_2 = 0.$$

Theorem 1.2. ([8]) Let $\Upsilon(t)$ be a second order stochastic process. The stochastic mean square Riemann–Liouville fractional integral of $\Upsilon(t)$, denoted by $\mathbf{I}^{\beta}_{0+}\Upsilon(t)$, of order $\beta \in (0,1]$ is defined by

$$\mathbf{I}^{\beta}_{0+}\Upsilon(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\Upsilon(s)ds,$$

where Γ is the Gamma function.

Definition 1.3. ([8]) Let $\Upsilon(t)$ be a second order stochastic process. The stochastic mean square Caputo fractional derivative of $\Upsilon(t)$, denoted by $\mathbf{D}^{\beta}_{0+}\Upsilon(t)$, of order $\beta \in (0,1]$ is defined by

$$\mathbf{D}^{\beta}_{0+}\Upsilon(t) := I^{1-\beta}_{0+}\frac{d}{dt}\Upsilon(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\Upsilon'(s)ds,$$

where $\Upsilon'(t)$ denotes the mean square derivative of $\Upsilon(t)$.

Theorem 1.4. ([8]) Let $\beta > 0$ and $t \in [0,a]$. If stochastic process $\Upsilon(t)$ is mean square differentiable with mean square integrable second order derivative, then

i) $\mathbf{I}^{\beta}_{0+}\mathbf{D}^{\beta}_{0+}\Upsilon(t) = \Upsilon(t) - \Upsilon(0)$;

ii) $\mathbf{D}^{\beta}_{0+}\mathbf{I}^{\beta}_{0+}\Upsilon(t) = \Upsilon(t)$.

Lemma 1.5. ([10], Lemma 7.1.1) Let $a(t)$ be continuous function and $b(t)$ is a positive, integrable function on $[0,T]$. Assume that there is constant $c > 0$ such that

$$a(t) \leq b(t) + c\int_0^t (t-s)^{\beta-1}a(s)ds, \quad \beta \in (0,1].$$

Then, there exists a constant $K = K_\beta$ such that

$$a(t) \leq b(t) + K_\beta c\int_0^t (t-s)^{\beta-1}b(s)ds, \quad \forall t \in [0,T].$$

2 Existence and Uniqueness of solution

Given $\sigma > 0$, we denote by $C_\sigma := C([-\sigma, 0], L_2(\Omega))$ is the Banach space of the mean square continuous functions from $[-\tau, 0]$ into $L_2(\Omega)$ with the distance metric as follows:

$$\|\theta\|_\sigma = \max_{t\in[-\tau,0]} \|\theta(t)\|_2.$$

Let $\theta \in C\big([-\tau, a], L_2(\Omega)\big)$. For any $t \in [0, a]$, then we denote by θ_t is an element of C_σ, given by $\theta_t(s) = \theta(t + s)$, for any $t \in [-\sigma, 0]$.

We consider the following random delay differential equation of fractional order:

$$\begin{cases} \mathbf{D}^\beta_{0+}\Phi(t) = \mathcal{F}\big(t, \Phi_t\big), & t \in [0, a], \\ \Phi(t) = \hat{\Phi}(t), & t \in [-\sigma, 0], \end{cases} \tag{2.1}$$

where $\mathbf{D}^\beta_{0+}\Phi(t)$ is stochastic mean square Caputo fractional derivative of $\Phi(t)$ of order $\beta \in (0, 1]$; $\mathcal{F} : [0, a]\times C_\sigma \to L_2(\Omega)$ is mean square continuous on $[0, a]$ and $\hat{\Phi} : [-\sigma, 0] \to L_2(\Omega)$ is a random variable satisfying $E(\hat{\Phi}^2) < \infty$.

Remark 2.1. We say that the second-order stochastic process $\Phi : [-\sigma, a] \to L_2(\Omega)$ is a mean square solution of (2.1) if Φ satisties $\Phi(t) = \hat{\Phi}(t)$ for $t \in [-\sigma, 0]$ and $\mathbf{D}^\beta_{0+}\Phi(t) = \mathcal{F}\big(t, \Phi_t\big)$ for $t \in [0, a]$.

Lemma 2.2. Let $\Phi : [-\sigma, a] \to L_2(\Omega)$ be a second-order stochastic process and $\mathcal{F} : [0, a] \times C_\sigma \to L_2(\Omega)$ is mean square continuous function, then the problem (2.1) is equivalent to the random fractional integral equation as follows:

$$\Phi(t) = \begin{cases} \hat{\Phi}(t), & t \in [-\sigma, 0], \\ \hat{\Phi}(0) + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\mathcal{F}\big(s, \Phi_s\big)ds, & t \in [0, a]. \end{cases} \tag{2.2}$$

To prove the below theorems, we introduce some assumptions for the function $\mathcal{F} : [0, a] \times C_\sigma \to L_2(\Omega)$ as follows:

(A1) There exists a positive constant M such that

$$\big\|\mathcal{F}\big(t, \Phi_1\big) - \mathcal{F}\big(t, \Phi_2\big)\big\|_2 \le M\big\|\Phi_1 - \Phi_2\big\|_\sigma,$$

for any $t \in [0, a]$ and $\Phi_1, \Phi_2 \in C_\sigma$;

(A2) There exists a positive constant K such that

$$\big\|\mathcal{F}\big(t, \Phi\big)\big\|_2 \le K,$$

for any $t \in [0, a]$ and $\Phi \in C_\sigma$.

Now, we will show the existence and uniqueness of the mean square solution of the problem (2.1) by using the method of Picard's successive approximation.

Theorem 2.3. Assume that $\mathcal{F} : [0, a] \times C_\sigma \to L_2(\Omega)$ is mean square continuous function and it satisfies the assumptions (A1)–(A2). Then the problem (2.1) has unique mean square solution $\Phi : [-\sigma, a] \to L_2(\Omega)$.

Proof . Now we construct a sequence mean square continuous function $\Phi_m : [-\sigma, a] \to L_2(\Omega)$, $m = 0, 1, \dots,$ as follows: for $m = 0$

$$\Phi^0(t) = \begin{cases} \hat{\Phi}(t), & t \in [-\sigma, 0], \\ \hat{\Phi}(0), & t \in [0, a] \end{cases} \tag{2.3}$$

and for $m \ge 1$,

$$\Phi^{m+1}(t) = \begin{cases} \hat{\Phi}(t), & t \in [-\sigma, 0], \\ \hat{\Phi}(0) + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\mathcal{F}\big(s, \Phi^m_s\big)ds, & t \in [0, a]. \end{cases} \tag{2.4}$$

For $m=0$, then by the assumption (A2) and (2.3), we obtain

$$\left\|\Phi^1(t)-\Phi^0(t)\right\|_2=0, \quad \forall t\in[-\sigma,0]$$

and

$$\begin{aligned}\left\|\Phi^1(t)-\Phi^0(t)\right\|_2 &\le \left\|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mathcal{F}\big(s,\Phi_s^0\big)ds\right\|_2\\ &\le \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\left\|\mathcal{F}\big(s,\Phi_s^0\big)\right\|_2 ds \le \frac{Kt^\beta}{\Gamma(1+\beta)}, \quad \forall t\in[0,a]. \end{aligned} \tag{2.5}$$

For $m\ge 1$. Let us suppose that

$$\left\|\Phi^m(t)-\Phi^0(t)\right\|_2 \le \frac{Kt^\beta}{\Gamma(1+\beta)}, \quad \forall t\in[0,a].$$

This yields that $\mathcal{F}\big(s,\Phi_s^m\big)$ is defined on $[0,a]$, and since $\mathcal{F}\big(s,\Phi_s^m\big)$ is mean square continuous function on $[0,a]$, we have

$$\begin{aligned}\left\|\Phi^{m+1}(t)-\Phi^0(t)\right\|_2 &\le \left\|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mathcal{F}\big(s,\Phi_s^m\big)ds\right\|_2\\ &\le \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\left\|\mathcal{F}\big(s,\Phi_s^m\big)\right\|_2 ds \le \frac{Kt^\beta}{\Gamma(1+\beta)}, \quad \forall t\in[0,a]. \end{aligned} \tag{2.6}$$

This mean that (2.3) and (2.4) are well-defined.

Combining (2.4) and the assumption (A1), for $m\ge 1$ we have

$$\begin{aligned}\left\|\Phi^{m+1}(t)-\Phi^m(t)\right\|_2 &\le \left\|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mathcal{F}\big(s,\Phi_s^m\big)ds-\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mathcal{F}\big(s,\Phi_s^{m-1}\big)ds\right\|_2\\ &\le \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\left\|\mathcal{F}\big(s,\Phi_s^m\big)-\mathcal{F}\big(s,\Phi_s^{m-1}\big)\right\|_2 ds\\ &\le \frac{M}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\left\|\Phi_s^m(\cdot)-\Phi_s^{m-1}(\cdot)\right\|_\sigma ds\\ &\le \frac{M}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\max_{r\in[-\sigma,0]}\left\|\Phi^m(s+r)-\Phi^{m-1}(s+r)\right\|_2 ds\\ &\le \frac{M}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\max_{\eta\in[s-\sigma,s]}\left\|\Phi^m(\eta)-\Phi^{m-1}(\eta)\right\|_2 ds, \quad \forall t\in[0,a].\end{aligned}$$

From the estimation (2.5) and assumption (A1), one obtain

$$\begin{aligned}\left\|\Phi^2(t)-\Phi^1(t)\right\|_2 &\le \frac{M}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\max_{\eta\in[s-\sigma,s]}\left\|\Phi^2(\eta)-\Phi^1(\eta)\right\|_2 ds\\ &\le \frac{M}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\max_{\eta\in[s-\sigma,s]}\left(\frac{K\eta^\beta}{\Gamma(1+\beta)}\right)ds\\ &\le \frac{MK}{\Gamma(\beta)\Gamma(1+\beta)}\int_0^t (t-s)^{\beta-1}s^\beta ds=\frac{K}{M}\times\frac{\big(Mt^\beta\big)^2}{\Gamma(1+2\beta)}, \quad \forall t\in[0,a].\end{aligned}$$

Furthermore, if we assume that

$$\left\|\Phi^m(t)-\Phi^{m-1}(t)\right\|_2 \le \frac{K}{M}\times\frac{\big(Mt^\beta\big)^m}{\Gamma(1+m\beta)}, \quad \forall t\in[0,a],$$

then

$$\begin{aligned}\left\|\Phi^{m+1}(t)-\Phi^m(t)\right\|_2 &\le \frac{M}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\max_{\eta\in[s-\sigma,s]}\left(\frac{K}{M}\times\frac{\big(M\eta^\beta\big)^m}{\Gamma(1+m\beta)}\right)ds\\ &\le \frac{KM^m}{\Gamma(\beta)\Gamma(1+m\beta)}\int_0^t (t-s)^{\beta-1}s^{m\beta}ds\\ &= \frac{K}{M}\times\frac{\big(Mt^\beta\big)^{m+1}}{\Gamma(1+(m+1)\beta)}, \quad \forall t\in[0,a].\end{aligned} \tag{2.7}$$

By the mathematical induction, the inequality (2.7) is true for any $m \geq 0$. Therefore, the series $\sum_{m=0}^{\infty} \left\|\Phi^{m+1}(t) - \Phi^{m}(t)\right\|_2$ is uniformly convergent in mean square for any $t \in [0, a]$. That means, the sequence $\{\Phi_m\}_{m=0}^{\infty}$ is uniformly convergent in mean square for any $t \in [0, a]$. It follows that there exists a mean square continuous function $\Phi : [0, a] \to L_2(\Omega)$ such that

$$\left\|\Phi^m - \Phi\right\|_C \to 0 \quad \text{as} \quad m \to \infty.$$

From the assumption (A1) and for any $t \in [0, a]$, we imply

$$\left\|\mathcal{F}\big(s, \Phi_s^m\big) - \mathcal{F}\big(s, \Phi_s\big)\right\|_2 \leq M\left\|\Phi_s^m - \Phi_s\right\|_\sigma \leq M \max_{t\in[0,a]} \left\|\Phi_s^m(t) - \Phi_s(t)\right\|_2.$$

So, we conclude that $\left\|\mathcal{F}\big(s, \Phi_s^m\big) - \mathcal{F}\big(s, \Phi_s\big)\right\|_2$ uniformly converges in mean square sense to zero, as $m \to \infty$.

On the other hand, we have the following estimation

$$\begin{aligned}
&\left\|\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathcal{F}\big(s, \Phi_s^m\big) ds - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathcal{F}\big(s, \Phi_s\big) ds\right\|_2 \\
&\qquad \leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left\|\mathcal{F}\big(s, \Phi_s^m\big) - \mathcal{F}\big(s, \Phi_s\big)\right\|_2 ds, \quad \forall t \in [0, a].
\end{aligned}$$

From here we infer

$$\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathcal{F}\big(s, \Phi_s^m\big) ds \to \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathcal{F}\big(s, \Phi_s\big) ds, \tag{2.8}$$

in mean square sense, for any $m \geq 0$ and $t \in [0, a]$.

Combining (2.4) and (2.8), we obtain

$$\Phi(t) = \begin{cases} \hat{\Phi}(t), & t \in [-\sigma, 0], \\ \hat{\Phi}(0) + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathcal{F}\big(s, \Phi_s\big) ds, & t \in [0, a], \end{cases}$$

and hence Φ is a mean square solution of the problem (2.1).

Finally, we prove the unique solution in the mean square of the problem (2.1). Assume that Ψ is another mean square solution of the problem (2.1).

For any $t \in [0, a]$ and by the assumption (A1), we obtain

$$\begin{aligned}
\left\|\Phi(t) - \Psi(t)\right\|_2 &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left\|\mathcal{F}\big(s, \Phi_s\big) - \mathcal{F}\big(s, \Psi_s\big)\right\|_2 ds \\
&\leq \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \max_{\eta\in[s-\sigma,s]} \left\|\Phi(\eta) - \Psi(\eta)\right\|_2 ds.
\end{aligned}$$

If we put $A(s) = \max_{\eta\in[s-\sigma,s]} \left\|\Phi(\eta) - \Psi(\eta)\right\|_2$ for any $s \in [\sigma, t]$, then by the Gronwall inequality (1.5), one get $A(t) \leq 0$ for any $t \in [0, a]$. The proof is completed. $\square$

3 Continuous Dependence of solution

In this section, in the mean square sense, we will study the dependence of solution of the problem (2.1) on the history condition and the right-hand side.

Let us consider the following two problems:

$$\begin{cases} \mathbf{D}_{0+}^{\beta} \Phi(t) = \mathcal{F}\big(t, \Phi_t\big), & t \in [0, a], \\ \Phi(t) = \hat{\Phi}(t), & t \in [-\sigma, 0], \end{cases} \tag{3.1}$$

and

$$\begin{cases} \mathbf{D}^{\beta}_{0+}\Phi^m(t) = \mathcal{F}^m\big(t, \Phi^m_t\big), & t \in [0, a], \\ \Phi^m(t) = \hat{\Phi}^m(t), & t \in [-\sigma, 0], \quad m = 0, 1, 2, \ldots, \end{cases} \tag{3.2}$$

where $\mathcal{F}, \mathcal{F}^m : [0, a] \times C_\sigma \to L_2(\Omega)$ are mean square continuous on $[0, a]$ and $\hat{\Phi}, \hat{\Phi}^m : [-\sigma, 0] \to L_2(\Omega)$ are a random variables satisfy $E(\hat{\Phi}^2) < \infty$ and $E((\hat{\Phi}^m)^2) < \infty$, respectively.

Theorem 3.1. Let $m = 0, 1, 2, \ldots$. Assume that the functions $\mathcal{F}, \mathcal{F}^m$ satisfy the assumptions (A1) and

(i) $\big\|\hat{\Phi}^m(t) - \hat{\Phi}(t)\big\|_2$ converges to zero in mean square sense, for any $t \in [-\sigma, 0]$;

(ii) $\big\|\Phi^m(0) - \hat{\Phi}(0)\big\|_2$ converges to zero in mean square sense;

(iii) $\big\|\mathcal{F}^m\big(t, \Phi^m_t\big) - \mathcal{F}\big(t, \Phi_t\big)\big\|_2$ converges to zero in mean square sense, for any $t \in [0, a]$.

Then, $\big\|\Phi^m(t) - \Phi(t)\big\|_2$ converges to zero in mean square sense, for any $t \in [-\sigma, a]$.

Proof . Let $\Phi^m(t), \Phi(t)$ be mean square solution of the problem (3.1) and (3.2), respectively. For any $t \in [-\sigma, 0]$, the assumptions (ii) and (A1), we have

$$\big\|\Phi^m(t) - \Phi(t)\big\|_2 = \big\|\hat{\Phi}^m(t) - \hat{\Phi}(t)\big\|_2 \to 0, \quad m = 0, 1, 2, \ldots, \tag{3.3}$$

and for any $t \in [0, a]$ and the assumption (iii), one get

$$\begin{aligned} \big\|\mathcal{F}^m\big(t, \Phi^m_t\big) - \mathcal{F}\big(t, \Phi_t\big)\big\|_2 &\le \big\|\mathcal{F}^m\big(t, \Phi^m_t\big) - \mathcal{F}^m\big(t, 0\big)\big\|_2 + \big\|\mathcal{F}^m\big(t, 0\big) - \mathcal{F}\big(t, 0\big)\big\|_2 \\ &\quad + \big\|\mathcal{F}\big(t, 0\big) - \mathcal{F}\big(t, \Phi_t\big)\big\|_2 \\ &\le M\big\|\Phi_t\big\|_2 + \big\|\mathcal{F}^m\big(t, 0\big) - \mathcal{F}\big(t, 0\big)\big\|_2, \quad m = 0, 1, 2, \ldots. \end{aligned} \tag{3.4}$$

Performing the calculations as in Theorem (2.3), and applying Lebesgue dominated convergence theorem and the estimation (3.4), for any $t \in [0, a]$ we obtain

$$\frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \big\|\mathcal{F}^m\big(s, \Phi_s\big) - \mathcal{F}\big(s, \Phi_s\big)\big\|_2 ds \to 0, \quad m = 0, 1, 2, \ldots. \tag{3.5}$$

Observe that

$$\begin{aligned} \big\|\Phi^m(t) - \Phi(t)\big\|_2 &\le \big\|\Phi^m(0) - \hat{\Phi}(0)\big\|_2 + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \big\|\mathcal{F}^m\big(s, \Phi^m_s\big) - \mathcal{F}\big(s, \Phi_s\big)\big\|_2 ds \\ &\le \big\|\Phi^m(0) - \hat{\Phi}(0)\big\|_2 + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \Big\{ \big\|\mathcal{F}^m\big(s, \Phi^m_s\big) - \mathcal{F}^m\big(s, \Phi_s\big)\big\|_2 \\ &\qquad + \big\|\mathcal{F}^m\big(s, \Phi_s\big) - \mathcal{F}\big(s, \Phi_s\big)\big\|_2 \Big\} ds \\ &\le \big\|\Phi^m(0) - \hat{\Phi}(0)\big\|_2 + \frac{M}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \Big\{ \max_{\eta \in [s - \sigma, s]} \big\|\Phi^m(\eta) - \Phi(\eta)\big\|_2 \\ &\qquad + \big\|\mathcal{F}^m\big(s, \Phi_s\big) - \mathcal{F}\big(s, \Phi_s\big)\big\|_2 \Big\} ds \end{aligned} \tag{3.6}$$

From the assumption (ii) and (3.6) and the Gronwall inequality (1.5), we infer that $\big\|\Phi^m(t) - \Phi(t)\big\|_2 \to 0$ in mean square sense, for any $m = 0, 1, 2, \ldots$ and $t \in [0, a]$. Together with the estimation (3.3), we can conclude that $\big\|\Phi^m(t) - \Phi(t)\big\|_2 \to 0$ in mean square sense, for any $m = 0, 1, 2, \ldots$ and $t \in [-\sigma, a]$. The proof is completed. □

Consider the following problem:

$$\mathbf{D}^{\beta}_{0+}\Phi(t) = \mathcal{F}\big(t, \Phi_t\big), \quad \forall t \in [0, a], \tag{3.7}$$

with respect to the history conditions.

Let us denote by $\Phi(\cdot; \hat{\Phi})$ the mean square solution of (3.7) with history condition $\Phi(t) = \hat{\Phi}(t)$ and $\Phi(\cdot; \hat{\Psi})$ the mean square solution of (3.7) with history condition $\Phi(t) = \hat{\Psi}(t)$.

Theorem 3.2. Assume that $\mathcal{F}$ satisfies the all assumption of Theorem (2.3). If

$$0 < \frac{Ma^\beta}{\Gamma(1+\beta)} < 1,$$

then there exists a non-negative constant $\widetilde{M}$ such that

$$\big\|\Phi(t;\hat{\Phi}) - \Phi(t;\hat{\Psi})\big\|_2 \le \widetilde{M}\big\|\hat{\Phi} - \hat{\Psi}\big\|_C, \quad \forall t \in [-\sigma, a].$$

Proof . Let $\Phi(\cdot;\hat{\Phi})$ and $\Phi(\cdot;\hat{\Psi})$ be mean square solution of the problem (3.7) with history conditions, respectively.

From Lemma 2.2, we obtain

$$\Phi(\cdot;\hat{\Phi}) = \begin{cases} \hat{\Phi}(t), & t \in [-\sigma, 0], \\ \hat{\Phi}(0) + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathcal{F}\big(s, \Phi_s^{\hat{\Phi}}\big) ds, & t \in [0, a]. \end{cases}$$

and

$$\Phi(t;\hat{\Psi}) = \begin{cases} \hat{\Psi}(t), & t \in [-\sigma, 0], \\ \hat{\Psi}(0) + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathcal{F}\big(s, \Phi_s^{\hat{\Psi}}\big) ds, & t \in [0, a]. \end{cases}$$

Observer that

$$\big\|\Phi(t;\hat{\Phi}) - \Phi(t;\hat{\Psi})\big\|_2 = \big\|\hat{\Phi}(t) - \hat{\Psi}(t)\big\|_2 \le \big\|\hat{\Phi} - \hat{\Psi}\big\|_C, \quad \forall t \in [-\sigma, 0] \tag{3.8}$$

and

$$\begin{aligned} &\big\|\Phi(t;\hat{\Phi}) - \Phi(t;\hat{\Psi})\big\|_2 \\ &\quad \le \big\|\hat{\Phi}(0) - \hat{\Psi}(0)\big\|_2 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \big\|\mathcal{F}\big(s, \Phi_s^{\hat{\Phi}}\big) - \mathcal{F}\big(s, \Phi_s^{\hat{\Psi}}\big)\big\|_2 ds \\ &\quad \le \big\|\hat{\Phi}(0) - \hat{\Psi}(0)\big\|_2 + \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \big\|\Phi_s^{\hat{\Phi}} - \Phi_s^{\hat{\Psi}}\big\|_2 ds \\ &\quad \le \big\|\hat{\Phi}(0) - \hat{\Psi}(0)\big\|_2 + \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \max_{\eta \in [s-\sigma, s]} \big\|\Phi(\eta;\hat{\Phi}) - \Phi(\eta;\hat{\Psi})\big\|_2 ds, \quad \forall t \in [0, a]. \end{aligned}$$

From this, for any $t \in [0, a]$, we get

$$\big\|\Phi(t;\hat{\Phi}) - \Phi(t;\hat{\Psi})\big\|_2 \le \big\|\hat{\Phi} - \hat{\Psi}\big\|_C + \frac{Mt^\beta}{\Gamma(1+\beta)} \big\|\Phi(t;\hat{\Phi}) - \Phi(t;\hat{\Psi})\big\|_2.$$

Since by the assumption $0 < \dfrac{Ma^\beta}{\Gamma(1+\beta)} < 1$, then we infer

$$\big\|\Phi(t;\hat{\Phi}) - \Phi(t;\hat{\Psi})\big\|_2 \le \left(1 - \frac{Ma^\beta}{\Gamma(1+\beta)}\right)^{-1} \big\|\hat{\Phi} - \hat{\Psi}\big\|_C, \quad \forall t \in [0, a]. \tag{3.9}$$

Combining (3.8) and (3.9), we can conclude that

$$\big\|\Phi(t;\hat{\Phi}) - \Phi(t;\hat{\Psi})\big\|_2 \le \widetilde{M}\big\|\hat{\Phi} - \hat{\Psi}\big\|_C, \quad \forall t \in [-\sigma, a],$$

where $\widetilde{M} = \max\left\{1; \left(1 - \dfrac{Ma^\beta}{\Gamma(1+\beta)}\right)^{-1}\right\}$. The proof is completed. □

4 $\epsilon-$ Approximate solution

In this section, we discuss the $\epsilon-$ solution of the problem (2.1) in the mean square sense.

Definition 4.1. A mean square solution of the random delay differential inequality of fractional order of the form

$$\left\|\mathbf{D}_{0^+}^{\beta}\Phi(t)-\mathcal{F}\big(t,\Phi_t\big)\right\|_2\leq\epsilon,\quad\forall t\in[0,a],$$

with the history conditions $\Phi(t)=\hat{\Phi}(t)$, $\forall t\in[-\sigma,0]$, is called a $\epsilon-$approximate solution in mean square of the problem (2.1) on $[0,a]$ with with the history conditions $\Phi(t)=\hat{\Phi}(t)$, $\forall t\in[-\sigma,0]$.

Theorem 4.2. Assume that $\mathcal{F}$ satisfies the all assumption of Theorem (2.3). Let $\Phi^{\epsilon_i}(t)$, $(i=1,2)$, be ϵ_i-approximate solution in mean square of the problem (2.1) on $t\in[0,a]$, corresponding to the history conditions $\Phi^{\epsilon_i}(t)=\hat{\Phi}^{\epsilon_i}(t)$, $\forall t\in[-\sigma,0]$. Then

$$\begin{aligned}\left\|\Phi^{\epsilon_2}(t)-\Phi^{\epsilon_1}(t)\right\|_2&\leq\big(\epsilon_1+\epsilon_2\big)\times\left(\frac{Mt^{2\beta}}{\Gamma(1+2\beta)}+\frac{t^{\beta}}{\Gamma(1+\beta)}\right)\\&\quad+\left(1+\frac{Mt^{\beta}}{\Gamma(1+\beta)}\right)\times\left\|\hat{\Phi}^{\epsilon_2}(0)-\hat{\Phi}^{\epsilon_1}(0)\right\|_2,\quad\forall t\in[0,a].\end{aligned}$$

Proof . Since $\Phi^{\epsilon_i}(t)$, $(i=1,2)$, be ϵ_i-approximate solution in mean square of the problem (2.1) on $t\in[0,a]$, then we have

$$\left\|\mathbf{D}_{0^+}^{\beta}\Phi^{\epsilon_i}(t)-\mathcal{F}\big(t,\Phi_t^{\epsilon_i}\big)\right\|_2\leq\epsilon_i,\quad\forall t\in[0,a].$$

Applying the fractional integral $\mathbf{I}_{0^+}^{\beta}(\cdot)$ on both sides of the above inequality, one obtain

$$\mathbf{I}_{0^+}^{\beta}\left\|\mathbf{D}_{0^+}^{\beta}\Phi^{\epsilon_i}(t)-\mathcal{F}\big(t,\Phi_t^{\epsilon}\big)\right\|_2\leq\mathbf{I}_{0^+}^{\beta}\epsilon_i,\quad\forall t\in[0,a].$$

Based on Lemma 2.2 and Theorem 2.3, we infer

$$\left\|\Phi^{\epsilon_i}(t)-\hat{\Phi}^{\epsilon_i}(0)-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\mathcal{F}\big(s,\Phi_s^{\epsilon_i}\big)ds\right\|_2\leq\frac{\epsilon_i t^{\beta}}{\Gamma(1+\beta)},\quad\forall t\in[0,a].$$

On the other hand, we have

$$\begin{aligned}&\left\|\Phi^{\epsilon_2}(t)-\hat{\Phi}^{\epsilon_2}(0)-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\mathcal{F}\big(s,\Phi_s^{\epsilon_2}\big)ds\right\|_2\\&+\left\|\Phi^{\epsilon_1}(t)-\hat{\Phi}^{\epsilon_1}(0)-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\mathcal{F}\big(s,\Phi_s^{\epsilon_1}\big)ds\right\|_2\leq\frac{(\epsilon_1+\epsilon_2)t^{\beta}}{\Gamma(1+\beta)},\quad\forall t\in[0,a].\end{aligned}$$

By the inequalities $|A-B|\leq|A|+|B|$ and $|A|-|B|\leq|A-B|$, we get

$$\begin{aligned}\left\|\Phi^{\epsilon_2}(t)-\Phi^{\epsilon_1}(t)\right\|_2&\leq\frac{(\epsilon_1+\epsilon_2)t^{\beta}}{\Gamma(1+\beta)}+\left\|\hat{\Phi}^{\epsilon_2}(0)-\hat{\Phi}^{\epsilon_1}(0)\right\|_2\\&\quad+\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\left\|\mathcal{F}\big(s,\Phi_s^{\epsilon_2}\big)-\mathcal{F}\big(s,\Phi_s^{\epsilon_1}\big)\right\|_2ds,\quad\forall t\in[0,a].\end{aligned}$$

Form the above estimation and the assumption (A1), we obtain

$$\begin{aligned}\left\|\Phi^{\epsilon_2}(t)-\Phi^{\epsilon_1}(t)\right\|_2&\leq\frac{(\epsilon_1+\epsilon_2)t^{\beta}}{\Gamma(1+\beta)}+\left\|\hat{\Phi}^{\epsilon_2}(0)-\hat{\Phi}^{\epsilon_1}(0)\right\|_2\\&\quad+\frac{M}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\max_{\eta\in[s-\sigma,s]}\left\|\Phi^{\epsilon_2}(\eta)-\Phi^{\epsilon_1}(\eta)\right\|_2ds,\quad\forall t\in[0,a].\end{aligned}$$

If we put $v(s)=\max_{\eta\in[s-\sigma,s]}\left\|\Phi^{\epsilon_2}(\eta)-\Phi^{\epsilon_1}(\eta)\right\|_2$ for any $s\in[\sigma,t]$ and $w(t)=\dfrac{(\epsilon_1+\epsilon_2)t^{\beta}}{\Gamma(1+\beta)}+\left\|\hat{\Phi}^{\epsilon_2}(0)-\hat{\Phi}^{\epsilon_1}(0)\right\|_2$ for any $t\in[0,a]$, then we obtain

$$v(t)\leq w(t)+\frac{M}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}v(s)ds,\quad\forall t\in[0,a].$$

By using Gronwall inequality (1.5) to the above estimation, one obtain

$$v(t) \leq w(t) + K_\beta \times \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} w(s) ds, \quad \forall t \in [0,a].$$

So we have

$$\begin{aligned} \left\| \Phi^{\epsilon_2}(t) - \Phi^{\epsilon_1}(t) \right\|_2 &\leq (\epsilon_1 + \epsilon_2) \times \left(\frac{Mt^{2\beta}}{\Gamma(1+2\beta)} + \frac{t^\beta}{\Gamma(1+\beta)} \right) \\ &\quad + \left(1 + \frac{Mt^\beta}{\Gamma(1+\beta)} \right) \times \left\| \hat{\Phi}^{\epsilon_2}(0) - \hat{\Phi}^{\epsilon_1}(0) \right\|_2, \quad \forall t \in [0,a]. \end{aligned}$$

□

5 Example

Let us consider the following random delay differential equation of fractional order:

$$\begin{cases} \mathbf{D}_{0^+}^{1/2} \Phi(t) = \dfrac{\Phi(t-1)}{9 + \Phi(t-1)}, & t \in [0,1], \\ \Phi(t) = 1, & t \in [-1,0]. \end{cases} \tag{5.1}$$

We see

$$\mathcal{F}(t, \Phi(t)) = \frac{\Phi(t-1)}{1 + \Phi(t-1)}, \quad \forall t \in [0,1],$$
$$\Phi(t) = 1, \quad \forall t \in [-1,0].$$

It is easy to see that $\mathcal{F}$ satisfies Lipschitz condition with $L = \dfrac{1}{81}$. Indeed, for any $\Phi_1, \Phi_2 \in C([-1,1], L_2(\Omega))$ we have

$$\begin{aligned} \left\| \mathcal{F}(t, \Phi_2(t)) - \mathcal{F}(t, \Phi_1(t)) \right\|_2 &= \left\| \frac{\Phi_2(t-1)}{9 + \Phi_2(t-1)} - \frac{\Phi_1(t-1)}{9 + \Phi_1(t-1)} \right\|_2 \\ &\leq \left\| \frac{\Phi_2}{9 + \Phi_2} - \frac{\Phi_1}{9 + \Phi_1} \right\|_C = \frac{\left\| \Phi_2 - \Phi_1 \right\|_C}{\left\| (9 + \Phi_1)(9 + \Phi_2) \right\|_C} \\ &\leq \frac{1}{81} \left\| \Phi_2 - \Phi_1 \right\|_C. \end{aligned}$$

Moreover, for any $\Phi \in C([-1,1], L_2(\Omega))$ we obtain

$$\left\| \mathcal{F}(t, \Phi(t)) \right\|_2 = \left\| \frac{\Phi(t-1)}{9 + \Phi(t-1)} \right\|_2 \leq \left\| \frac{\Phi}{9 + \Phi} \right\|_C \leq \frac{1}{9}.$$

All the assumptions of Theorem 2.3 are satisfied. It follows that the problem (5.1) has a unique mean square solution on $[-1,1]$.

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