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$H - \mu_e^*$-essential-supplemented modules

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Abstract

Let R be a ring and M be a unital left R-module. We define μ^*-essential extension relation on the set of submodules of M and investigate its properties. Moreover, we define H-μ^*-essential-supplemented on M and investigate the relations between M and direct summand of its submodules.

Keywords: μ^*-essential–relation, $H - \mu^*$-essential–supplemented, completely $H - \mu^*$-essential–supplemented, μ^* co–essential submodule, μ^* co–closed submodule
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1 Introduction

In this research, the rings are with identity and all the modules are unital left R-modules, where R denoted such a "ring" and M denotes such a module. A sub-module L of R-module M is called "small" sub.module of M, if $M = L + K$ for any sub.module K of M, implies that $M = K$, it is written as $(L \ll M)$, See [2]. M is said to be μ^*-essential extension to L or L is "μ^*-essential" sub.modul of M if any non-zero singular submodule K of M, $L \cap K \neq 0$, denoted by $(L \leq_{\mu_e^*} M)$ [3]. This concept leads as to introduce the "μ^*-essential small" a submodule L of M is called "μ^*-essential small denoted as $(L \ll_{\mu_e^*} M)$, if whenever $M = L + K$ and L. is μ^*-essential –submodule of M implies $M = K$ [4]. M is called μ^*-essential –lifting module if for every submodule A of M there exists a direct summand submodule D of M such that $M = D \bigoplus D'$, $D', \leq M$ and $A \cap D'$, $\ll_{\mu e}^* D'$ [6]. For R-module M we define μ^*-essential relation on the set of submodules of M as follows: $A\ \mu^*\ B$ if $\frac{A+B}{A} \ll_{\mu e}^* \frac{M}{A}$ and $\frac{A+B}{B} \ll_{\mu e}^* \frac{M}{B}$. Let X and A be submodules of M such that $X \leq A \leq M$, then X is called μ^* co–essential sub.module of A in M (briefly $X \leq_{\mu ce}^* A$ in M) if $\frac{A}{X} \ll_{\mu^* e} \frac{M}{X}$, T is called μ_e^*– co-closed –essential sub-module of L in M (denoted by $T \leq_{\mu_{cc}^*} L$ in M), if $\frac{L}{M} \ll_{\mu_e^*} \frac{M}{T}$ implies $T = L$ [6]. We will mentioned the most important characteristic that related to the research. We will use all of these concepts to introduce "$H - \mu^*$-essential–supplemented modules" and touching to the most important and prominent propositions in this topic, and we set a condition that make μ^*-essential– lifting modules and $H - \mu^*$-essential– supplemented modules equivalent. We give the main properties of this concept and the necessary condition that make the direct summand and infinite sum of $H - \mu^*$-essential– supplemented modules are $H - \mu^*$-essential– supplemented modules.

2 μ^*-essential -relation

Definition 2.1. Let M be an R-module we define a μ^*-essential relation on the set of submodules of M as follows: $A\ \mu^*\ B$ if $\frac{A+B}{A} \ll_{\mu e}^* \frac{M}{A}$ and $\frac{A+B}{B} \ll_{\mu e}^* \frac{M}{B}$.

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Lemma 2.2. μ^*-essential is an equivalent relation:

Proof . Clearly that μ^* is reflexive and symmetric. To show that μ^* is transitive, let A, B and C be a submodules of M such that $A\mu^*B$, and $B\mu^*C$, then $\frac{A+B}{A} \ll^*_{\mu e} \frac{M}{A}$ and $\frac{A+B}{B} \ll^*_{\mu e} \frac{M}{B}$, also $\frac{B+C}{B} \ll^*_{\mu e} \frac{M}{B}$ and $\frac{B+C}{C} \ll^*_{\mu e} \frac{M}{C}$. Let $\frac{U}{A}$ be a μ^*-essential submodule of M containing A, such that $\frac{M}{A} = \frac{U}{A} + \frac{C+A}{A}$, $\frac{U}{A}$ is μ^*-essential submodule by [6], then $M = A + C + U = C + U$ and hence $\frac{M}{B} = \frac{C+U}{B} = \frac{U+B}{B} + \frac{C+B}{B}$, $\frac{U+B}{B}$, is μ^*-essential submodule by [6], and $\frac{C+B}{B} \ll^*_{\mu e} \frac{M}{B}$, then $\frac{M}{B} = \frac{U+B}{B}$. Hence $M = U + B$and$\frac{M}{A} = \frac{U}{A} + \frac{A+B}{A}$, but $\frac{A+B}{A} \ll^*_{\mu e} \frac{M}{A}$ therefore $M = U$ which mean that $\frac{C+A}{A} \ll^*_{\mu e} \frac{M}{A}$ similarly $\frac{C+A}{C} \ll^*_{\mu e} \frac{M}{C}$, then $A\mu^*B$. □

Example 2.3.

1. Let A and B be a submodules of an R-module M such that $A \leq B$, then $A\mu^*B$ if and only if $A \leq^*_{\mu ce} B$ in M, for example Z_8 as a Z-module, it is easy to see that $\{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}\mu^*\{\{\overline{0}, \overline{4}\}\}$, where $\{\{\overline{0}, \overline{4}\}\} \leq^*_{\mu ce} \{\{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}\}$.
2. Z_12 is a Z-module, $< \overline{2} > \mu^*$, $< \overline{6} >$ and $< \overline{6} > \mu^* < \overline{2} >$ are Z–modules, but $< \overline{3} >$ is not $\mu^* < \overline{4} >$, and $< \overline{4} >$, is not $\mu^* < \overline{3} >$.
3. Consider Z as a Z-module. Let $A = 6Z$, $B = 4Z$. One can easily to show that A has a relation with B by μ^*.
4. Let A be a submodule of an R-module M, then $A\ \mu^*0$ if and only if $A \ll^*_{\mu e} M$.

The following definition appeared in [6]:

Definition 2.4. Let M be an R-module and let X and A be a submodules of M such that $X \leq A \leq M$, then X is called μ^* co–essential sub.module of A in M (briefly $X \leq^*_{\mu ce} A$ in M) if $\frac{A}{X} \ll_{\mu^* e} \frac{M}{X}$.

The following theorem gives a characterization of the relation μ^*:

Theorem 2.5. Let A, B be a submodule of an R-module M. The following statements are equivalent:

1. $A\mu^*B$.
2. $A \leq^*_{\mu ce} A + B$, in M and $B \leq^*_{\mu ce} A + B$ in M.
3. For each submodule X of M such that $M = A + B + X$, X is μ^*-essential, then $M = A + X$ and $M = B + X$.
4. If $M = K + A$, for any submodule K of M such that K is μ^*-essential submodule, then $M = K + B$ and if $M = B + L$, for any submodule L of M such that L is μ^*-essential submodule, then $M = A + L$.

Proof . $(1 \to 2)$: Clearly holds.

$(2 \to 3)$: Assume that $A \leq^*_{\mu ce} A + B$ in M and $B \leq^*_{\mu ce} A + B$ in M, let X be a μ^*-essential submodule of M such that $M = A + B + X$, $X \leq M$, then $\frac{M}{A} = \frac{A+B}{A} + \frac{X+B}{A}$, $\frac{X+A}{A}$ is μ^*-essential submodule by [3], but $A \leq^*_{\mu ce} A + B$ in M, therefore $M = A + X$. Similarly $M = B + X$.

$(3 \to 4)$: Let K be a submodule of M such that $M = A + K$, K is a μ^*–essential submodule, then $M = A + B + K$, by (3) $M = B + K$, similarly one can easily prove that the second part.

$(4 \to 1)$: To show that $\frac{A+B}{B} \ll^*_{\mu e} \frac{M}{B}$ and $\frac{A+B}{A} \ll^*_{\mu e} \frac{M}{A}$. Let U be a submodule of M containing A such that $\frac{M}{A} = \frac{A+B}{B} + \frac{U}{A}$, and $\frac{U}{A}$ is a μ^*–essential submodule, then U is μ^*-essential submodule of M by [3], so $M = A + B + U = B + U$ by (4) $M = A + U = U$, hence $\frac{A+B}{A} \ll^*_{\mu e} \frac{M}{A}$ similarly $\frac{A+B}{B} \ll^*_{\mu e} \frac{M}{B}$. □

Corollary 2.6. Let A and B be a submodules of an R-module M such that $A \leq B + K$, and $B \leq A + L$, where K, X are μ^*–essential small submodules of M, then $A\ \mu^*B$.

Proof . Let $M = A + B + X$, X be a μ^*-essential, for some submodule X of M, then $M = B + K + X$ and $\frac{M}{B+X}$ a μ^*-essential. Since $K \leq^*_{\mu ce} M$, $M = B + X$, similarly $M = A + X$. Thus by (3) $A\ \mu^*B$. □

Let A, B and K be submodules of M such that $M = A + K = B + K$, but A is not related with B, by μ^*-essential for example; consider Z as a Z module and let $K = 3K$, $A = 2Z$, $B = 5Z$. Clearly $Z = 2Z + 3Z = 5Z + 3Z$, but $2Z$ is not related to $5Z$.

Proposition 2.7. Let M be an R-module and let A, B and C be submodules of M then:

1. If $A\ \mu^*B$,then $A \ll^*_{\mu e} M$ if and only if $B \ll^*_{\mu e} M$.

2. If $C \ll_{\mu e}^* M$ and $A \leq B + C$, then $A\ \mu^* B$.

Proof .

1. Assume that $A\ \mu^* B$ and $A \ll_{\mu e}^* M$. Let U be a submodule of M such that $M = B + U$, U is aμ^*–essential submodule of M, since $A\ \mu^* B$, $M = A + U$ by (theorem 2.5), but $A \ll_{\mu e}^* M$, therefore $M = U$. Hence $B \ll_{\mu e}^* M$. The converse is clear.
2. Let $M = A + X$, X is μ^*–essential submodule of M, then $M = A + B + C + X = B + C + X$, but $C \ll_{\mu e}^* M$, and $B + X$ is μ^*–essential, therefore $M = B + X$, similarly if $M = B + L$, for some submodule L of M, L is μ^*–essential, then $M = A + L$. Thus $A\ \mu^* B$.

□

Proposition 2.8. Let $M = D \bigoplus D'$, and let A, B be a submodule of D, then $A\ \mu^* B$ in M if and only if $A\ \mu^* B$ in D.

Proof . Suppose that $A\ \mu^* B$ in M and let $D = A + B + X$, X is μ^*-essential submodule of M, then $M = D \bigoplus D'$, $= A + B + X \bigoplus D'$, $X + D'$ is μ^*-essential, but $A\ \mu^* B$ in M, then $M = A + X + D = B + X + D$. Note that $D = D \cap M = D \cap (A + X + D) = A + X$, similarly $D = B + X$. Thus $A\ \mu^* B$ in D. For the converse assume that $A\ \mu^* B$ in D, then $\frac{A+B}{A} \ll_{\mu e}^* \frac{D}{A}$ and $\frac{A+B}{B} \ll_{\mu e}^* \frac{D}{A}$. Hence $\frac{A+B}{A} \ll_{\mu e}^* \frac{M}{A}$ and $\frac{A+B}{B} \ll_{\mu e}^* \frac{M}{B}$ by [7]. □

Proposition 2.9. Let M be an R-module, and let A, B be a submodules of M, then $A\ \mu^* B$ if and only if $\frac{A}{L}\mu^*\frac{B}{L}$, for every submodules L of M contained in A and B.

Proof .($\Longleftarrow$) Suppose that $\frac{A}{L}\mu^* B\frac{B}{L}$, for every L of M contained in A and B, then $\frac{A}{L} \leq_{\mu ce}^* \frac{A}{L} + \frac{B}{L} = \frac{A+B}{L}$ in $\frac{M}{L}$ and $\frac{B}{L} \leq_{\mu ce}^* \frac{A}{L} + \frac{B}{L} = \frac{A+B}{L}$ in $\frac{M}{L}$ by [6] $A \leq_{\mu ce}^* A + B$ in M $B \leq_{\mu ce}^* A + B$ in M. Thus $A\ \mu^* B$ by (theorem 2.5). □

Proof .($\Longrightarrow$) Suppose that $A\ \mu^* B$, and let L be a submodule of M contained in A and B, then by 2.5 $A \leq_{\mu ce}^* A + B$ in M and $B \leq_{\mu ce}^* A + B$ in M. By [6] $\frac{A}{L} \leq_{\mu ce}^* \frac{A}{L} + \frac{B}{L} = \frac{A+B}{L}$ in $\frac{M}{L}$ and $\frac{B}{L} \leq_{\mu ce}^* \frac{A}{L} + \frac{B}{L} = \frac{A+B}{L}$ in $\frac{M}{L}$. Thus $\frac{A}{L}\mu^*\frac{B}{L}$. □

Proposition 2.10. Let A_1, $A_2 B_1$ and B_2 be a submodules of an R-module M such that $A_1\mu^* B_1$ and $A_2\mu^* B_2$, then $(A_1 + A_2)\mu^*(B_1 + B_2)$.

Proof . Assume that $A_1\mu^* B_1$ and $A_2\mu^* B_2$. Then $A_1 \leq_{\mu ce}^* A_1 + B_1$ in M, $A_2 \leq_{\mu ce}^* A_2 + B_2$ in M, $B_1 \leq_{\mu ce}^* A_1 + B_1$ in M and $B_2 \leq_{\mu ce}^* A_2 + B_2$ in M. So $(A_1 + A_2) \leq_{\mu ce}^* (A_1 + A_2) + (B_1 + B_2)$ in M and $(B_1 + B_2) \leq_{\mu ce}^* (A_1 + A_2) + (B_1 + B_2)$ in M, by theorem 2.5. Thus $(A_1 + A_2)\mu^*(B_1 + B_2)$. □

By induction, one can easily prove the following corollary.

Corollary 2.11. Let A, B_1, B_2, $B_3, ..., B_n$ be submodules of a module M if $A\ \mu^* B_i$, for all $i = 1, 2, ..., n$. Then $A\ \mu^* B$, where $B = \sum_{i=1}^{n} B_i$.

Corollary 2.12. Let M be an R-module, if $A\ \mu^* B$ and C is any submodule of M, then $(A + C)\mu^*(B + C)$. The converse is true when $C \ll_{\mu e}^* M$.

Proof . Assume that $A\ \mu^* B$, since $C\mu^* C$, by proposition 2.10, we have $(A + C)\mu^*(B + C)$. Conversely assume that $C \ll_{\mu e}^* M$, and $(A + C)\mu^*(B + C)$, then $A + C \leq_{\mu ce}^* A + B + C$ in M, and $B + C \leq_{\mu ce}^* A + B + C$ in M by (theorem 2.5), since $C \ll_{\mu e}^* M$, $A \leq_{\mu ce}^* A + B$ in M and $B \leq_{\mu ce}^* A + B$ in M. By [6]. Thus, by theorem 2.5, we have $A\ \mu^* B$. □

Proposition 2.13. Let $f : M \longrightarrow M'$ be an R-epimorphism module, If A, B are submodules of M such that $A\ \mu^* B$, then $f(A)\mu^* f(B)$.

Proof . Suppose that $f(A)\mu^* f(B)$, then $A \leq_{\mu ce}^* A + B$ in M and $B \leq_{\mu ce}^* A + B$ in M, hence $f(A) \leq_{\mu ce}^* f(A + B) = f(A) + f(B)$ in M and $f(B) \leq_{\mu ce}^* f(A + B) = f(A) + f(B)$ in M' by [6]. Thus $f(A)\mu^* f(B)$. □

Proposition 2.14. Let $M = M_1 \bigoplus M_2$ be an R- module and let $A \leq M$, $B \leq M$, then $A\mu^* M_1$ and $B\mu^* M_2$ if and only if $(A \bigoplus B)\mu^*(M_1 \bigoplus M_2)$.

Proof . ($\Longrightarrow$) by proposition 2.10. □

Proof . ($\Longleftarrow$) Let $P_1 : M \longrightarrow M_1$ and $P_2 : M \longrightarrow M_2$ be the projection homomorphisims on M_1 and M_2 respectively, since $(A \bigoplus B)\mu^*(M_2 \bigoplus M_2)$ and $A\mu^* M_1$, by proposition 2.13, we have $P_1(A \bigoplus B)\mu^*(P(M_1 \bigoplus M_2)$. Since $B\mu^* M_2$, $P_1(A \bigoplus B)\mu^* P_2(M_1 \bigoplus M_2)$. Thus we get the result. □

3 $H - \mu^*$-essential -supplemented module

By using the concept of μ^*-essential- relation on the set of submodules of M we define the following:

Definition 3.1. Let M be an R- module, M is said to be $H - \mu^*$-essential -supplemented if every submodule A of M there exists a direct summand D of M such that $A\mu^*D$.

Example 3.2.

1. Z_4 as Z-module is $H - \mu^*$-essential–supplemented.
2. Z as Z–module is not $H - \mu^*$-essential–supplemented.
3. Z_6 as Z_6-module is $H - \mu^*$-essential–supplemented.
4. Z_12 as Z_12 is $H - \mu^*$-essential $-$ supplemented.
5. Its easy to show that Q as Z-module is not $H - \mu^*$-essential– supplemented, since the only direct summand submodules of Q is Q and $\{0\}$.
6. $H - \mu^*$-essential– supplemented modules is closed under isomorphisim.
7. Every μ^*-essential-lifting module is $H - \mu^*$-essential–supplemented to show that

Proof . Let A be a submodule of M, since M is μ^*-essential-lifting module, there exists a direct summand D of M such that $M = D \bigoplus D'$, $D \leq A$, $D' \leq M$. And $A \cap D' \ll^*_{\mu e} M$. $A = A \cap M = A \cap (D \bigoplus D') = D \bigoplus (A \cap D')$, by modular law. Now $\frac{A+D}{A} \cong 0 \ll^*_{\mu e} M$, and $\frac{A+D}{D} \cong (A \cap D') \ll^*_{\mu e} M$, Hence $A\mu^*D$, then M is $H - \mu^*$-essential–supplemented module. □

The converse is not true in general for Examples:

Example 3.3. Consider the Z- module $M = Z_2 \bigoplus Z_8$. The submodules of M are:

$A_1 = \{(\bar{0}, \bar{0}), (\bar{1}, \bar{0}), (\bar{2}, \bar{0}), (\bar{3}, \bar{0}), (\bar{4}, \bar{0}), (\bar{5}, \bar{0}), (\bar{6}, \bar{0}), (\bar{7}, \bar{0})\}$.

$A_2 = \{(\bar{0}, \bar{0}), (\bar{2}, \bar{0}), (\bar{4}, \bar{0}), (\bar{6}, \bar{0})\}$.

$A_3 = \{(\bar{0}, \bar{0}), (\bar{4}, \bar{0})\}$.

$A_4 = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{1})\}$.

$A_5 = \{(\bar{0}, \bar{0}), (\bar{1}, \bar{1}), (\bar{2}, \bar{0}), (\bar{3}, \bar{1}), (\bar{4}, \bar{0}), (\bar{5}, \bar{1}), (\bar{6}, \bar{0}), (\bar{7}, \bar{1})\}$.

$A_6 = \{(\bar{0}, \bar{0}), (\bar{2}, \bar{1}), (\bar{4}, \bar{0}), (\bar{6}, \bar{1})\}$.

$A_7 = \{(\bar{0}, \bar{0}), (\bar{4}, \bar{1})\}$.

$A_8 = \{(\bar{0}, \bar{0}), (\bar{2}, \bar{0}), (\bar{4}, \bar{0}), (\bar{6}, \bar{0}), (\bar{2}, \bar{1}), (\bar{4}, \bar{1}), (\bar{6}, \bar{1}), (\bar{0}, \bar{1})\}$.

$A_9 = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{1}), (\bar{4}, \bar{0}), (\bar{4}, \bar{1})\}$.

$A_{10} = \{(\bar{0}, \bar{0})\}$.

$A_{11} = M$

Clearly, $M = A_1 \bigoplus A_4 = A_1 \bigoplus A_7 = A_4 \bigoplus A_5$ and the μ^*-essential-small submodules of M are A_2 and A_3. It enough to check that A_6, A_8, and A_9 satisfy the definition. For A_6, the only submodules A of M satisfy $A_6 + A = M$ is A_1. Since A_1 is a direct summand of M, $A_6\mu^*A_4$ and $A_6\mu^*A_7$. For A_8, since A_1 and A_5 are satisfy $M = A_8 + A_1 = A_8 + A_5$ and booth is a direct summand of M, $A_8\mu^*A_4$, by the same argument one can see that $A_9\mu^*A_4$. Thus M is $H - \mu^*$-essential–supplemented module. But not μ^*-lifting to show that consider the submodule A_6, the only direct summand of M in A_6 is $\{0\}$, then $A_6 \cap M = A_6$ is not small in M. Hence M is not μ^*-lifting.

We say the submodule A of an R-module M is a μ^*-essential-co–closed submodule of M denoted by $A \leq^*_{\mu cc} M$, if whenever $X \leq^*_{\mu ce} A$ in M for some X of A, implies that $X = A$ [6].

Lemma 3.4. Let M be an R- module. The following statement are equivalent:

1. Every submodule of M, has a unique μ^*-essential-co- closed
2. Given a submodule A of M, then there exists a μ^*-essential-co- closed A' of A such that $A' \leq B$ where $B \leq^*_{\mu ce} A$ in M.

Proof . $(1 \Longrightarrow 2)$: Let A be a submodule of M, by (1) A has a unique μ^*-essential-co-closed say A', hence $A' \leq^*_{\mu ce} A$ in M and $A \leq^*_{\mu ce} A'$, let B be a submodule of M such that $B \leq^*_{\mu ce} A$ in M and let B' be a μ^*-essential-co- closed of

B, hence $B' \leq^*_{\mu ce} B$ in M, and $B' \leq^*_{\mu ce} M$, so $B' \leq^*_{\mu ce} A$ in M by [6], hence B' is a μ^*-essential-co- closed of A by (1) we get $A'B' \leq B$. □

Proof . ($2 \Longrightarrow 1$): Let A be a submodule of M and assume that A has a μ^*-essential-co-closed B and C in M, hence $B \leq^*_{\mu ce} A$ in M, and $C \leq^*_{\mu ce} A$ in M and B, C are μ^*-essential-co-closed submodule of M, to show that $B = C$, by (2) we have $B \leq C$. Since $B \leq^*_{\mu ce} A$ in M, $B \leq^*_{\mu ce} C$ in M, but $C \leq^*_{\mu ce} A$. Therefore $B = C$. □

The following proposition gives a condition under which μ^*-essential–lifting modules and $H-\mu^*$-essential–supplemented modules be equivalent:

Proposition 3.5. Let M be an R-module such that every submodule of M has a unique μ^*-essential-co-closed. M is μ^*-essential–lifting module if and only if M is $H - \mu^*$-essential–supplemented module.

Proof . Let M be an $H - \mu^*$-essential–supplemented module, and let A be a submodule of M then there exists a direct summand D of M such that $A\ \mu^* D$. Now D is a unique μ^*-essential-co-closed of $A + D$ in M, by lemma 3.4 $D \leq A$. Thus M is a μ^*-essential–lifting module. The converse is clear. □

Proposition 3.6. Let M be an R-module. Then the following statements are equivalent:

1. M is $H - \mu^*$-essential–supplemented module.
2. For every submodule A of M there exists a direct summand D of M such that $M = D \bigoplus D'$, $D' \leq M$, and $(A + D) \cap D' \ll^*_{\mu e} D'$.
3. For every submodule A of M, there exists a direct summand D of M such that $A + D = D \bigoplus S$, $S \ll^*_{\mu e} M$.

Proof . ($1 \Longrightarrow 2$): Assume that M is a $H - \mu^*$-essential–supplemented module, and let $A \leq M$, so there exists a direct summand D of M such that $A \mu^* D$. Let $M = D \bigoplus D'$, $D' \leq M$. To show that $(A + D) \cap D' \ll^*_{\mu e} D'$. Let $U \leq D'$ such that $[(A + D) \cap D'] + U = D'$, U is a μ^*-essential-submodule, so $M = D + D' = D + [(A + D) \cap D'] + U$ now $\frac{M}{D} \cong \frac{D+U}{D} + \frac{[(A+D)\cap D']+D}{D}$, but $D \leq [(A + D) \cap D'] + D \leq A + D$, and $D \leq^*_{\mu ce} A + D$ in M. Therefore $D \leq^*_{\mu ce} [(A + D) \cap D'] + D$ in M. By [6], and $M = D + U$, $D \cap U \leq D \cap D' = 0$, then $D \cap U = 0$. Hence $M = D \bigoplus U$. So $U = D'$. Thus $[(A + D) \cap D'] \ll^*_{\mu e} D'$. □

Proof . ($2 \Longrightarrow 3$): Let A be a submodule of M, by (2) there exists a direct summand D of M such that $M = D \bigoplus D'$, $D' \leq M$ and $[(A + D) \cap D'] \ll^*_{\mu e} D'$. Now $A + D = (A + D) \cap M = (A + D) \cap (D \bigoplus D') = D \bigoplus [(A + D) \cap D']$, $(A + D) \cap D' \ll^*_{\mu e} D'$. □

Proof . ($3 \Longrightarrow 1$): Let A be a submodule of M, by (3) there exists a direct summand D of M such that $A + D = D \bigoplus S$, $S \ll^*_{\mu e} M$. Let $\frac{M}{D} = \frac{A+D}{D} \frac{U}{D}$, $\frac{U}{D}$ be a μ^*-essential-submodule and by [3], U is μ^*-essential-submodule. Now $M = A + D + U = D + S + U = S + U = U$, hence $\frac{A+D}{A} \ll^*_{\mu e} \frac{M}{D}$. Similarly. One can show that $\frac{A+D}{A} \ll^*_{\mu e} \frac{M}{A}$. Thus $A \mu^* D$. □

Corollary 3.7. Let M be an $H - \mu^*$-essential–supplemented module, then for each submodule A of M, there exists a direct summand D of M such that $M = D \bigoplus D'$, where $D' \leq M$, and $A \cap D' \ll^*_{\mu e} D'$.

Proof . Since $A \cap D' \leq (A + D) \cap D' \ll^*_{\mu e} D'$, we have $(A \cap D') \ll^*_{\mu e} D'$. □

One can easily prove the following characterization:

Proposition 3.8. Let M be an R-module. M is $H - \mu^*$-essential–supplemented module if and only if for each submodule A of M, there exists an idempotent $f \in (\text{End }(M))$ such that $A \mu^* f(M)$,

The following proposition gives another characterization of $H - \mu^*$-essential–supplemented module.

Proposition 3.9. Let M be an R-module. M is $H - \mu^*$-essential–supplemented module if and only if each submodule A of M, there exists a direct summand D of M and submodule B of M such that $A \leq^*_{\mu ce} B$, $D \leq^*_{\mu ce} B$.

Proof . suppose that M is $H - \mu^*$-essential–supplemented module, let $A \leq M$, so there exists a direct summand D of M such that $A \mu^* D$, hence $A \leq^*_{\mu ce} A + D$, and $D \leq^*_{\mu ce} A + D$ in M. Put $B = A + D$. Thus we get the result. □

Proof . Let $A \leq M$, by our assumption, there exists a direct summand D of M, and $B \leq M$ such that $A \leq^*_{\mu ce} B$ in M, and $D \leq^*_{\mu ce} B$, in M. Since $D \leq A + D \leq B$, and $D \leq^*_{\mu ce} B$ in M, $D \leq^*_{\mu ce} A + D$ in M, by [6] Similarly $A \leq^*_{\mu ce} A + D$ in M. Thus M is $H - \mu^*$-essential–supplemented module,

Recall that an R-module M is called distributive module if for all A, B and C submodules of M $A \cap (B + C) = (A \cap B) + (A \cap C)$ [1]. □

Proposition 3.10. Let M be an R-module and let A be a submodule of M. Then $\frac{M}{A}$ is $H - \mu^*$-essential–supplemented module in each of the following cases:

1. For every direct summand D of M, $\frac{D+A}{A}$ is a direct summand of $\frac{M}{A}$
2. M is distributive module.

Proof .

1. Suppose that M is an $H - \mu^*$-essential–supplemented R-module and let $\frac{X}{A}$ be a submodule of $\frac{M}{A}$, since M is $H - \mu^*$-essential–supplemented, there exists a direct summand D of M such that $M = D \bigoplus D'$, $D' \leq M$, and $X \mu^* D$, since $\frac{D+A}{A}$ is a direct summand of $\frac{M}{A}$ and $\frac{D+A}{A} \mu^* \frac{X}{A}$ by proposition 2.9. Thus $\frac{M}{A}$ is $H - \mu^*$-essential–supplemented.
2. Suppose that M is a distributive module, we use (1) to show that $\frac{M}{A}$ is $H - \mu^*$-essential–supplemented. Let D be a direct summand of M, since M is a distributive module, $\frac{D+A}{A}$ is a direct summand of $\frac{X}{A}$. So by (1) M is a $H - \mu^*$-essential–supplemented.

□

Proposition 3.11. Let M be an $H - \mu^*$-essential–supplemented R-module. If A is fully invariant submodule of M, then $\frac{M}{A}$ is $H - \mu^*$-essential–supplemented module.

Proof . Let $\frac{X}{A}$ be a submodule of $\frac{M}{A}$. Since M is $H - \mu^*$-essential–supplemented module, there is a direct summand D of M such that $X \mu^* A$, where $M = D \bigoplus D'$ and $D' \leq M$. By lemma 3.4 [5] we have $\frac{M}{A} = \frac{D+A}{A} \bigoplus \frac{D'+A}{A}$, since $X \mu^* A$, by proposition 2.9, we have $\frac{X}{A} \mu^* \frac{D+A}{A}$. Thus $\frac{M}{A}$ is $H - \mu^*$-essential–supplemented module. □

Proposition 3.12. Let $M = M_1 \bigoplus M_2$ be an R-module such that $ann(M_1) + ann(M_2)$ if M_1 and M_2 are $H - \mu^*$-essential–supplemented. Then M is $H - \mu^*$-essential–supplemented module.

Proof . Let A be a submodule of M by [2], $A = A_1 \bigoplus A_2$ where $A_1 \leq M_1$ and $A_2 \leq M_2$, since M_1 and M_2 are $H - \mu^*$-essential–supplemented modules, there is a direct summand D_1 and D_2 of M_1 and M_2 respectively such that $A_1 \mu^* D_1$ and $A_2 \mu^* D_2$ then $A = (A_1 \bigoplus A_2) \mu^* (D_1 \bigoplus D_2)$, where $(D_1 \bigoplus D_2)$ is a direct Summand of M. Thus M is a $H - \mu^*$-essential–supplemented module. □

Proposition 3.13. Let $M = M_1 \bigoplus M_2$ be a due module such that M_1 and M_2 are $H - \mu^*$-essential–supplemented module. Then M is $H - \mu^*$-essential–supplemented module.

Proof . Let $M = M_1 \bigoplus M_2$ be a due module, and let A be a submodule of M, then A is fully invariant. Hence $A = A \cap M = A \cap (M_1 \bigoplus M_2) = (A \cap M_1) \bigoplus (A \cap M_2)$, since M_1 and M_2 are $H - \mu^*$-essential–supplemented module. Then there is a direct summand D_1 and D_2 of M_1 and M_2 respectively such that $A_1 \mu^* D_1$ and $A_2 \mu^* D_2$, then $A = (A \cap M_1) \bigoplus (A \cap M_2) \mu^* (D_1 \bigoplus D_1)$. Where $(D_1 \bigoplus D_2)$ is a direct summand of M. Thus M is a $H - \mu^*$-essential–supplemented module. □

Proposition 3.14. Let $M = M_1 \bigoplus M_2$ be a distributive module such that M_1 and M_2 are $H - \mu^*$-essential–supplemented modules, then M is a $H - \mu^*$-essential–supplemented module.

Proof . Let $M = M_1 \bigoplus M_2$ be a distributive module and let A be a submodule of M. $A = A \cap M = A \cap (M_1 \bigoplus M_2) = (A \cap M_1) \bigoplus (A \cap M_2)$, since M_1 and M_2 are $H - \mu^*$-essential–supplemented module, there is a direct summand D_1 and D_2 of M_1 and M_2 respectively such that $A_1 \mu^* D_1$ and $A_2 \mu^* D_2$, then $A = (A \cap M_1) \bigoplus (A \cap M_2) \mu^* (D_1 \bigoplus D_1)$. Where $(D_1 \bigoplus D_2)$ is a direct summand of M. Thus M is a $H - \mu^*$-essential–supplemented module. □

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