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Superstability of the p-radical sine functional equation

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Abstract

In this paper, we investigate the transferred superstability of the p-radical functional equation

$$f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = f(x)f(y)$$

with respect to the sine functional quation from the Pexider type p-radical functional equation $f\left(\sqrt[p]{x^p+y^p}\right) + g\left(\sqrt[p]{x^p-y^p}\right) = \lambda \cdot h(x)k(y)$, where p is an odd positive integer and f is a complex valued function. Furthermore, the results are applied to the stability of the cosine type p-radical functional equations.

Keywords: superstability, p-radical equation, cosine functional equation, sine functional equation, Wilson equation, Kim equation.
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1 Introduction

In 1940, the stability problem of the functional equation was conjectured by Ulam [25]. Next year, this problem was affirmatively solved by Hyers [14], which is the case of additive mapping.

Thereafter, the stability of the functional equation was improved by Bourgin [9], Aoki [3], Th. M. Rassias [24], Găvruta [12], Badora[4] Badora and R. Ger[5], Baker[6]

In 1979, Baker,*et al.*[7] announced the *superstability* as the new concept as follows: If f satisfies $|f(x+y) - f(x)f(y)| \leq \epsilon$ for some fixed $\epsilon > 0$, then either f is bounded or f satisfies the exponential functional equation $f(x+y) = f(x)f(y)$.

D'Alembert [1] in 1769 (see Kannappen's book [15]) introduced the cosine functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y), \tag{C}$$

and which superstability was proved by Baker [6] in 1980.

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The cosine (d'Alembert) functional equation (C) was generalized to the following:

$$f(x+y)+f(x-y)=2f(x)g(y), \tag{W}$$
$$f(x+y)+f(x-y)=2g(x)f(y), \tag{K}$$

in which (W) is called the Wilson equation, and (K) raised by Kim was appeared in Kannappan and Kim's paper ([16]).

The superstability of the cosine (C), Wilson (W) and Kim (K) function equations were founded in Badora, Ger, Kannappan and Kim (see [8, 16, 21]).

In 2009, Eshaghi Gordji and Parviz [13] introduced the radical functional equation related to the quadratic functional equation

$$f\big(\sqrt{x^2+y^2}\big)=f(x)+f(y). \tag{R}$$

Recently, Almahalebi*et al.*[2] and Kim[17] obtained the superstability bounded by constant and bounded by the function, respectively, for the p-radical functional equations related to Wilson and Kim equation as following:

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda f(x)f(y), \tag{C_r^λ}$$
$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda f(x)g(y), \tag{W_r^λ}$$
$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda g(x)f(y). \tag{K_r^λ}$$

In 1983, Cholewa [10] investigated the superstability of the sine functional equation

$$f(x)f(y)=f\big(\frac{x+y}{2}\big)^2-f\big(\frac{x-y}{2}\big)^2, \tag{S}$$

under the condition bounded by constant.

The aim of this paper is to solve and investigate the transferred superstability of the p-radical functional equation

$$f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2-f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2=f(x)f(y) \tag{S_r}$$

with respect to the sine functional equation from the Pexider type p-radical functional equation

$$f\left(\sqrt[p]{x^p+y^p}\right)+g\left(\sqrt[p]{x^p-y^p}\right)=\lambda h(x)k(y), \tag{P_r^λ}$$

Furthermore, the results are extended to Banach spaces.

In this paper, let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_+=[0,\infty)$ and $\mathbb{C}$ be the field of complex numbers. We may assume that f is a complex valued function, ε is a nonnegative real number, $\varphi:\mathbb{R}\to\mathbb{R}_+$ is a given nonnegative function and p is an odd positive integer.

2 Superstability of the p-radical sine functional equation (S_r)

In this section, we will investigate the transferred superstability of the radical sine functional equation (S_r) with respect to the sine functional quation from the Pexider type p-radical functional equation (P_r^λ).

Theorem 2.1. Suppose that $f,g,h,k:\mathbb{R}\to\mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p+y^p}\right)+g\left(\sqrt[p]{x^p-y^p}\right)-\lambda\cdot h(x)k(y)|\le\varphi(x)\qquad\forall\, x,y\in\mathbb{R}. \tag{2.1}$$

Then, either k is bounded or h satisfies (S_r) under one of the cases $h(0)=0$ or $f(-x)=-g(x)$,

Proof . Let k be unbounded solution of the inequality (2.1). Then, there exists a sequence $\{y_n\}$ in G such that $0\neq|k(y_n)|\to\infty$ as $n\to\infty$.

Putting $y=y_n$ in (2.1) and dividing both sides by $\lambda f(y_n)$, we have

$$\left|\frac{f\left(\sqrt[p]{x^p+y_n^p}\right)+g\left(\sqrt[p]{x^p-y_n^p}\right)}{\lambda k(y_n)}-h(x)\right|\le\frac{\varphi(x)}{\lambda k(y_n)}. \tag{2.2}$$

As $n \to \infty$ in (2.2), we get

$$h(x) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x^p + y_n^p}\right) + g\left(\sqrt[p]{x^p - y_n^p}\right)}{\lambda k(y_n)} \tag{2.3}$$

for all $x \in \mathbb{R}$.

Replacing y by $\sqrt[p]{y^p + y_n^p}$ and $\sqrt[p]{-y^p + y_n^p}$ in (2.1), we obtain

$$|f\left(\sqrt[p]{x^p + (y^p + y_n^p)}\right) + g\left(\sqrt[p]{x^p - (y^p + y_n^p)}\right) - \lambda h(x)k(\sqrt[p]{y^p + y_n^p})| \le \varphi(x),$$

$$|f\left(\sqrt[p]{x^p + (-y^p + y_n^p)}\right) + g\left(\sqrt[p]{x^p - (-y^p + y_n^p)}\right) - \lambda h(x)k(\sqrt[p]{-y^p + y_n^p})| \le \varphi(x).$$

The above two inequalities summed and divided implies that

$$\begin{aligned} &|\frac{f\left(\sqrt[p]{(x^p + y^p) + y_n^p}\right) + g\left(\sqrt[p]{(x^p + y^p) - y_n^p}\right)}{\lambda k(y_n)} \\ &+ \frac{f\left(\sqrt[p]{(x^p - y^p) + y_n^p}\right) + g\left(\sqrt[p]{(x^p - y^p) - y_n^p}\right)}{\lambda k(y_n)} \\ &- \lambda h(x)\frac{k(\sqrt[p]{y^p + y_n^p}) + k(\sqrt[p]{-y^p + y_n^p})}{\lambda k(y_n)}| \le \frac{2\varphi(x)}{\lambda k(y_n)} \end{aligned} \tag{2.4}$$

for all $x, y, y_n \in \mathbb{R}$.

Letting $n \to \infty$ in (2.4), by applying (2.3), then, for every $y \in G$, there exists a limit function $L_k : \mathbb{R} \to \mathbb{C}$ such that

$$L_k(y) := \lim_{n\to\infty} \frac{k(\sqrt[p]{y^p + y_n^p}) + k(\sqrt[p]{-y^p + y_n^p})}{\lambda \cdot k(y_n)}, \quad \forall y \in \mathbb{R}. \tag{2.5}$$

And also we conclude that the function h with L_k satisfies

$$h\left(\sqrt[p]{x^p + y^p}\right) + h\left(\sqrt[p]{x^p - y^p}\right) = \lambda h(x)L_k(y). \tag{2.6}$$

Applying the case $h(0) = 0$ in (2.6), it implies that h is odd.

Putting $y = x$ in (2.6), we get the equation

$$h(\sqrt[p]{2}x) = \lambda \cdot h(x)L_k(x), \quad x \in \mathbb{R}.$$

Keeping this in mind, by means of (2.6), we infer the equality

$$\begin{aligned} h\left(\sqrt[p]{x^p + y^p}\right)^2 - h\left(\sqrt[p]{x^p - y^p}\right)^2 &= \lambda h(x)L_k(y)[h\left(\sqrt[p]{x^p + y^p}\right) - h\left(\sqrt[p]{x^p - y^p}\right)] \\ &= \lambda h(x)L_k(y)h\left(\sqrt[p]{y^p + x^p}\right) + h\left(\sqrt[p]{y^p - x^p}\right) \\ &= \lambda h(x)L_k(y)\lambda h(y)L_k(x) \\ &= h\left(\sqrt[p]{2}x\right)h\left(\sqrt[p]{2}y\right). \end{aligned} \tag{2.7}$$

Replacing x to $\frac{x}{\sqrt[p]{2}}$ and y to $\frac{y}{\sqrt[p]{2}}$in (2.7), this, in return, leads to the equation

$$h\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - h\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 = h(x)h(y)$$

valid for all $x, y \in \mathbb{R}$, which states nothing else but (S_r).

For the other case $f(-x) = -g(x)$, it is enough to show that $h(0) = 0$. Suppose that this is not the case.

Putting $x = 0$ in (2.1), due to $h(0) \neq 0$ and $f(-x) = -g(x)$, we obtain the inequality

$$|k(y)| \leq \frac{\varphi(0)}{\lambda \cdot |h(0)|}, \qquad y \in \mathbb{R}. \tag{2.8}$$

This inequality means that k is globally bounded, which is a contradiction. Thus, since the claimed $h(0) = 0$ holds, we know that h satisfies (S_r). □

Theorem 2.2. Suppose that $f, g, h, k : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot h(x)k(y)| \leq \varphi(y) \qquad \forall\, x, y \in \mathbb{R}. \tag{2.9}$$

Then, either h is bounded or k satisfies (S_r) under one of the cases $k(0) = 0$ or $g(x) = -f(x)$.

Proof . (i) Taking $x = x_n$ in the inequality (2.9), dividing both sides by $|\lambda \cdot h(x_n)|$, and passing to the limit as $n \to \infty$, we obtain that

$$k(y) = \lim_{n \to \infty} \frac{f\left(\sqrt[p]{x_n^p + y^p}\right) + g\left(\sqrt[p]{x_n^p - y^p}\right)}{\lambda h(x_n)} \tag{2.10}$$

for all $y \in \mathbb{R}$.

Replace x by $\sqrt[p]{x_n^p + x^p}$ and $\sqrt[p]{x_n^p - x^p}$ in (2.9), dividing by $\lambda \cdot h(x_n)$, then it gives us the existence of a limit function

$$l_h(x) := \lim_{n \to \infty} \frac{h(\sqrt[p]{x_n^p + x^p}) + h(\sqrt[p]{x_n^p - x^p})}{\lambda \cdot h(x_n)}, \tag{2.11}$$

where the function $l_h : \mathbb{R} \to \mathbb{C}$ satisfies the equation

$$k\left(\sqrt[p]{x^p + y^p}\right) + k\left(\sqrt[p]{-x^p + y^p}\right) = \lambda \cdot l_h(x)k(y) \qquad \forall x, y \in \mathbb{R}. \tag{2.12}$$

Applying the case $k(0) = 0$ in (2.12), it implies that k is odd.

A similar procedure to that applied after (2.6) of Theorem 2.1 in equation (2.12) allows us to show that k satisfies (S_r).

The case $g(x) = -f(x)$ is also the same reason as Theorem 2.1. □

Corollary 2.3. Suppose that $f, g, h, k : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot h(x)k(y)| \leq \begin{cases} \min\{\varphi(x), \varphi(y)\} \\ \varepsilon \end{cases} \qquad \forall\, x, y \in \mathbb{R}.$$

(a) Then, either k is bounded or h satisfies (S_r) under one of the cases $h(0) = 0$ or $f(-x) = -g(x)$.

(b) Then, either h is bounded or k satisfies (S_r) under one of the cases $k(0) = 0$ or $g(x) = -f(x)$.

Replacing and reducing for the functions f, g, h, k in (P_r^λ) arrives the functional equations ($fgfk$, $fghf$, $fghh$, $ffhk$, $fffg$,$ffgf$, $ffgg$, $ffff$). Hence, as corollaries, the stabilities of these equations follow smoothly. We will illustrate some results.

Corollary 2.4. Suppose that $f, g, k : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot f(x)k(y)| \leq \begin{cases} (i)\ \varphi(x) \\ (ii)\ \varphi(y) \end{cases} \qquad \forall\, x, y \in \mathbb{R}.$$

(i) Then, either k is bounded or f satisfies (S_r) under one of the cases $f(0) = 0$ or $f(-x) = -g(x)$.

(ii) Then, either f is bounded or k satisfies (S_r) under one of the cases $k(0) = 0$ or $g(x) = -f(x)$.

Corollary 2.5. Suppose that $f, g, k : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot h(x)f(y)| \leq \begin{cases} (i)\ \varphi(x) \\ (ii)\ \varphi(y) \end{cases} \qquad \forall\, x, y \in \mathbb{R}.$$

(i) Then, either f is bounded or h satisfies (S_r) under one of the cases $h(0) = 0$ or $f(-x) = -g(x)$.

(ii) Then, either h is bounded or f satisfies (S_r) under one of the cases $f(0) = 0$ or $g(x) = -f(x)$.

Corollary 2.6. Suppose that $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot h(x)h(y)| \leq \begin{cases} (i)\ \varphi(x) \\ (ii)\ \varphi(y) \end{cases} \qquad \forall\, x, y \in \mathbb{R}.$$

If h fails to be bounded, then, under one of the cases $h(0) = 0$ or $f(-x) = -g(x)$, h satisfies (S_r).

Corollary 2.7. Suppose that $f, g, h, k : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot h(x)k(y)| \leq \begin{cases} (i)\ \varphi(x) \\ (ii)\ \varphi(y) \end{cases} \qquad \forall\, x, y \in \mathbb{R}.$$

(i) Then, either k is bounded or h satisfies (S_r) under one of the cases $h(0) = 0$ or f is odd.

(ii) Then, either h is bounded or k satisfies (S_r) under the case $k(0) = 0$.

Corollary 2.8. Suppose that $f, g, h, k : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot f(x)g(y)| \leq \begin{cases} (i)\ \varphi(x) \\ (ii)\ \varphi(y) \end{cases} \qquad \forall\, x, y \in \mathbb{R}.$$

(a) Then, either g is bounded or f satisfies (S_r) under one of the cases $f(0) = 0$ or $f(-x) = -g(x)$.

(b) Then, either f is bounded or g satisfies (S_r) under one of the cases $g(0) = 0$ or $g(x) = -f(x)$.

Corollary 2.9. Suppose that $f, g, h, k : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot g(x)f(y)| \leq \begin{cases} (i)\ \varphi(x) \\ (ii)\ \varphi(y) \end{cases} \qquad \forall\, x, y \in \mathbb{R}.$$

(a) Then, either f is bounded or g satisfies (S_r) under one of the cases $g(0) = 0$ or $f(-x) = -g(x)$.

(b) Then, either g is bounded or f satisfies (S_r) under one of the cases $f(0) = 0$ or $g(x) = -f(x)$.

Corollary 2.10. Suppose that $f, g, h, k : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot g(x)g(y)| \leq \begin{cases} (i)\ \varphi(x) \\ (ii)\ \varphi(y) \end{cases} \qquad \forall\, x, y \in \mathbb{R}.$$

(a) Then, either g is bounded or g satisfies (S_r) under one of the cases $g(0) = 0$ or f is odd.

(b) Then, either g is bounded or g satisfies (S_r) under the case $g(0) = 0$.

2.1 Stability of (C_r^λ)

In this subsection, let us study the stability of the cosine type p-radical functional equation

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda f(x)f(y). \qquad (C_r^\lambda)$$

Note that $\widetilde{f}(x):=f(x)f(0)^{-1}$. The following lemma which is easy to verify shows that the similar argument holds without assuming the continuity. To make it easy to write, we continue using this notation $\widetilde{f}$ and note that it is legel only when $f(0)\neq 0$.

Lemma 2.11. Let $f,g:\mathbb{R}\to\mathbb{C}$ be functions satisfying

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda f(x)g(y)\ \forall x,y\in\mathbb{R}.$$

If f is an even function, then either $f\neq 0$ or $\widetilde{f}$ satisfies (C_r^λ).

We now apply the proceeding lemma to obtain the following theorem.

Theorem 2.12. Suppose that $f,g,h,k:\mathbb{R}\to\mathbb{C}$ satisfy the inequality

$$\left|f\left(\sqrt[p]{x^p+y^p}\right)+g\left(\sqrt[p]{x^p-y^p}\right)-\lambda\cdot h(x)k(y)\right|\leq\begin{cases}(i)\ \varphi(x)\\(ii)\ \varphi(y)\end{cases}\qquad\forall\, x,y\in\mathbb{R}.$$

(i) If k is unbounded, and $h\neq 0$ is an even function, then $\widetilde{h}$ satisfies (C_r^λ).

(ii) If h is unbounded, and $k\neq 0$ is an even function, then $\widetilde{k}$ satisfies (C_r^λ).

Proof . (i) Since k is unbounded, there exists a sequence $\{y_n\}$ in $\mathbb{R}$ such that $\lim_{n\to\infty}|k(y_n)^{-1}|=0$. Then, by the Theorem 2.1, there exists an limit function (2.5): L_k with $L_k(0)=2\lambda^{-1}$ and

$$h\left(\sqrt[p]{x^p+y^p}\right)+h\left(\sqrt[p]{x^p-y^p}\right)=\lambda\cdot h(x)L_k(y)$$

for all $x,y\in\mathbb{R}$.

Then, by the Lemma 2.11, we deduce that $\widetilde{h}$ satisfies (C_r^λ).

(ii) Similarly to (i), by the Lemma 2.11 and Theorem 2.2, we deduce that $\widetilde{k}$ satisfies (C_r^λ). □

Corollary 2.13. Suppose that $f,g,h,k:\mathbb{R}\to\mathbb{C}$ satisfy the inequality

$$\left|f\left(\sqrt[p]{x^p+y^p}\right)+g\left(\sqrt[p]{x^p-y^p}\right)-\lambda\cdot f(x)g(y)\right|\leq\begin{cases}(i)\ \varphi(x)\\(ii)\ \varphi(y)\end{cases}\qquad\forall\, x,y\in\mathbb{R}.$$

(i) If g is unbounded, and $f\neq 0$ is even, then $\widetilde{f}$ satisfies (C_r^λ).

(ii) If f is unbounded, and $g\neq 0$ is even, then $\widetilde{g}$ satisfies (C_r^λ).

Corollary 2.14. Suppose that $f,g,h,k:\mathbb{R}\to\mathbb{C}$ satisfy the inequality

$$\left|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda\cdot h(x)k(y)\right|\leq\begin{cases}(i)\ \varphi(x)\\(ii)\ \varphi(y)\end{cases}\qquad\forall\, x,y\in\mathbb{R}.$$

(i) If k is unbounded, and $h\neq 0$ and $f(or h)$ is even, then $\widetilde{h}$ satisfies (C_r^λ).

(ii) If h is unbounded, and $k\neq 0$, then $\widetilde{k}$ satisfies (C_r^λ).

Proof . (i) Take $g=f$ in Theorem 2.12. The evenness of f and the definition of h in (2.3) lead to the evenness of h.

(ii) Taking $g=f$ in Theorem 2.12, we infer the evenness of k from its definition in (2.10). □

Corollary 2.15. Suppose that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot g(x)f(y)| \leq \begin{cases} (i)\ \varphi(x) \\ (ii)\ \varphi(y) \end{cases} \qquad \forall\, x, y \in \mathbb{R}.$$

(i) If f is unbounded, and $g \neq 0$ and f is even, then $\widetilde{g} = (\lambda/2) \cdot g$ satisfies (C_r^λ).

(ii) If g is unbounded, and $f \neq 0$, then $\widetilde{f}$ satisfies (C_r^λ).

Corollary 2.16. Suppose that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot f(x)g(y)| \leq \begin{cases} (i)\ \varphi(x) \\ (ii)\ \varphi(y) \end{cases} \qquad \forall\, x, y \in \mathbb{R}.$$

(i) If g is unbounded, and $f \neq 0$ is even, then $\widetilde{f}$ satisfies (C_r^λ).

(ii) If f is unbounded, and $g \neq 0$, then $\widetilde{g} = (\lambda/2) \cdot g$ satisfies (C_r^λ).

Corollary 2.17. Suppose that $f : \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot f(x)f(y)| \leq \begin{cases} (i)\ \varphi(x) \\ (ii)\ \varphi(y) \end{cases} \qquad \forall\, x, y \in \mathbb{R}.$$

Then either f is bounded or $\widetilde{f} = (\lambda/2) \cdot f$ satisfies (C_r^λ).

Remark 2.18. (i) As like Corollary 2.3, we can apply an stability inequalities to two cases ($\min\{\varphi(\mathrm{x}), \varphi(\mathrm{y})\}$ or ε) in all results of the Section 2.

(ii) As Corollaries, the stability of the p-radical functional equations ($fgfk$,$fghf$, $fghh$,$ffhk$,$fffg$,$ffgf$,$ffgg$,$ffff$) are obtained.

(iv) Setting with $p = 1$ in all results contained the above (i) and (ii), then the obtained stability for the p-radical equations arrive to the stability results of the same type's trigonometric functional equations(cosine, sine, Wilson, Kim), which is founded in papers ([5], [11], [12], [18], [16], [19], [23]).

3 Extension to the Banach space

In all results presented in Section 2, the range of functions can be extended to the semisimple commutative Banach space. We will represent just for the main equation (P_r^λ).

Theorem 3.1. Let $(E, \|\cdot\|)$ be a semisimple commutative Banach space. Assume that $f, g, h, k : \mathbb{R} \to E$ satisfy the inequalities

$$\|f\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) - \lambda \cdot h(x)k(y)\| \leq \begin{cases} (i)\ \varphi(x), \\ (ii)\ \varphi(y) \quad \forall x, y \in \mathbb{R}. \end{cases} \tag{3.1}$$

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) either k is bounded or h satisfies (S_r) under one of the cases $h(0) = 0$ or $f(-x) = -g(x)$.

(ii) Then, either h is bounded or k satisfies (S_r) under one of the cases $k(0) = 0$ or $g(x) = -f(x)$.

Proof . Case (i), assume that (3.1) holds and arbitrarily fixes a linear multiplicative functional $x^* \in E^*$. As is well known, we have $\|x^*\| = 1$, hence, for every $x, y \in \mathbb{R}$, we have

$$\begin{aligned}\varphi(x) &\geq \left\| f\left(\sqrt[p]{x^p+y^p}\right) + g\left(\sqrt[p]{x^p-y^p}\right) - \lambda \cdot h(x)k(y) \right\| \\ &= \sup_{\|y^*\|=1} \left| y^*\left(f\left(\sqrt[p]{x^p+y^p}\right) + g\left(\sqrt[p]{x^p-y^p}\right) - \lambda \cdot h(x)k(y)\right) \right| \\ &\geq \left| x^*\left(f(\sqrt[p]{x^p+y^p})\right) + x^*\left(g(\sqrt[p]{x^p-y^p})\right) - \lambda \cdot x^*\left(h(x)\right)x^*\left(k(y)\right) \right|,\end{aligned}$$

which states that the superpositions $x^* \circ f$, $x^* \circ g$, $x^* \circ h$ and $x^* \circ k$ yield a solution of inequality (2.1) in Theorem 2.1. Since, by assumption, the superposition $x^* \circ k$ with $(x^* \circ h)(0) = 0$ is unbounded, an appeal to Theorem 2.1 shows that the result holds.

The superposition $x^* \circ h$ solves (S_r), that is

$$(x^* \circ h)\Big(\sqrt[p]{\frac{x^p+y^p}{2}}\Big)^2 - (x^* \circ h)\Big(\sqrt[p]{\frac{x^p-y^p}{2}}\Big)^2 = (x^* \circ h)(x)(x^* \circ h)(y).$$

Since x^* is a linear multiplicative functional, we get

$$x^*\left(h\Big(\sqrt[p]{\frac{x^p+y^p}{2}}\Big)^2 - h\Big(\sqrt[p]{\frac{x^p-y^p}{2}}\Big)^2 - h(x)h(y)\right) = 0.$$

Hence a unrestricted choice of x^* implies that

$$h\Big(\sqrt[p]{\frac{x^p+y^p}{2}}\Big)^2 - h\Big(\sqrt[p]{\frac{x^p-y^p}{2}}\Big)^2 - h(x)h(y) \in \bigcap\{\ker x^* : x^* \in E^*\}$$

Since the space E is semisimple, $\bigcap\{\ker x^* : x^* \in E^*\} = 0$, which means that h satisfies the claimed equation (S_r).

For second case $(x^* \circ f)(-x) = -(x^* \circ g)(x)$, it is enough to show that $(x^* \circ h)(0) = 0$, which can be easily check as Theorem 2.1. Hence, the proof (i) of (a) is completed.

For (ii), by appeal to Theorem 2.2, this also runs along the proof of case (i). An appeal to Theorem 2.2 shows that if $x^* \circ k$ satisfies (**??**), then $x^* \circ h$ and $x^* \circ k$ satisfies the equation

$$(x^* \circ h)\left(\sqrt[p]{x^p+y^p}\right) + (x^* \circ h)\left(\sqrt[p]{x^p-y^p}\right) = \lambda(x^* \circ h)(x)(x^* \circ k)(y).$$

This means by a linear multiplicativity of x^* that

$$\mathcal{D}C_{hk}^{\lambda}(x,y) := h\left(\sqrt[p]{x^p+y^p}\right) + h\left(\sqrt[p]{x^p-y^p}\right) - \lambda h(x)k(y),$$

falls into the kernel of x^*. As the above process, since x^* is a linear multiplicative, we obtain

$$\mathcal{D}C_{hk}^{\lambda}(x,y) = 0 \quad \text{for all} \quad x, y \in \mathbb{R}$$

as claimed. □

Theorem 3.2. Let $(E, \|\cdot\|)$ be a semisimple commutative Banach space. Assume that $f, g, h, k : \mathbb{R} \to E$ satisfy one of each inequalities

$$\|f\left(\sqrt[p]{x^p+y^p}\right) + g\left(\sqrt[p]{x^p-y^p}\right) - \lambda \cdot h(x)k(y)\| \leq \begin{cases}\min\{\varphi(x), \varphi(y)\} & or \\ \varepsilon\end{cases}$$

for all $x, y \in \mathbb{R}$. For an arbitrary linear multiplicative functional $x^* \in E^*$,

(a) Suppose that $x^* \circ k$ fails to be bounded, then

h satisfies (S_r) under one of the cases $x^* \circ h(0) = 0$ or $x^* \circ f(-x) = -x^* \circ g(x)$,

(b) Suppose that $x^* \circ h$ fails to be bounded, then

k satisfies (S_r) under one of the cases $x^* \circ k(0) = 0$ or $x^* \circ f(-x) = -x^* \circ g(x)$,

Remark 3.3. As Remarks 2.18, we can be applied all results of the Section 2 to the Banach space.

Namely, we obtain the stability results of $14 \times 4(\varphi(x), \varphi(y), \min\{\varphi(x), \varphi(y)\}, \varepsilon)$ numbers for the other 14 equations except for Eq.(P_r^{λ}). Some of them are found in papers ([5], [11], [12], [16], [19]).

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