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Continuity of homomorphisms on complete metrizable topological algebras

C. Ganesa Moorthy, G. Siva*

Department of Mathematics, Alagappa University, Karaikudi-630 003, India

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Abstract

Let A be an F-algebra over the complex field. Let B be an F-algebra such that the intersection of kernels of all continuous multiplicative linear functionals on B is singleton zero. If any nonempty open subset of the collection $M(B)$ of all continuous multiplicative linear functionals in the Gelfand topology contains uncountably many functionals or if B is a commutative Frechet algebra such that $M(B)$ has no isolated points, then any homomorphism from A onto a dense finitely generated subalgebra of B is continuous. This result has been proved in this article which is similar to a result derived by R.L. Carpenter.

Keywords: Homomorphism, Closed graph theorem, Frechet algebra
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1 Introduction

R.L. Carpenter [2] proved the following Theorem 1.1 to establish uniqueness of Frechet algebra topology for a commutative semisimple Frechet algebra over the complex field. The purpose of this present article is to consider two possible variations of this theorem. All topologies which are to be considered are Hausdorff. All algebras which are to be considered will be over the field of complex numbers, and they contain multiplicative identity element. Multiplication is jointly continuous in every topological algebra. For a topological algebra A, let $M(A)$ denote the collection of all nonzero continuous multiplicative linear functionals on A, and let it be endowed with the Gelfand topology. A complete metrizable topological algebra is called an F-algebra. A topological algebra A is said to be a LMC algebra, if its topology is induced by a family of submultiplicative seminorms. A Frechet algebra is a LMC algebra which is also an F-algebra. A metrizable topological algebra is written in the form (A, d), where d is an addition invariant metric which induces the topology of the topological algebra A. A metrizable LMC algebra is written in the form $(A, (p_n)_{n=1}^{\infty})$, where each p_n is a submultiplicative seminorm (i.e. $p_n(xy) \le p_n(x)p_n(y), \forall x, y \in A$) satisfying $p_n(x) \le p_{n+1}(x), \forall n, \forall\ x \in A$, in which the topology on A is induced by the seminorms $p_n, n = 1, 2, \ldots$. A topological algebra A is said to be finitely generated, if the smallest algebra containing some finitely many elements of A is dense in A. The following Theorem 1.1 was proved in [2], in which R.L. Carpenter uses the terminology F-algebra to refer to a Frechet algebra.

*Corresponding author
Email addresses: ganesamoorthyc@gmail.com (C. Ganesa Moorthy), gsivamaths2012@gmail.com (G. Siva)

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Theorem 1.1. [2] Let A be a Frechet algebra. Let B be a commutative semisimple Frechet algebra such that every point in $M(B)$ is not isolated point in $M(B)$. Let $\varphi : A \to B$ be a surjective homomorphism. Then φ is continuous.

Let $(A, (p_n)_{n=1}^{\infty})$ be a commutative Frechet algebra. To each $n, A/{p_n}^{-1}(0)$ is a normed algebra with respect to the quotient norm induced by p_n. Let the completion of the normed algebra $A/{p_n}^{-1}(0)$ be denoted by A_n and the continuous extension of the quotient norm to A_n also be denoted by p_n. If $\varphi \in M(A)$, then it can also be identified as an element of $\varphi \in M((A/{p_n}^{-1}(0)))$, for some n; and in this case φ has a unique continuous extension to A_n so that φ is identified as a member of $M(A_n)$. Thus $M(A) = \cup_{n=1}^{\infty} M(A_n)$, and $M(A_n) \subseteq M(A_{n+1})$, $\forall n$. A version of Shilov idempotent theorem [14] implies that a subset E of $M(A)$ is both open and closed in $M(A)$ if and only if $E \cap M(A_n)$ is both open and closed in $M(A_n)$, for every n.

2 Technical Lemmas

The results of this article are based on the following technical lemmas.

Lemma 2.1. Let $(\varphi_n)_{n=1}^{\infty}$ be an infinite sequence of distinct nonzero multiplicative linear functionals on an algebra A such that $(\varphi_n(z))_{n=1}^{\infty}$ is a sequence of distinct nonzero scalars, for some $z \in A$. Then there is a sequence $(y_n)_{n=1}^{\infty}$ in A such that $\varphi_i(y_j) \neq 0$, whenever $i \geq j$ and $\varphi_i(y_{i+1}) = 0$, $\forall i \geq 1$.

Proof . Let e denote the multiplicative identity element in A. Let $y_1 = z$ so that $\varphi_i(y_1) \neq 0, \forall i \geq 1$, and they are distinct. In general, let us define $y_2, y_3, ...$ by using the relations $\lambda_i = \varphi_i(y_i), y_{i+1} = y_i - \lambda_i e$, $\forall i \geq 1$. Then $\varphi_i(y_{i+1}) = \varphi_i(y_i) - \lambda_i \varphi_i(e) = 0$, $\forall i \geq 1$. Also, $\varphi_i(y_2) = \varphi_i(y_1) - \varphi_1(y_1) \neq 0$, $\forall i \geq 2$, and they are distinct. Similarly, $\varphi_i(y_3) = \varphi_i(y_2) - \varphi_2(y_2) \neq 0$, $\forall i \geq 3$, and they are distinct. In general, for every fixed j, $\varphi_i(y_j) \neq 0$, $\forall i \geq j$, and they are distinct. This proves the lemma. □

Lemma 2.2. Let $(\psi_n)_{n=1}^{\infty}$ be a sequence of distinct nonzero continuous multiplicative linear functionals on a topological algebra A with a multiplicative identity element e. Suppose that there are finitely many elements $z_1, z_2, ..., z_m$ in A such that the smallest algebra containing $\{e, z_1, z_2, ..., z_m\}$ is dense in A. Then there is a z_k and a there is a subsequence $(\varphi_n)_{n=1}^{\infty}$ of $(\psi_n)_{n=1}^{\infty}$ such that $(\varphi_n(z_k))_{n=1}^{\infty}$ is a sequence of distinct nonzero scalars.

Proof . Consider the sets $A_k = \{\psi_1(z_k), \psi_2(z_k), ...\}$, $k = 1, 2, ..., m$. If each A_k is a finite set, then $\{\psi_i(z_j) : i = 1, 2, ...; j = 1, 2, ..., m\}$is a finite set, and hence $\{\psi_1, \psi_2, ...\}$ should be a finite set. This proves that some A_k is an infinite set, and the lemma follows. □

3 Main Results

Let us now prove the main results of this article.

Theorem 3.1. Let (A, d) be an F-algebra and $(B, (p_n)_{n=1}^{\infty})$ be a commutative semi-simple Frechet algebra such that every point in $M(B)$ is not an isolated point in $M(B)$. Let $\Psi : A \to B$ be a homomorphism such that $\Psi(A)$ is a dense finitely generated subalgebra of B. Then Ψ is continuous on A.

Proof . Let $S = \{\varphi \in M(B) : \varphi \circ \Psi$ is continuous on $A\}$. If $\overline{S} = M(B)$, then Ψ is continuous on A, by the closed graph theorem, where $\overline{S}$ is the closure of S in $M(B)$. So, let us assume that $\overline{S} \neq M(B)$, to reach a contradiction. Let $U = M(B) \setminus \overline{S}$. Then U is an infinite set, because $M(B)$ is Hausdorff, and every point in U is not an isolated point of $M(B)$. If $U \cap M(B_i)$ is finite, for every i, then every point in $U \cap M(B_i)$ is an isolated point in $M(B_i)$, for every i, because $M(B)$ is Hausdorff. By the Shilov idempotent theorem, every point in U is an isolated point in $M(B)$. This is impossible. So, $U \cap M(B_l)$ is an infinite set, for some l. By Lemma 2.1 and Lemma 2.2, it is possible to assume that there is an infinite sequence $(\varphi_n)_{n=1}^{\infty}$ of distinct multiplicative linear functionals in $U \cap M(B_l)$, and there is an infinite sequence $(y_n)_{n=1}^{\infty}$ in $\Psi(A)$ such that $(i)\varphi_i(y_j) \neq 0$, whenever $i \geq j$, and $(ii)\varphi_i(y_{i+1}) = 0$, $\forall i \geq 1$. Let $(z_n)_{n=1}^{\infty}$ be a sequence in A such that $\Psi(z_i) = y_i$,$\forall i$. Find a sequence $(x_n)_{n=1}^{\infty}$ in A such that

$$\max_{1 \leq j \leq i} d(z_j z_{j+1} ... z_i x_i) < 2^{-i}$$

and such that

$$|\varphi_i(\Psi(x_i))| > \frac{|\varphi_i(\Psi(\sum_{j=1}^{i-1} z_1 z_2 ... z_j x_j))| + i}{|\varphi_i(\Psi(z_1 z_2 ... z_i))|}, \tag{3.1}$$

for every i, with a convention that empty sum is zero. Let $x = \sum_{i=1}^{\infty} z_1 z_2 ... z_i x_i$. Then there is a finite constant $K_l > 0$ such that $|\varphi_i(\Psi(x))| \leq K_l$, $\forall i = 1, 2, ...$, because $\varphi_i \in M(B_l)$, $\forall\ i = 1, 2, ...$, and x is fixed here. Then,

$$\begin{aligned}\varphi_k(\Psi(x)) = \varphi_k\Big(\Psi\Big(\sum_{i=1}^{k-1} z_1 z_2 ... z_i x_i\Big)\Big) + \varphi_k(\Psi(z_1 z_2 ... z_k x_k)) + \varphi_k(\Psi(z_1 z_2 ... z_{k+1} x_{k+1}))\\ + \varphi_k\Big[(\Psi(z_1 z_2 ... z_{k+1}))\Big(\Psi\Big(\sum_{i=k+2}^{\infty} z_{k+2} ... z_i x_i\Big)\Big)\Big]\end{aligned}$$

$$= \varphi_k\Big(\Psi\Big(\sum_{i=1}^{k-1} z_1 z_2 ... z_i x_i\Big)\Big) + \varphi_k(\Psi(z_1 z_2 ... z_k x_k)).$$

Thus, by (3.1),

$$K_l \geq |\varphi_k(\Psi(x))| > |\varphi_k(\Psi(z_1 z_2 ... z_k x_k))| - \Big|\varphi_k\Big(\Psi\Big(\sum_{i=1}^{k-1} z_1 z_2 ... z_i x_i\Big)\Big)\Big| > k, \forall k = 1, 2, ...$$

This is a contradiction. This contradiction proves the theorem. □

Definition 3.2. A topological algebra A is said to be functionally semisimple, if $\cap_{f \in M(A)} f^{-1}(0) = \{0\}$.

Theorem 3.3. Let (A, d) be an F-algebra and (B, d_B) be a functionally semisimple F-algebra. Suppose every nonempty open subset of $M(B)$ contains uncountable many elements. Let $\Psi : A \to B$ be a homomorphism such that $\Psi(A)$ is a dense finitely generated subalgebra of B. Then Ψ is continuous on A.

Proof . Let $S = \{\varphi \in M(B) : \varphi \circ \Psi$ is continuous on $A\}$. If $\overline{S} = M(B)$, then Ψ is continuous on A, by the closed graph theorem, where $\overline{S}$ is the closure of S in $M(B)$. So, let us assume that $\overline{S} \neq M(B)$, to reach a contradiction. Let $U = M(B) \setminus \overline{S}$. Then U is an uncountable set. To each $\varphi \in M(B)$, there is an integer n such that $|\varphi(x)| < 1$, whenever $d_B(0, x) < \frac{1}{n}$ in B. Since U is uncoutable, there is an integer l such that $\{\varphi \in U : |\varphi(x)| < 1$, whenever $d_B(0, x) < \frac{1}{l}$ in $B\}$=M(say) is an uncountable set. By Lemma 2.1 and Lemma 2.2, there is an infinite sequence $(\varphi_n)_{n=1}^{\infty}$ in $M(\subseteq M(B))$ such that $|\varphi_i(x)| < 1$ whenever $d_B(0, x) < \frac{1}{l}$ in B, for every i, and there is an infinite sequence $(y_n)_{n=1}^{\infty}$ in $\Psi(A)$ such that $(i)\varphi_i(y_j) \neq 0$, whenever $i \geq j$, and $(ii)\varphi_i(y_{i+1}) = 0$, $\forall i \geq 1$. Let $(z_n)_{n=1}^{\infty}$ be a sequence in A such that $\Psi(z_i) = y_i$,$\forall i$. Find a sequence $(x_n)_{n=1}^{\infty}$ in A such that

$$\max_{1 \leq j \leq i} d(z_j z_{j+1} ... z_i x_i) < 2^{-i}$$

and such that

$$|\varphi_i(\Psi(x_i))| > \frac{|\varphi_i(\Psi(\sum_{j=1}^{i-1} z_1 z_2 ... z_j x_j))| + i}{|\varphi_i(\Psi(z_1 z_2 ... z_i))|}, \tag{3.2}$$

for every i, with a convention that empty sum is zero. Let $x = \sum_{i=1}^{\infty} z_1 z_2 ... z_i x_i$. Then there is a finite constant $K_l > 0$ such that $|\varphi_i(\Psi(x))| \leq K_l$, $\forall i = 1, 2, ...$, because $\varphi_i \in M$, $\forall\ i = 1, 2, ...$, and x is fixed here. Then,

$$\begin{aligned}\varphi_k(\Psi(x)) = \varphi_k\Big(\Psi\Big(\sum_{i=1}^{k-1} z_1 z_2 ... z_i x_i\Big)\Big) + \varphi_k(\Psi(z_1 z_2 ... z_k x_k)) + \varphi_k(\Psi(z_1 z_2 ... z_{k+1} x_{k+1}))\\ + \varphi_k\Big[(\Psi(z_1 z_2 ... z_{k+1}))\Big(\Psi\Big(\sum_{i=k+2}^{\infty} z_{k+2} ... z_i x_i\Big)\Big)\Big]\end{aligned}$$

$$= \varphi_k\Big(\Psi\Big(\sum_{i=1}^{k-1} z_1z_2...z_ix_i\Big)\Big) + \varphi_k(\Psi(z_1z_2...z_kx_k)).$$

Thus, by (3.2),

$$K_l \geq |\varphi_k(\Psi(x))| > |\varphi_k(\Psi(z_1z_2...z_kx_k))| - \Big|\varphi_k\Big(\Psi\Big(\sum_{i=1}^{k-1} z_1z_2...z_ix_i\Big)\Big)\Big| > k, \forall k = 1, 2, ...$$

This is a contradiction. This contradiction proves the theorem. □

One may find good examples for algebras B satisfying the conditions of Theorem 3.1 and Theorem 3.3 in [13] and in references given in [13]. Let us consider the following two simple examples, which are very particular cases.

Example 3.4. Let B be the collection of all complex valued analytic functions on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. Then B is an algebra under point wise operations. Let $p_n(f) = \sup\limits_{|z|\leq 1-\frac{1}{n+1}} |f(z)|$, $\forall n = 1, 2, ...$, $\forall f \in B$. Then $(B, (p_n)_{n=1}^{\infty})$ is a Frechet algebra. To each $\varphi \in M(B)$, there is a unique $z \in \Delta$ such that $\varphi(f) = f(z)$, $\forall f \in B$. On the other hand, to each $z \in \Delta$, there is a unique $\varphi \in M(B)$ such that $\varphi(f) = f(z)$, $\forall f \in B$. Thus, Δ may be considered as $M(B)$ through this relation, and the usual topology in Δ coincides with the Gelfand topology under this identification. Let us note that Δ has no isolated point and every nonempty open set in Δ is an uncountable set. The smallest algebra containing the functions g_1, g_2 is dense in B, where $g_1(z) = z$ and $g_2(z) = 1$, $\forall z \in \Delta$.

Example 3.5. Let $B = \Big\{\sum\limits_{j=0}^{\infty}\sum\limits_{i=0}^{\infty} a_{ij}{z_1}^i{z_2}^j : a_{ij} \in \mathbb{C}, \forall i, \forall j, \text{ and } \sum\limits_{j=0}^{\infty}\sum\limits_{i=0}^{\infty} |a_{ij}| < \infty\Big\}$. Then B is an algebra of formal power series with two formal variables z_1, z_2 with respect to usual function operations. It is also a Banach algebra with an identity element with respect to the norm given by $\Big\|\sum\limits_{j=0}^{\infty}\sum\limits_{i=0}^{\infty} a_{ij}{z_1}^i{z_2}^j\Big\| = \sum\limits_{j=0}^{\infty}\sum\limits_{i=0}^{\infty} |a_{ij}|$. Then $M(B)$ can be identified with the set $\{(\xi_1, \xi_2) : \xi_1 \in \mathbb{C}, \xi_2 \in \mathbb{C}, |\xi_1| \leq 1, |\xi_2| \leq 1\}$ such that its usual topology coincides with the Gelfand topology of $M(B)$. Thus $M(B)$ has no isolated points and every nonempty subset of $M(B)$, which are relatively open in $M(B)$, is an uncountable set. The smallest algebra containing the formal functions ${z_1}^1, {z_2}^1$ and 1 is dense in B.

4 conclusion

Continuity of homomorphisms implies uniqueness of algebra topology. There are many recent research articles for continuity of homomorphisms [6, 7, 12, 19], and for uniqueness of algebra topologies [4, 13, 15, 17, 18]. There are many recent research articles for continuity of multiplicative linear functionals [1, 3, 5, 8, 10, 11, 16]. An indirect aim of this article is to try to find some methods with an expectation of finding a chance to solve the Michael's open problem [9]: Is every multiplicative linear functional on a commutative complex Frechet algebra continuous?; and to solve the general open problem: Is every multiplicative linear functional on a complex F-algebra continuous?

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