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The optimal control of an evolutionary boundary value problem

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Abstract

We consider a nonlinear initial boundary value problem in a two-dimensional rectangle. We derive variational formulation of the problem which is in the form of an evolutionary variational inequality in a product Hilbert space. Then, we establish the existence of a unique weak solution to the problem and prove the continuous dependence of the solution with respect to some paremeters. We proceed with the study of an associated control problem for which we prove the existence of an optimal pair. Finally, we consider a perturbed optimal control problem for which we prove a convergence result.

Keywords: nonlinear boundary value problem, evolutionary inequality, weak solution, convergence results, optimal control
2020 MSC: 74M15, 49J40, 49J45, 49J20

1 Introduction

Our aim in this paper is to provide the variational analysis of an initial boundary value problem by using arguments of evolutionary variational inequalities and history-dependent operators. The theory of variational inequalities started in early sixty, based on arguments of monotonicity and convexity. Classical references in the mathematical and numerical analysis of variational inequalities are [3, 12, 10, 13, 5, 6], for instance. Various applications in Mechanics and, more specifically, in Contact Mechanics could be found in the books [1, 2, 4, 10, 21, 16, 11, 15, 8, 7] and in the special issue [14]. Evolutionary variational inequalities are inequalities which involve the time derivative of the solution and, therefore, they require an initial condition. Existence and uniqueness results for such inequalities can be found in the books [19, 21, 8, 9], for instance. Recently, there is an interest in the study of a special class of inequalities, the so-called history-dependent variational inequalities. There are inequalities in which various functions or operators depend on the history of the solution. Their study is motivated by important application in problems involving constitutive laws for materials with memory, total slip or total slip rate friction laws. Existence, uniqueness and regularity uniqueness results for such kind of inequalities can be found in [20, 21, 22], for instance.

The problem we are interested in this paper leads, in a primal variational formulation, to an evolutionary variational inequality. In contrast, its dual variational formulation is in a form of a histroy-dependent variational inequality. To introduce this problem let L, h and T be given positive constants and denote $\Omega = (0, L) \times (-h, h)$. From now on we

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use the notation (x, y) for a generic point in Ω and the subscripts x and y will represent the partial derivative with respect to the variables. The problem under consideration is the following [18].

Problem $\mathcal{P}$. *Find the functions $u = u(x, y, t) : [0, L] \times [-h, h] \times [0, T] \to \mathbb{R}$ and $w = (x, t) : [0, L] \times [0, T] \to \mathbb{R}$ such that*

$$\lambda \dot{u}_{xx} + E u_{xx} + \mu \dot{u}_{yy} + G u_{yy} + q_B = 0 \qquad \text{for all } (x, y) \in \Omega,\ t \in [0, T], \tag{1.1}$$

$$\mu \dot{w}_{xx} + G w_{xx} + (\lambda - \mu)\dot{u}_{xy} + (E - G)u_{xy} + f_B = 0 \qquad \text{for all } (x, y) \in \Omega,\ t \in [0, T], \tag{1.2}$$

$$u(0, y, t) = w(0, t) = 0 \quad \text{for all } y \in [-h, h],\ t \in [0, T], \tag{1.3}$$

$$\lambda \dot{u}_x(L, y, t) + E u_x(L, y, t) = 0 \qquad \text{for all } y \in [-h, h], \quad t \in [0, T], \tag{1.4}$$

$$\mu(\dot{u}_y(L, y, t) + \dot{w}_x(L, y, t)) + G(u_y(L, y, t) + w_x(L, y, t)) = 0 \qquad \text{for all } y \in [-h, h], \quad t \in [0, T]. \tag{1.5}$$

$$\mu(\dot{u_y}(x, h, t) + \dot{w}_x(x, t)) + G(u_y(x, h, t) + w_x(x, t)) = q_N(x, t) \qquad \text{for all } x \in [0, L], \quad t \in [0, T], \tag{1.6}$$

$$(\lambda - 2\mu)\dot{u}_x(x, h, t) + (E - 2G)u_x(x, h, t) = f_N(x, t) \qquad \text{for all } x \in [0, L], \quad t \in [0, T]. \tag{1.7}$$

$$\begin{aligned} &|(\lambda - 2\mu)(\dot{u}_x(x, -h, t) + (E - 2G)(u_x(x, -h, t)| \le g, \\ &-(\lambda - 2\mu)(\dot{u}_x(x, -h, t) - (E - 2G)(u_x(x, -h, t) = g\frac{\dot{w}(x, t)}{|\dot{w}(x, t)|} \quad \text{if } \dot{w}(x, t) \neq 0, \quad \text{for all } x \in [0, L], \quad t \in [0, T], \end{aligned} \tag{1.8}$$

$$\mu(\dot{u}(x, -h, t) + \dot{w}(x, t)) + G(u_y(x, -h, t) + w_x(x, t)) = 0 \quad \text{for all } x \in [0, L], \quad t \in [0, T], \tag{1.9}$$

$$u(x, y, 0) = u_0(x, y), \qquad w(x, 0) = w_0(x), \qquad \text{for all } x \in [0, L],\ y \in [-h, h]. \tag{1.10}$$

Problem $\mathcal{P}$ describes the equilibrium of a viscoelastic plate submitted to the action of body forces and tractions and to nonlinear contact conditions on part of its boundary. Here Ω represents the cross section of the plate, u is the horizontal displacement and w is the vertical displacement. The constants λ and μ are positive viscosity coefficients and E and G are positive elastic coefficients. A brief description of equations and boundary condition in Problem $\mathcal{P}$, including their mechanical significance, follows.

First, equations (1.1) and (1.2) represent the equilibrium equation in which the functions $q_B = q_B(x, y, t) : \Omega \times [0, T] \to \mathbb{R}$ and $f_B = f_B(x, y, t) : \Omega \times [0, T] \to \mathbb{R}$ are the horizontal and the vertical components of the body forces. Condition (1.3) shows that the plate is fixed on the boundary $x = 0$ and conditions (1.4), (1.5) show that the boundary $x = L$ is free of tractions. Next, conditions (1.6), (1.7) represent the traction conditions. Here, the functions $q_N = q_N(x, t) : [0, L] \times [0, T] \to \mathbb{R}$ and $f_N = f_N(x, t) : [0, L] \times [0, T] \to \mathbb{R}$ denote the horizontal and the vertical components of the traction forces which act on the top $y = h$ of the plate. Condition (1.8) represents the a multivalued contact condition on the bottom $x = -h$ in which $g \ge 0$ is given. Condition (1.9) represents the frictionless condition and, finally, (1.10) represents the initial condition, in which the functions u_0 and w_0 are the initial horizontal and vertical displacement, respectively.

The rest of paper is structured as follows. In Section 2 we list the assumptions on the data and derive the variational formulation of problem $\mathcal{P}$. In Section 3 we state and prove our main result, Theorem 2.1, which states the unique weak solvability of the problem, see Theorem 11.3 in [9]. The proof is based on arguments of evolutionary variational inequalities. In Section 3 we state and prove a convergence result, Theorem 3.1. It states the continuous dependence of the solution with respect to the data. Finally, in Section 4 we state and prove the solvability of an optimal control problem associated the contact Problem $\mathcal{P}$. Then, we derive a convergence result related to this optimal problem, Theorem 4.3.

2 Variational formulation

We start with some notation and preliminaries. Given a real Hilbert space Y we denote by $\langle \cdot, \cdot \rangle_Y$ its inner product and by $\| \cdot \|_Y$ the associate norm, i.e. $\|y\|_Y^2 = \langle u, u \rangle_Y$ for all $y \in Y$. For a normed space Y we denote by $C([0, T]; Y)$ the space of the continuous functions defined on $[0, T]$ with values to Y, equiped with the canonic norm. Moreover, $\| \cdot \|_{\mathcal{L}(Y,Z)}$ denotes the norm in the space of linear continuous operators on Y with values on the normed space Z.

Everywhere below we use the standard notation for Lebesgue and Sobolev spaces. In addition, recalling that $\Omega = (0, L) \times (-h, h)$, we introduce the spaces

$$V = \{u \in H^1(\Omega) : u(0, \cdot) = 0\}, \quad W = \{w \in H^1(0, L) : w(0) = 0\}. \tag{2.1}$$

Note that equalities $u(0, \cdot) = 0$ and $w(0) = 0$ in the definitions of the spaces V and W are understood in the sense of traces. The spaces V and W are real Hilbert spaces with the canonical inner products defined by

$$\langle u, \psi\rangle_V = \iint_\Omega (u\psi + u_x\psi_x + u_y\psi_y)\, dxdy \qquad \forall\, u, \psi \in V, \tag{2.2}$$

$$\langle w, \varphi\rangle_W = \int_0^L (w\varphi + w_x\varphi_x)\, dx \qquad \forall\, w, \varphi \in W. \tag{2.3}$$

We also consider the product space $X = V \times W$ equipped with the cannonical inner product given by

$$\langle \mathbf{u}, \mathbf{v}\rangle_X = \langle u, \psi\rangle_V + \langle w, \varphi\rangle_W \qquad \forall\, \mathbf{u} = (u, w),\ \mathbf{v} = (\psi, \varphi) \in X, \tag{2.4}$$

On the data of Problem $\mathcal{P}$ we make the following hypotheses.

$$\lambda > 0, \quad E > 0, \quad \mu > 0, \quad G > 0. \tag{2.5}$$

$$f_B \in L^2(0, T; L^2(\Omega)), \quad q_B \in L^2(0, T; L^2(\Omega)). \tag{2.6}$$

$$f_N \in L^2(0, T; L^2(0, L)), \quad q_N \in L^2(0, T; L^2(0, L)). \tag{2.7}$$

$$g \geq 0. \tag{2.8}$$

$$u_0 \in V, \quad w_0 \in W. \tag{2.9}$$

Under these assumptions we define the operators $A, B : X \to X$, functional $j : X \to \mathbb{R}$, the function $\boldsymbol{f} : [0, T] \to X$ by equalities

$$\langle A\mathbf{u}, \mathbf{v}\rangle_X = \lambda \iint_\Omega u_x\psi_x\, dxdy + \mu \iint_\Omega (u_y + w_x)(\psi_y + \varphi_x)\, dxdy, \tag{2.10}$$

$$\langle B\mathbf{u}, \mathbf{v}\rangle_X = E \iint_\Omega u_x\psi_x\, dxdy + G \iint_\Omega (u_y + w_x)(\psi_y + \varphi_x)\, dxdy, \tag{2.11}$$

$$j(\mathbf{v}) = g \int_0^L |\varphi|\, dx, \tag{2.12}$$

$$\langle \boldsymbol{f}(t), \mathbf{v}\rangle_X = \iint_\Omega q_B(t)\psi\, dxdy + \iint_\Omega f_B(t)\varphi\, dxdy + \int_0^L q_N(t)\psi\, dx + \int_0^L f_N(t)\varphi\, dx, \tag{2.13}$$

for all $\mathbf{u} = (u, w)$, $\mathbf{v} = (\psi, \varphi) \in X$, $t \in [0, T]$. We also consider the initial data $\mathbf{u}_0 \in X$ given by

$$\mathbf{u}_0 = (u_0, v_0). \tag{2.14}$$

Note that the definitions above we do not specify the dependence of various functions on the variables x and y.

The variational formulation of the Problems $\mathcal{P}$ follows from a tedious calculus, based on standard arguments. For this reason we skip the details and we restrict ourselves to describe the main steps of this calculus. We proceed formally. Thus, we assume in what follows that $\mathbf{u} = (u(x, y, t), w(x, t))$ represents a regular solution to the problem $\mathcal{P}$, $\mathbf{v} = (\psi(x, y), \varphi(x))$ is an arbitraty element of $\mathbf{X}$ and $t \in [0, T]$ is fixed. Then, multiplying (1.1) by $\psi - \dot{u}$, integrating the result over Ω and using the boundary conditions (1.3), (1.4) and the definition (2.1) of the space V we deduce that

$$\begin{aligned}
&\lambda \iint_\Omega \dot u_x(x,y,t)(\psi_x(x,y)-\dot u_x(x,y,t))\,dxdy + \mu \iint_\Omega \dot u_y(x,y,t)(\psi_y(x,y)-\dot u_y(x,y,t))\,dxdy \\
&+E \iint_\Omega u_x(x,y,t)(\psi_x(x,y)-\dot u_x(x,y,t))\,dxdy + G \iint_\Omega u_y(x,y,t)(\psi_y(x,y)-\dot u_y(x,y,t))\,dxdy \\
=\;& \int_0^L (Gu_y(x,h,t)+\mu\dot u_y(x,h,t))(\psi(x,h)-\dot u(x,h,t))\,dx \\
&-\int_0^L (Gu_y(x,-h,t)+\mu\dot u_y(x,-h,t))(\psi(x,-h)-\dot u(x,-h,t))\,dx \\
&+\iint_\Omega q_B(t)(\psi(x,t)-\dot u(x,y,t))dxdy. \qquad (2.15)
\end{aligned}$$

Assume now that $x \in [0,L]$ is fixed. We integrate equation (1.2) with respect to y on $[-h,h]$ and deduce that

$$2h\mu\,\dot w_{xx}(x,t) + 2h\,Gw_{xx}(x,t) + (\lambda-\mu)\int_{-h}^{h} \dot u_{xy}(x,y,t)\,dy + (E-G)\int_{-h}^{h} u_{xy}(x,y,t)\,dy + \int_{-h}^{h} f_B(t)dy = 0. \qquad (2.16)$$

Then, using the boundary conditions (1.7), (1.8) and notation

$$\sigma(x,-h,t) = (\lambda-2\mu)\dot u_x(x,-h,t) + (E-2G)u_x(x,-h,t), \qquad (2.17)$$

after some elementary calculus we find that

$$\begin{aligned}
(\lambda-\mu)\int_{-h}^{h} \dot u_{xy}(x,y,t)\,dy + (E-G)\int_{-h}^{h} u_{xy}(x,y,t)\,dy \;=\;& f_N(x,t) - \sigma(x,-h,t) + \mu(\dot u_x(x,h,t)-\dot u_x(x,-h,t)) \\
&+G(u_x(x,h,t)-u_x(x,-h,t)). \qquad (2.18)
\end{aligned}$$

Next, we subtract equalities (2.18) and (2.16) to deduce that

$$\begin{aligned}
-2hG\,w_{xx}(x,t) - 2h\mu\,\dot w_{xx}(x,t) \;=\;& f_N(x,t) - \sigma(x,-h,t) + \mu(\dot u_x(x,h,t)-\dot u_x(x,-h,t)) \\
&+G(u_x(x,h,t)-u_x(x,-h,t)) + \int_{-h}^{h} f_B(t)\,dy. \qquad (2.19)
\end{aligned}$$

To proceed, we multiply equality (2.19) with $\varphi - \dot w$, then we integrate the result on $[0,L]$ and perform integration by parts to obtain that

$$\begin{aligned}
&G \iint_\Omega w_x(x,t)(\varphi_x(x,t)-\dot w_x(x,t))\,dxdy + \mu \iint_\Omega \dot w_x(x,t)(\varphi_x(x,t)-\dot w_x(x,t))\,dxdy, \\
=\;& \int_0^L -\sigma(x,-h,t)(\varphi(x,t)-\dot w(x,t))\,dx + \mu\int_0^L (\dot u_x(x,h,t)-\dot u_x(x,-h,t))(\varphi(x,t)-\dot w(x,t))\,dx \\
&+G\int_0^L u_x(x,h,t)-u_x(x,-h,t))(\varphi(x,t)-\dot w(x,t))\,dx \\
&+2hG\,w_x(L,t)(\varphi(L,t)-\dot w(L,t)) + 2h\mu\,\dot w_x(L,t)(\varphi(L,t)-\dot w(L,t)) \\
&+\int_0^L f_N(\varphi(x,t)-\dot w(x,t))\,dx + \iint_\Omega f_B(\varphi(x,t)-\dot w(x,t))\,dxdy. \qquad (2.20)
\end{aligned}$$

We now add equalities (2.15) and (2.20) and use integration by parts and the boundary conditions (1.6), (1.9) to obtain

$$\begin{aligned}
& E\iint_\Omega u_x(x,y,t)(\psi_x(x,y)-\dot u_x(x,y,t))\,dxdy + G\iint_\Omega u_y(x,y,t)(\psi_y(x,y)-\dot u_y(x,y,t))\,dxdy \\
& +\lambda\iint_\Omega \dot u_x(x,y,t)(\psi_x(x,y)-\dot u_x(x,y,t))\,dxdy + \mu\iint_\Omega \dot u_y(x,y,t)(\psi_y(x,y)-\dot u_y(x,y,t))\,dxdy \\
& +G\iint_\Omega w_x(x,t)(\varphi_x(x)-\dot w_x(x,t))\,dxdy + \mu\iint_\Omega \dot w_x(x,t)(\varphi_x(x,t)-\dot w_x(x,t))\,dxdy \\
& +G\iint_\Omega w_y(x,t)(\varphi_x(x)-\dot w_x(x,t))\,dxdy + \mu\iint_\Omega \dot w_y(x,t)(\varphi_x(x,t)-\dot w_x(x,t))\,dxdy, \\
= \;& \iint_\Omega q_B(t)(\psi(x,t)-\dot u(x,y,t))\,dxdy + \int_0^L f_N(\varphi(x,t)-\dot w(x,t))\,dx \\
& +\iint_\Omega f_B(\varphi(x,t)-\dot w(x,t))dxdy - \int_0^L \sigma(x,-h,t)(\varphi(x,t)-\dot w(x,t))\,dx \\
& +\int_0^L q_N(x,t)(\psi(x,t)-\dot u(x,h,t)\,dx - \mu\int_0^L \dot w_x(x,t)(\psi(x,t)-\dot u(x,h,t)\,dx \\
& -G\int_0^L w_x(x,t)(\psi(x,t)-\dot u(x,h,t)\,dx + \mu\int_0^L \dot w_x(x,t)(\psi(x,t)-\dot u(x,-h,t)\,dx \\
& +G\int_0^L w_x(x,t)(\psi(x,t)-\dot u(x,-h,t)\,dx. \qquad (2.21)
\end{aligned}$$

Also, note that

$$-\mu\int_0^L \dot w_x(x,t)(\psi(x,t)-\dot u(x,h,t)\,dx + \mu\int_0^L \dot w_x(x,t)(\psi(x,t)-\dot u(x,-h,t) = -\mu\iint_\Omega \dot w_x(x,t)(\psi_y(x,t)-\dot u_y(x,y,t))\,dxdy \qquad (2.22)$$

and

$$\begin{aligned}
& -G\int_0^L w_x(x,t)(\psi(x,t)-\dot u(x,h,t)\,dx + G\int_0^L w_x(x,t)(\psi(x,t)-\dot u(x,-h,t)\,dx \\
= \;& -G\iint_\Omega w_x(x,t)(\psi_y(x,t)-\dot u_y(x,y,t))\,dxdy. \qquad (2.23)
\end{aligned}$$

Substituting (2.22) and (2.23) in (2.21) and using the definitions (2.10), (2.11), (2.13) we obtain

$$\langle A\dot{\mathbf u}(t),\mathbf v-\dot{\mathbf u}(t)\rangle_X + \langle B\mathbf u(t),\mathbf v-\dot{\mathbf u}(t)\rangle_X + \int_0^L \sigma(x,-h,t)(\varphi(x,t)-\dot w(x,t))\,dx = \langle \boldsymbol f(t),\mathbf v-\dot{\mathbf u}(t)\rangle_X\,, \qquad (2.24)$$

for all $\mathbf v\in X, t\in[0,T]$. Finally, using the boundary condition (1.8) and notation (2.17), it is easy to check that

$$\sigma(x,-h,t)(\varphi(x,t)-\dot w(x,t))\,dx \le g|\varphi(x,t)| - g|\dot w(x,t)| \quad \forall\, x\in[0,L].$$

We integrate this inequality on $[0,L]$ and use notation (2.12) to deduce that

$$\int_0^L \sigma(x,-h,t)(\varphi(x,t)-\dot w(x,t))\,dx \le j(\mathbf v) - j(\dot{\mathbf u}(t)). \qquad (2.25)$$

We now combine equality (2.24) with inequality (2.25) and then use the initial conditions (1.10) and notation (2.14). As a result we obtain the variational formulation of problem $\mathcal P$.

Problem $\mathcal P_V$. *Find a function* $\mathbf u:[0,T]\to X$ *such that*

$$\langle A\dot{\mathbf u}(t),\mathbf v-\dot{\mathbf u}(t)\rangle_X + \langle B\mathbf u(t),\mathbf v-\dot{\mathbf u}(t)\rangle_X + j(\mathbf v) - j(\dot{\mathbf u}(t)) \ge \langle \boldsymbol f(t),\mathbf v-\dot{\mathbf u}(t)\rangle_X\,, \qquad (2.26)$$

$$\mathbf u(0)=\mathbf u_0. \qquad (2.27)$$

for all $\mathbf{v} \in X$, $t \in [0,T]$. Note that Problem $\mathcal{P}_V$ represents an evolutionary variational inequality. Its unique solvability will presented in the next section. Here we restrict ourselves to mention that the solution of this inequality will be called a *weak solution* to Problem $\mathcal{P}$. We also mention that in Section 4 we provide a second variational formulation of Problem $\mathcal{P}$, the so-called dual variational formulation, which, in fact, is equivalent with Problem $\mathcal{P}_V$.

Our existence and uniqueness result in the study of Problem $\mathcal{P}_V$ is the following.

Theorem 2.1. Assume (2.5)–(2.9). Then Problem $\mathcal{P}_V$ has a unique solution with regularity $\mathbf{u} \in C^1([0,T];X)$.

The proof is carried out in several steps. The first one consists to investigate the properties of the operators A and B and, with this concern, we have the following results.

Lemma 2.2. Assume that (2.5) holds. Then the operator A is linear, symmetric continuous and coercive, i.e. it satisfies

$$\langle A\mathbf{v},\mathbf{v}\rangle_X \geq m_A \|\mathbf{v}\|_X^2 \qquad \text{for all } \mathbf{v} \in X, \quad \text{with } \ m_A > 0. \tag{2.28}$$

Lemma 2.3. Assume that (2.5) holds. Then the operator B is linear, symmetric and coercive, i.e. it satisfies

$$\langle B\mathbf{v},\mathbf{v}\rangle_X \geq m_B \|\mathbf{u}\|_X^2 \qquad \text{for all } \mathbf{v} \in X, \quad \text{with } \ m_B > 0. \tag{2.29}$$

The proof of Lemmas 2.2 and 2.3 are identical and are based on standard arguments. Nevertheless, for the convenience of the reader we present, for instance, the proof of Lemma 2.2.

Proof . The linearity and symmetry of the operator A are obvious. Moreover, an elementary computation shows that

$$\langle A\mathbf{v},\mathbf{v}\rangle_X \leq (\lambda+2\mu)\,\|\mathbf{u}\|_X\|\mathbf{v}\|_X \qquad \forall\, \mathbf{u},\, \mathbf{v} \in X. \tag{2.30}$$

which implies that A is continuous. Inequality (2.28) is a direct consequence of the two-dimensional version of Korn's inequality. Indeed, consider an arbitrary element $\mathbf{v} = (\psi(x,y), \varphi(x)) \in X$. Then, the small strain tensor associated to the two-dimensional displacement field $\mathbf{v}$ is given by

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \begin{pmatrix} \psi_x & \frac{1}{2}\left(\psi_y+\varphi_x\right) \\ \frac{1}{2}\left(\psi_y+\varphi_x\right) & 0 \end{pmatrix}.$$

We have

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2 = \boldsymbol{\varepsilon}(\mathbf{v})\cdot\boldsymbol{\varepsilon}(\mathbf{v}) = \psi_x^2 + \frac{1}{2}(\psi_y+\varphi_x)^2 \qquad \text{a.e. on } \Omega. \tag{2.31}$$

Note also that the function $\mathbf{v}$ vanishes on the boundary $x = 0$ of the rectangle Ω which is, obviously, of positive one-dimensional measure and, in addition, since X can be identified as a subspace of $H^1(\Omega)^2$, we have $\mathbf{v} \in H^1(\Omega)^2$. Therefore, using Korn's inequality we obtain that there exists a constand $c_K > 0$ which depends on h such that

$$\iint_\Omega \|\boldsymbol{\varepsilon}(\mathbf{v})\|^2\, dxdy \geq c_K\, \|\mathbf{v}\|^2_{H^1(\Omega)^2}. \tag{2.32}$$

We now combine (2.31) and (2.32) to deduce that

$$\iint_\Omega \left(\psi_x^2 + \frac{1}{2}(\psi_y+\varphi_x)^2\right) dxdy \geq c_K \iint_\Omega \left(\psi^2+\psi_x^2+\psi_y^2+\varphi^2+\varphi_x^2\right) dxdy$$

and then, using (2.2)–(2.4), we obtain that

$$\iint_\Omega \left(\psi_x^2 + \frac{1}{2}(\psi_y+\varphi_x)^2\right) dxdy \geq \widetilde{c}_K \|\mathbf{v}\|_X^2. \tag{2.33}$$

where $\widetilde{c}_K$ depends on c_K and L. On the other hand, using the definition (2.10) of the operator A and inequality (2.33) we deduce that

$$\langle A\mathbf{v},\mathbf{v}\rangle_X \geq \min(\lambda, 2\mu) \iint_\Omega \left(\psi_x^2 + \frac{1}{2}(\psi_y+\varphi_x)^2\right) dxdy. \tag{2.34}$$

We now combine (2.33), (2.34) and assumption (2.5) to see that inequality (2.28) holds with $m_A = \widetilde{c}_K \min(\lambda, 2\mu) > 0$, which concludes the proof. □

We are now in a position to provide the proof of Theorem 2.1.

Proof . Using assumption (2.8) it is easy to see that the functional j is a continuous seminorm on the space X. Therefore, it follows from here that j is a convex lower semicontinuous function on X. In addition, assumptions (2.6), (2.7) and definition (2.13) imply that $\boldsymbol{f} \in C([0,T];X)$. Moreover, assumption (2.9) shows that the initial data satisfy $\mathbf{u}_0 \in V$. Finally, Lemma 2.2 shows that $A : X \to X$ is a strongly monotone Lipschitz continuous operator and Lemma 2.3 implies that $B : X \to X$ is Lipschitz continuous operator. Theorem 2.1 is now a direct consequence of Theorem 11.3 in [9]. □

3 A continuous dependence result

In this section we study the dependence of the solution with respect the parametres E, G and g. To this end we assume that (2.5)–(2.9) hold and we consider some positive constants E_ρ, G_ρ and g_ρ which represent a perturbation of E, G and g, respectively. Here ρ denotes a positive parameter which will converge to zero. We define the operator B_ρ and the function j_ρ by equalities

$$\langle B_\rho \mathbf{u}, \mathbf{v}\rangle_X = E_\rho \iint_\Omega u_x \psi_x \, dxdy + G_\rho \iint_\Omega (u_y + w_x)(\psi_y + \varphi_x)\, dxdy, \tag{3.1}$$

$$j_\rho(\mathbf{v}) = g_\rho \int_0^L |\varphi(x)|\, dx \tag{3.2}$$

for all $\mathbf{u} = (u, w)$, $\mathbf{v} = (\psi, \varphi) \in X$. Then, we consider the following variational problem.

Problem $\mathcal{P}_V^\rho$. *Find a function* $\mathbf{u}_\rho : [0,T] \to X$ *such that*

$$\langle A\dot{\mathbf{u}}_\rho(t), \mathbf{v} - \dot{\mathbf{u}}_\rho(t)\rangle_X + \langle (B_\rho \mathbf{u}_\rho)(t), \mathbf{v} - \dot{\mathbf{u}}_\rho(t)\rangle_X + j_\rho(\mathbf{v}) - j_\rho(\dot{\mathbf{u}}_\rho(t)) \geq \langle \boldsymbol{f}(t), \mathbf{v} - \dot{\mathbf{u}}_\rho(t))\rangle_X \tag{3.3}$$

$$\mathbf{u}_\rho(0) = \mathbf{u}_0, \tag{3.4}$$

for all $\mathbf{v} \in X, t \in [0,T]$. Using Theorem 2.1 it follows that Problem $\mathcal{P}_V$ has a unique solution $\mathbf{u} \in C^1(0,T;X)$ and, in addition, Problem $\mathcal{P}_V^\rho$ has a unique solution $\mathbf{u}_\rho \in C^1([0,T];X)$. Our main result in this section is the following.

Theorem 3.1. Assume (2.5)–(2.9) and, moreover, assume that

$$E_\rho \to E, \quad G_\rho \to G, \quad g_\rho \to g \quad \text{as} \quad \rho \to 0. \tag{3.5}$$

Then the solution $\mathbf{u}_\rho$ of problem $\mathcal{P}_V^\rho$ converges to the solution $\mathbf{u}$ of the problem $\mathcal{P}_V$ *i.e*

$$\mathbf{u}_\rho \longrightarrow \mathbf{u} \quad \text{in} \quad C^1([0,T];X) \quad \text{as} \quad \rho \to 0. \tag{3.6}$$

Proof . Let $\rho > 0$ and let $t \in [0,T]$ be given. We use inequalities (2.26) and (3.3) to deduce that

$$\langle A\dot{\mathbf{u}}(t), \dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\rangle_X + \langle B\mathbf{u}(t), \mathbf{u}_\rho(t) - \dot{\mathbf{u}}(t)\rangle_X + j(\dot{\mathbf{u}}_\rho(t)) - j(\dot{\mathbf{u}}(t)) \geq \langle \boldsymbol{f}(t), \dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t))\rangle_X,$$
$$\langle A\dot{\mathbf{u}}_\rho(t), \dot{\mathbf{u}}(t) - \dot{\mathbf{u}}_\rho(t)\rangle_X + \langle B_\rho\mathbf{u}_\rho(t), \dot{\mathbf{u}}(t) - \dot{\mathbf{u}}_\rho(t)\rangle_X + j_\rho(\dot{\mathbf{u}}(t)) - j_\rho(\dot{\mathbf{u}}_\rho(t)) \geq \langle \boldsymbol{f}(t), \dot{\mathbf{u}}(t) - \dot{\mathbf{u}}_\rho(t))\rangle_X.$$

We now add these inequalities and use the property (2.28) of the operator A to obtain that

$$m_A\|\dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\|_X^2 \leq \langle B_\rho\mathbf{u}_\rho(t) - B\mathbf{u}(t), \dot{\mathbf{u}}(t) - \dot{\mathbf{u}}_\rho(t)\rangle_X + j_\rho(\dot{\mathbf{u}}(t)) - j_\rho(\dot{\mathbf{u}}_\rho(t)) + j(\dot{\mathbf{u}}_\rho(t)) - j(\dot{\mathbf{u}}(t)). \tag{3.7}$$

Next, we use the definitions (3.2) and (2.12) to see that

$$j_\rho(\dot{\mathbf{u}}(t)) - j_\rho(\dot{\mathbf{u}}_\rho(t)) + j(\dot{\mathbf{u}}_\rho(t)) - j(\dot{\mathbf{u}}(t)) \leq c\,|g_\rho - g|\,\|\dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\|_X \tag{3.8}$$

where, here and below, c represents a constant wich does not depend on ρ and whose value may change from line to line. We now combine inequalities (3.7) and (3.8) to find that

$$m_A\|\dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\|_X \leq \|B_\rho\mathbf{u}_\rho(t) - B\mathbf{u}(t)\|_X + c\,|g_\rho - g| \tag{3.9}$$

On the other hand, using definitions (2.11) and (3.1) it is easy to see that

$$\|B_\rho \mathbf{u}_\rho(t) - B\mathbf{u}(t)\|_X \leq (E_\rho + G_\rho)\|\mathbf{u}_\rho(t) - \mathbf{u}(t)\|_X + (|E_\rho - E| + |G_\rho - G|)\|\mathbf{u}(t)\|_X. \tag{3.10}$$

It follows now from assumption (3.5) that $E_\rho + G_\rho \leq c$ and, therefore, inequalities (3.9), (3.10) imply

$$\|\dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\|_X \leq c\,\|\mathbf{u}_\rho(t) - \mathbf{u}(t)\|_X + (|E_\rho - E| + |G_\rho - G|) \max_{r\in[0,T]} \|\mathbf{u}(r)\|_X + c\,|g_\rho - g|. \tag{3.11}$$

Next, we use the initial conditions (2.27) and (3.3) to see that

$$\|\mathbf{u}_\rho(t) - \mathbf{u}(t)\|_X \leq \int_0^t \|\dot{\mathbf{u}}_\rho(s) - \dot{\mathbf{u}}(s)\|_X \, ds, \tag{3.12}$$

then we substitute this inequality in (3.11) and use the Gronwall's Lemma to obtain that

$$\|\dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\|_X \leq c\,(|E_\rho - E| + |G_\rho - G|) \max_{r\in[0,T]} \|\mathbf{u}(r)\|_X + \,|g_\rho - g|). \tag{3.13}$$

The convergence (3.6) follows now from inequalities (3.12), (3.13) and assumption (3.5). □

4 An optimal control problem

We now turn to an optimal control problem associated to Problem $\mathcal{P}_V$ [17]. and, to this end, we assume that (2.5)–(2.9) hold. We know that $\Omega = (0, L) \times (-h, h)$ such that $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ et $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3 = \varnothing$, and $W \subset V \subset X$. We consider $Z = (L^2(\Omega))^2 \times (L^2(0, L))^2 \subset (L^2(\Omega))^4$, and we note $V \times Z$ the Hilbert space undowed by the canonical inner product.
Let $M > 0$ and $\mathbf{u}_0 \in V$ a given element. Also, we define the set of admissible pairs $\mathcal{V}_{ad} \subset X \times Z$ by :

$$\mathcal{V}_{ad} = \{(\mathbf{u}, \boldsymbol{f}) \in V \times Z, \quad \text{such that (2.26) holds}\}. \tag{4.1}$$

We consider the cost functional $J\colon V \times Z \longrightarrow \mathbb{R}$ defined by :

$$J(\mathbf{u}, \boldsymbol{f}) = \frac{1}{2}\|\mathbf{u} - \mathbf{u}_0\|_V^2 + \frac{M}{2}\|\, \boldsymbol{f}\|_Z^2, \quad \forall\, M > 0. \tag{4.2}$$

for all $\mathbf{u} = (u, w) \in V$ and $\boldsymbol{f} = (q_B, f_B, q_N, f_N) \in Z$,
Then, the optimal control problem we study in this section is the following.
Problème $\mathcal{O}$. Find the couple $(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}) \in \mathcal{V}_{ad}$ such that :

$$J(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}) = \min_{(\mathbf{u},\boldsymbol{f})\in\mathcal{V}_{ad}} J(\mathbf{u}, \boldsymbol{f}), \tag{4.3}$$

An element $(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op})$ which solves Problem $\mathcal{O}$ is called an *optimal pair* and the corresponding traction $\boldsymbol{f}_{op}$ is called an *optimal control*.

Our first result in the study of Problem $\mathcal{O}$ is the following.

Theorem 4.1. Assume that (2.5)-2.9) hold. Then, there exists at least one solution $(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}) \in \mathcal{V}_{ad}$ of Problem $\mathcal{O}$.

Proof . Let

$$\omega = \inf_{(\mathbf{u},\boldsymbol{f})\in\mathcal{V}_{ad}} J(\mathbf{u}, \boldsymbol{f}) \in \mathbb{R}, \tag{4.4}$$

and let $\{(\mathbf{u}_n, \boldsymbol{f}_n)\} \subset \mathcal{V}_{ad}$ be a minimizing sequence for the functional J, i.e.

$$\lim_{n\to+\infty} J(\mathbf{u}_n, \boldsymbol{f}_n) = \omega. \tag{4.5}$$

We claim that the sequence $\{\, \boldsymbol{f}_n\}$ is bounded in Z. Arguing by contradiction, assume that $\{\, \boldsymbol{f}_n\}$ is not bounded in Z. Then, passing to a subsequence still denoted $\{\, \boldsymbol{f}_n\}$, we have :

$$\|\, \boldsymbol{f}_n\|_Z \to +\infty \;\text{ in }\; Z \;\text{ as }\; n \to +\infty. \tag{4.6}$$

We now use the definition (4.2) to see that

$$J(\mathbf{u}_n, \boldsymbol{f}_n) \geq \frac{M}{2} \| \boldsymbol{f}_n \|_Z^2.$$

Therefore, passing to the limit as $n \to +\infty$ in this inequality and using (4.6), we deduce that :

$$\lim_{n \to +\infty} J(\mathbf{u}_n, \boldsymbol{f}_n) = +\infty. \tag{4.7}$$

The convergences (4.5) and (4.7) lead to a contradiction, then the sequence $\{ \boldsymbol{f}_n \}$ is bounded in Z. Therefore there exists $\boldsymbol{f}_{op} \in Z$ such that, passing to a subsequence still denoted $\{ \boldsymbol{f}_n \}$, we have

$$\boldsymbol{f}_n \rightharpoonup \boldsymbol{f}_{op} \quad \text{in} \quad Z \quad \text{as} \quad n \to +\infty. \tag{4.8}$$

Let $\mathbf{u}_{\boldsymbol{f}}$ be the solution of the variational inequality (2.26) for $\boldsymbol{f} = \boldsymbol{f}_{op}$, i.e.

$$\mathbf{u}_{\boldsymbol{f}} \in V, \quad \langle A\dot{\mathbf{u}}_{\boldsymbol{f}} + B\mathbf{u}_{\boldsymbol{f}}, \mathbf{v} - \dot{\mathbf{u}}_{\boldsymbol{f}} \rangle_V + j(\mathbf{v}) - j(\dot{\mathbf{u}}_{\boldsymbol{f}}) \geq \langle \boldsymbol{f}_{op}, \mathbf{v} - \dot{\mathbf{u}}_{\boldsymbol{f}} \rangle_Z, \quad \forall \mathbf{v} \in V.$$

Then, by the defenition (4.1) of the set $\mathcal{V}_{ad}$ we have

$$(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}) \in \mathcal{V}_{ad}. \tag{4.9}$$

Moreover, using the convergence (4.8) we have

$$\mathbf{u}_n \longrightarrow \mathbf{u}_{\boldsymbol{f}} \quad \text{in} \quad X \quad \text{as} \quad n \to +\infty. \tag{4.10}$$

We now use the convergences (4.8) and (4.10) and the weakly lower semicontinuity of the functional J to deduce that

$$\lim_{n \to +\infty} \inf J(\mathbf{u}_n, \boldsymbol{f}_n) \geq J(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}). \tag{4.11}$$

It follows now from (4.5) and (4.11) that

$$\omega \geq J(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}). \tag{4.12}$$

In addition, (4.9) et (4.4) yield

$$\omega \leq J(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}). \tag{4.13}$$

We combine now inequalities (4.12) and (4.13) to see that (4.3) holds, which concludes the proof. □

The uniqueness result of the solution of Problem $\mathcal{O}$ is given by the theorem below.

Theorem 4.2. Assume that $J \colon V \times Z \longrightarrow \mathbb{R}$ is strictly convex and lower semicontinuous l.s.c, for all $(\mathbf{u}, \boldsymbol{f}) \in \mathcal{V}_{ad}$, we have

$$\lim_{\|(\mathbf{u}, \boldsymbol{f})\| \to +\infty} J(\mathbf{u}, \boldsymbol{f}) = +\infty. \tag{4.14}$$

Then there exists a unique $(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}) \in \mathcal{V}_{ad}$ solution for Problem $\mathcal{O}$, and conversely.

Proof . Let $\eta_0 = (\mathbf{u}_0, \boldsymbol{f}_0)$ fixed in $\mathcal{V}_{ad}$. We put $\eta^* = (\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op})$ then

$$K_{\eta_0} = \{ \eta^* \in \mathcal{V}_{ad}, \quad J(\eta^*) \leq J(\eta_0) \}.$$

It is easy to show that any solution of (4.3) is also solution for

$$J(\eta^*) \leq J(\omega), \quad \forall \omega = (\mathbf{u}_\omega, \boldsymbol{f}_\omega) \in K_{\eta_0}, \tag{4.15}$$

and coversely. Therefor, we have J is strictly convex and (4.14) holds, K_{η_0} is convex and bounded in a Banach X. It is weakly closed because J is l.s.c. for the weak topology. Then K_{η_0} is weakly compact. We deduce that there exists an element $\eta^* \in K_{\eta_0}$ which realizes the lower bound of $J \colon K_{\eta_0} \to \mathbb{R}$. The strict convexity of J results in the uniqueness of the lower bound. □

We now investigate the dependence of the optimal pair $(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op})$. Assume that (2.5)-(2.9) hold. (3.1) and (3.2) represents the perturbations of the operator B and the function j repectively. And we define the perturbed set of admisible pairs by

$$\mathcal{V}_{ad}^{\rho} = \{(\mathbf{u}_\rho, \boldsymbol{f}) \in V \times Z \colon \langle A\dot{\mathbf{u}}_\rho + B_\rho \mathbf{u}_\rho, \mathbf{v}_\rho - \dot{\mathbf{u}}_\rho\rangle_X + j_\rho(\mathbf{v}) - j_\rho(\dot{\mathbf{u}}_\rho) \geq \langle \boldsymbol{f}, \mathbf{v}_\rho - \dot{\mathbf{u}}_\rho\rangle_Z, \quad \forall \mathbf{v}_\rho \in X\},$$

Then, we consider the following perturbed optimal control problem.

Problème $\mathcal{O}^\rho$. Find $(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho}) \in \mathcal{V}_{ad}^{\rho}$ such that

$$J(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho}) = \min_{(\mathbf{u}_\rho, \boldsymbol{f}) \in \mathcal{V}_{ad}^{\rho}} J(\mathbf{u}_\rho, \boldsymbol{f}). \tag{4.16}$$

It follows from Theorem 4.1 and Theorem 4.2 that, for each $\rho > 0$, Problem $\mathcal{O}^\rho$ has at least one solution $(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho}) \in \mathcal{V}_{ad}^{\rho}$. Moreover, we have the following convergence result.

Theorem 4.3. Let $\{(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho})\}$ be a sequence of solutions of Problem $\mathcal{O}^\rho$ and assume that (3.5) holds. Then, there exists a subsequence of the sequence $\{(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho})\}$, again denoted $\{(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho})\}$, and a solution $(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op})$ of Problem $\mathcal{O}$, such that

$$\mathbf{u}_{\boldsymbol{f}_\rho} \longrightarrow \mathbf{u}_{\boldsymbol{f}} \ \text{ in } \ X \ \text{ et } \ \boldsymbol{f}_{op_\rho} \rightharpoonup \boldsymbol{f}_{op} \ \text{ dans } \ Z \ \text{ as } \ \rho \to 0. \tag{4.17}$$

Proof . Let $\rho > 0$ and denote $\mathbf{u}_{\boldsymbol{f}_\rho} = (u_{\boldsymbol{f}_\rho}, w_{\boldsymbol{f}_\rho})$, $\boldsymbol{f}_{op_\rho} = (q_{Bop_\rho}, f_{Bop_\rho}, q_{Nop_\rho}, f_{Nop_\rho})$. We use the definition (4.2) of the functional J to obtain

$$J(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho}) \geq \frac{M}{2} \| \boldsymbol{f}_{op_\rho}\|_Z^2 \Longleftrightarrow \| \boldsymbol{f}_{op_\rho}\|_Z^2 \leq \frac{2}{M} J(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho}). \tag{4.18}$$

and, since $(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho})$ is a solution of Problem $\mathcal{O}^\rho$, we have

$$\| \boldsymbol{f}_{op_\rho}\|_Z^2 \leq \frac{2}{M} J(\mathbf{u}_\rho, \boldsymbol{f}), \quad \forall\, (\mathbf{u}_\rho, \boldsymbol{f}) \in \mathcal{V}_{ad}. \tag{4.19}$$

Next, since $A\mathbf{0}_X = \mathbf{0}_X$, it follows that $\mathbf{u}_\rho = \mathbf{0}_X$ is a solution of Problem $\mathcal{P}_V^\rho$ we have for $\boldsymbol{f}_\rho = \mathbf{0}_Z$. and, on the other hand, it is easy to see that

$$J(\mathbf{0}_X, \mathbf{0}_Z) = \frac{1}{2}\|\mathbf{u}_0\|_X^2. \tag{4.20}$$

We now take $(\mathbf{u}_\rho, \boldsymbol{f}) = (\mathbf{0}_X, \mathbf{0}_Z)$ in (4.19) then use (4.20) to see that the sequence $\{ \boldsymbol{f}_{op_\rho}\}$ is bounded in Z. Therefore, passing to a subsequence again denoted $\{ \boldsymbol{f}_{op_\rho}\}$, it follows that there exists $\boldsymbol{f}_{op} \in Z$ such that

$$\boldsymbol{f}_{op_\rho} \rightharpoonup \boldsymbol{f}_{op} \ \text{ in } \ Z \ \text{ as } \ \rho \to 0. \tag{4.21}$$

Denote by $\mathbf{u}_{\boldsymbol{f}}$ the solution of Problem $\mathcal{P}_V$ for $\boldsymbol{f} = \boldsymbol{f}_{op}$. Then, we have

$$(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}) \in \mathcal{V}_{ad}, \tag{4.22}$$

and, moreover, Theorem 3.1 yields

$$\mathbf{u}_{\boldsymbol{f}_\rho} \longrightarrow \mathbf{u}_{\boldsymbol{f}} \ \text{ in } \ X \ \text{ as } \ \rho \longrightarrow 0. \tag{4.23}$$

We now prove that $(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op})$ is a solution to the optimal control problem $\mathcal{O}$. To this end we use the convergences (4.21), (4.23) and the weakly lower semicontinuity of the functional J to see that

$$\liminf_{\rho \to 0} J(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho}) \geq J(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}). \tag{4.24}$$

Next, we fix a solution $(\widetilde{\mathbf{u}}_{\boldsymbol{f}}, \widetilde{\boldsymbol{f}}_{op})$ for Problem $\mathcal{O}$ and, for each $\rho > 0$ we denote by $\widetilde{\mathbf{u}}_\rho$ the solution of Problem $\mathcal{P}_V^\rho$ for $\boldsymbol{f}_\rho = \widetilde{\boldsymbol{f}}_{op}$ It follows from here that $(\widetilde{\mathbf{u}}_\rho, \widetilde{\boldsymbol{f}}_{op}) \in \mathcal{V}_{ad}^\rho$. and, by the optimality of the pair $(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho})$ du Problème $\mathcal{O}^\rho$, we have that

$$J(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho}) \leq J(\widetilde{\mathbf{u}}_\rho, \widetilde{\boldsymbol{f}}_{op}), \quad \forall\, \rho > 0.$$

We pass to the upper limit in this inequality to see that

$$\limsup_{\rho \longrightarrow 0} J(\mathbf{u}_{\boldsymbol{f}_\rho}, \boldsymbol{f}_{op_\rho}) \leq \limsup_{\rho \longrightarrow 0} J(\widetilde{\mathbf{u}}_\rho, \widetilde{\boldsymbol{f}}_{op}). \tag{4.25}$$

Now, remember that $\widetilde{\mathbf{u}}_{\boldsymbol{f}}$ is a solution of Problem $\mathcal{P}_V$ for $\boldsymbol{f} = \widetilde{\boldsymbol{f}}_{op}$ and $\widetilde{\mathbf{u}}_\rho$ is a solution of Problem $\mathcal{P}_V^\rho$ for $\boldsymbol{f}_\rho = \widetilde{\boldsymbol{f}}_{op}$. Therefore, assumption (3.5) allows to use Theorem 3.1. As a result, we deduce that

$$\widetilde{\mathbf{u}}_\rho \longrightarrow \widetilde{\mathbf{u}}_{\boldsymbol{f}} \quad \text{in} \quad X \quad \text{as} \quad \rho \longrightarrow 0.$$

and, therefore, the continuity of the functional $J\colon X \longrightarrow \mathbb{R}$ telle que $\mathbf{u} \longmapsto J(\mathbf{u}, \widetilde{\boldsymbol{f}}_{op})$ yields

$$\lim_{\rho\to 0} J(\widetilde{\mathbf{u}}_\rho, \widetilde{\boldsymbol{f}}_{op}) = J(\widetilde{\mathbf{u}}_{\boldsymbol{f}}, \widetilde{\boldsymbol{f}}_{op}). \tag{4.26}$$

We now combine (4.24)-(4.26) to see that

$$J(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}) \leq J(\widetilde{\mathbf{u}}_{\boldsymbol{f}}, \widetilde{\boldsymbol{f}}_{op}). \tag{4.27}$$

On the other hand, since $(\widetilde{\mathbf{u}}_{\boldsymbol{f}}, \widetilde{\boldsymbol{f}}_{op})$ is a solution of Problem $\mathcal{O}$, inclusion (4.22) yields

$$J(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}) \geq J(\widetilde{\mathbf{u}}_{\boldsymbol{f}}, \widetilde{\boldsymbol{f}}_{op}). \tag{4.28}$$

We now combine (4.26)-(4.28), to see that

$$J(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}) = J(\widetilde{\mathbf{u}}_{\boldsymbol{f}}, \widetilde{\boldsymbol{f}}_{op}). \tag{4.29}$$

Therefore

$$(\mathbf{u}_{\boldsymbol{f}}, \boldsymbol{f}_{op}) \text{ is a solution of Problem } \mathcal{O}. \tag{4.30}$$

Theorem 4.3 is now a consequence of (4.21), (4.23) et (4.30). □

5 Conclusion

The purpose of this paper is to introduce the reader a mathematical model which arise in Contact Mechanics. Our aim is: first, to present a sound and rigorous description of the way in which the mathematical model is constructed; second, to present the mathematical analysis of this model which includes the variational formulation, existence, uniqueness and convergence results. To this end, we used results on various classes of variational inequalities in Hilbert spaces. Also, we used various functional methods, including monotonicity, compactness, penalization, regularization and duality methods. Moreover, we paid a particular attention to the mechanical interpretation of our results. On the other hand, we associated an control problem for which we proved an optimal pair.

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