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Approximation properties in fuzzy cone normed linear space

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Abstract

The main purpose of this paper is to consider the best approximation in fuzzy cone normed linear space and study its related results. We introduce quotient fuzzy cone normed linear space and proved some results of approximation in such spaces. We also discuss the relation in proximity and Chebyshevity of a given space and its quotient space.

Keywords: fuzzy cone norm, best c-approximation, c-proximal, c-Chebyshev
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1 Introduction

After the introduction of fuzzy set by L.A.Zadeh[16] in 1965, the theory of fuzzy sets has become an area of active research. Several authors have developed various mathematical structures on this theory. One of the most important problems in fuzzy functional analysis is to obtain an appropriate concept of fuzzy metric and fuzzy normed spaces. It was Katsaras[8] who in 1984, first introduced the idea of fuzzy norm on a linear space. Since then many mathematicians have defined fuzzy metric and fuzzy norm on a linear space from various point of views ([1], [2], [3], [5], [7], [10]). After that many researchers developed the results of functional analysis in fuzzy setting. S.M. Vaezpour and F. Karimi [15] introduced the concept of t-best approximation in fuzzy normed linear spaces in 2008. In 2010, M. Goudarzi and S. M. Vaezpour [6] also studied best simultaneous approximation in fuzzy normed spaces.

On the other hand, in 2007, Huang and Zhang [11] re-introduced the concept of K-normed space under the name of cone metric space. They included the use of interior points of the cone and went further, defining convergent and Cauchy sequence in such spaces. Using the concept of cone introduced by Huang and Zhang, we [13] introduced the idea of fuzzy cone normed linear space which generalizes Bag and Samanta type fuzzy norm. In this paper, we consider the set of all best c-approximations on fuzzy cone normed linear space and obtained several results pertaining to the set. Idea of quotient fuzzy cone normed linear space is introduced and studied many results on best c-approximation in such spaces.

2 Preliminaries

Definition 2.1.[11] Let E be a real Banach space and P be a subset of E. P is called a cone if and only if:
(i) P is closed, nonempty and $P \neq \{\theta_E\}$; (the zero element of E)

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(ii) $a, b \in R,\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$;
(iii) $x \in P$ and $-x \in P \quad \Rightarrow x = \theta$.
Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ iff $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$ while $x << y$ will stand for $y - x \in$ Int$P$, where Int$P$ denotes the interior of P.
The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$, with $\theta_E \preceq x \preceq y$ implies $\|x\| \leq M\|y\|$. The least positive number satisfying above is called the normal constant of P.
The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence in E such that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if every decreasing sequence is bounded below is convergent. It is clear that a regular cone is a normal cone.

Definition 2.2.[4] The cone P is called strongly minihedral if every subset of E which is bounded above via the partial ordering obtained by P, must have a least upper bound. Hence, every subset which is bounded below must have greatest lower bound.
Throughout the paper, we consider a strongly minihedral cone P and $\bigwedge(\bigvee)$ denotes the infimum(supremum) respectively.

Definition 2.3.[9] A binary operation $* : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$ is a t-norm if it satisfies the following conditions:
(1) $*$ is associative and commutative;
(2) $a * 1 = a \quad \forall a \in [0, 1]$;
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0\ ,\ 1]$.

If $*$ is continuous then it is called continuous t-norm.

Definition 2.4.[14] Let X be a linear space over the field K and E be a real Banach space with cone P, $*$ is a t-norm. A fuzzy subset $N_c : X \times E \longrightarrow [0, 1]$ is said to be a fuzzy cone norm if
(FCN1) $\forall\ t \in E$ with $t \preceq \theta_E,\ N_c(x, t) = 0$;
(FCN2)($\forall\ \theta_E \prec t,\ N_c(x, t) = 1$) iff $x = \theta_X$;
(θ_X denotes the zero element of X)
(FCN3) $\forall\ \theta_E \prec t$ and $0 \neq c \in K,\ N_c(cx, t) = N_c(x, \frac{t}{|c|})$;
(FCN4) $\forall\ x, y \in X$ and $s, t \in E, \quad N_c(x + y, s + t) \geq N_c(x, s) * N_c(y, t)$;
(FCN5) $N_c(x, t) = 1$ if $s \prec t\ \forall\ s \in P$;
Then $(X, N_c, *)$ is said to be a fuzzy cone normed linear space (FCNLS) w.r.t. E.

Definition 2.5.[14] Let $(X, N_c, *)$ be a fuzzy cone normed linear space with a strongly minihedral cone P. A sequence $\{x_n\}$ is said to be α-fuzzy convergent and converges to x if for some $\alpha \in (0, 1)$,
$\lim\limits_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\} = \theta_E$.
If $\lim\limits_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\} = \theta_E$ holds $\forall \alpha \in (0, 1)$, then $\{x_n\}$ is said to be l-fuzzy convergent and converges to x.

Definition 2.6.[14] Let $(X, N_c, *)$ be a fuzzy cone normed linear space with a strongly minihedral cone P. A sequence $\{x_n\}$ is said to be α-fuzzy Cauchy sequence if for some $\alpha \in (0, 1)$, $\lim\limits_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} = \theta_E$ for each $p = 1, 2, 3, ...$
If $\lim\limits_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} = \theta_E$ holds $\forall \alpha \in (0, 1)$ and for each $p = 1, 2, 3, ...$, then $\{x_n\}$ is said to be l-fuzzy Cauchy sequence.

Definition 2.7.[14] Let $(X, N_c, *)$ be a fuzzy cone normed linear space with a strongly minihedral cone P and $\alpha \in (0, 1)$. Then X is said to be α-fuzzy complete if every α-fuzzy Cauchy sequence is α-fuzzy convergent to some element in X.

Definition 2.8.[14] Let $(X, N_c, *)$ be a fuzzy cone normed linear space with a strongly minihedral cone P and $\alpha \in (0, 1)$. Then X is said to be l-fuzzy complete if every l-fuzzy Cauchy sequence is l-fuzzy convergent.

Definition 2.9.[14] Let $(X, N_c, *)$ be a fuzzy cone normed linear space with a strongly minihedral cone P and $\alpha \in (0, 1)$. A subset M of X is said to be l-fuzzy closed, if any sequence $\{x_n\}$ in M l-fuzzy converges to x implies that $x \in M$.

Remark 2.10.[12] If A is a subset of $[0, 1]$ and sup$A = \beta$ then for $\epsilon \in [0, \beta)$ there exists α_0 in A such that $\alpha_0 \geq \beta * \epsilon$.

3 Main results

Definition 3.1. Let $(X, N_c, *)$ be a FCNLS and $A \subset X$. Then a point $x_0 \in A$ is said to be a best c-approximate point of $x \in X$ if $\forall \alpha \in (0,1)$,

$$\bigwedge\{t \succ \theta_E; N_c(x - x_0, t) \geq \alpha\} = \bigwedge_{a \in A} \bigwedge\{t \succ \theta_E; N_c(x - a, t) \geq \alpha\}$$

We denote the set of all best c-approximate points by $P_A(x)$.
Define $Q_A(x_0) = \{x \in X; x_0 \in P_A(x)\}$

Definition 3.2. If for each $x \in X$ has atleast one(exactly one) best c-approximation in $A \subset X$, then A is called a c-proximal(c-Chebyshev) set.

Example 3.3. Let $(X, \| \|)$ be a normed linear space over field K and take $E = R^2$. Then $P = \{(t_1, 0) : t_1 \geq 0\} \subset E$ is a strongly minihedral normal cone. Define a function $N_c : X \times E \longrightarrow [0,1]$ by

$$N_c(x,t) = \begin{cases} \frac{t_1}{t_1 + \|x\|}, & \text{if } t \in P \text{ i.e, for } t = (t_1, 0), \\ 0, & \text{if } t \notin P. \end{cases}$$

If we choose $*$ =min, then $(X, N_c, *)$ is a fuzzy cone normed linear space.

(i) For all $t \in E$ with $t \preceq \theta_E$, i.e, $-t \in P$. If $t \in P$ then $t = \theta_E$, So $N_c(x,t) = 0$. If $t \notin P$, then by the definition, $N_c(x,t) = 0$.

(ii) For all $t \succ \theta_E$, $N_c(x,t) = 1$. Then

$$\frac{t_1}{t_1 + \|x\|} = 1,$$

where $t = (t_1, 0) \in P$. This implies that $\|x\| = 0$. Thus, $x = \theta_X$.

(iii). $N_c(\lambda x, t) = \frac{t_1}{t_1 + \|\lambda x\|} = \frac{t_1}{t_1 + |\lambda| \|x\|} = \frac{\frac{t_1}{|\lambda|}}{\frac{t_1}{|\lambda|} + |\lambda| \|x\|} = N_c(x, \frac{t}{|\lambda|})$ where $0 \neq \lambda \in K$.

(iv) We have to show that
$N_c(x + u, s + t) \geq \min\{N_c(x,s), N_c(u,t)\}$ $\forall x, y \in X$ and $s, t \in E$.

Case I. $s + t \prec \theta_E$.

Case II. $s = \theta_E, t = \theta_E$.

Case III. $s + t \succ \theta_E, s = \theta_E, t \succ \theta_E, s \succ \theta_E, t = \theta_E$.

Case IV. $s + t \succ \theta_E, s \prec \theta_E, t \succ \theta_E, s \succ \theta_E, t \prec \theta_E$.

In all the above cases, result holds.

Case V. $s + t \succ \theta_E, s \succ \theta_E, t \succ \theta_E, s \succ \theta_E, t \succ \theta_E$. Then

$$N_c(x + u, s + t) = \frac{s_1 + t_1}{s_1 + t_1 + \|x + u\|}$$

where $s = (s_1, 0), t = (t_1, 0) \in P$. So

$$N_c(x + u, s + t) \geq \frac{s_1 + t_1}{s_1 + t_1 + \|x\| + \|u\|}.$$

Now if $N_c(x,s) \geq N_c(u,t)$, i.e, $\frac{s_1}{s_1 + \|x\|} \geq \frac{t_1}{t_1 + \|u\|}$, then $s_1\|u\| - t_1\|x\| \geq 0$. Thus

$$\begin{aligned} N_c(x + u, s + t) - N_c(u,t) &= \frac{s_1 + t_1}{s_1 + t_1 + \|x + u\|} - \frac{t_1}{t_1 + \|u\|} \\ &\geq \frac{s_1 + t_1}{s_1 + t_1 + \|x\| + \|u\|} - \frac{t_1}{t_1 + \|u\|} \\ &= \frac{s_1\|u\| - t_1\|x\|}{(s_1 + t_1 + \|x\| + \|u\|)(t_1 + \|u\|)} \geq 0. \end{aligned}$$

So, $N_c(x+u,s+t) \geq N_c(u,t)$. Similarly if $N_c(u,t) \geq N_c(x,s)$, then it can be shown that $N_c(x+u,s+t) \geq N_c(x,s)$. Thus in any cases, $N_c(x+u,s+t) \geq \min\{N_c(x,s), N_c(u,t)\}$ holds.

(v) It follows from the definition of N_c, $N_c(x,t) = 1$ if $t \succ s\ \forall\ s \in P$. Hence $(X, N_c, *)$ is a fuzzy cone normed linear space.

Example 3.4. Consider a fuzzy cone normed linear space $(X, N_c, *)$ as in Example 3.3. Take $X = R^2$ and $\|(a,b)\|$ =sup$\{|a|,|b|\}$, $(a,b) \in R^2$. Let $A = \{(a,0), a \in R\}$ and $x = (0,1) \in R^2$. Now for $\alpha \in (0,1)$,

$$\begin{aligned}
\bigwedge_{(a,0)\in A} \bigwedge\{t \succ \theta_E; N_c((0,1)-(a,0),t) \geq \alpha\} &= \bigwedge_{(a,0)\in A} \bigwedge\{t \succ \theta_E; N_c((-a,1),t) \geq \alpha\} \\
&= \bigwedge_{(a,0)\in A} \bigwedge\{(t_1,0); \frac{t_1}{t_1 + \|(-a,1)\|} \geq \alpha\} \text{ for some } t_1 > 0 \\
&= \bigwedge_{(a,0)\in A} \bigwedge\{(t_1,0); t_1 \geq \frac{\alpha}{1-\alpha}\|(-a,1)\|\} \\
&= \bigwedge_{(a,0)\in A} \{(\frac{\alpha}{1-\alpha}\sup\{|a|,1\},0)\} \\
&= (\frac{\alpha}{1-\alpha},0)
\end{aligned}$$

On the other hand, for $(u,0) \in A$ with $|u| \leq 1$, we have

$$\begin{aligned}
\bigwedge\{t \succ \theta_E; N_c((0,1)-(u,0),t) \geq \alpha\} &= \bigwedge\{t \succ \theta_E; N_c((-u,1),t) \geq \alpha\} \\
&= \bigwedge\{(t_1,0); \frac{t_1}{t_1 + \|(-u,1)\|} \geq \alpha\} \text{ where } t = (t_1,0) \in P \text{ for some } t_1 > 0 \\
&= \bigwedge\{(t_1,0); t_1 \geq \frac{\alpha}{1-\alpha}\|(-u,1)\|\} \\
&= (\frac{\alpha}{1-\alpha}\sup\{|u|,1\},0) \\
&= (\frac{\alpha}{1-\alpha},0).
\end{aligned}$$

Thus any points $(u,0) \in A$ with $|u| \leq 1$ are best c-approximate point of $(0,1)$. It is easy to see that any point $(a,b) \in R^2$ has best c-approximate points $(u,0)$ in A satisfying $|a-u| \leq |b|$. Thus A is not c-Chebyshev set but c-proximal set. If we consider $X = R^2$ with $\|(a,b)\| = \sqrt{a^2+b^2}$, $(a,b) \in R^2$. Then $(0,0)$ is the only best c-approximate point of $(0,1)$ in A. It can be shown that for any points $(a,b) \in R^2$, $(a,0)$ is the only best c-approximate point in A. Thus in this case, A is a c-Chebyshev set.

Lemma 3.5. Let $(X, N_c, *)$ be a FCNLS and $A \subset X$. Then for $x, y \in X$,

(i) $P_A(x) + y = P_{A+y}(x+y)$.

(ii) $P_{\lambda A}(\lambda x) = \lambda P_A(x)$, $0 \neq \lambda \in K$.

Proof . (i) Let $y_0 \in P_{A+y}(x+y)$. Then

$$\begin{aligned}
\bigwedge\{t \succ \theta_E; N_c(x+y-y_0,t) \geq \alpha\} &= \bigwedge_{z\in A+y} \bigwedge\{t \succ \theta_E; N_c(x+y-z,t) \geq \alpha\} \\
&= \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(x-a,t) \geq \alpha\}
\end{aligned}$$

where $z = \lambda a$ for some $a \in A$. i.e,

$$\bigwedge\{t \succ \theta_E; N_c(x-(y_0-y),t) \geq \alpha\} = \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(x-a,t) \geq \alpha\}.$$

So $y_0 - y \in P_A(x)$. Then $y_0 \in P_A(x) + y$. Thus $P_{A+y}(x+y) \subset P_A(x) + y$.

Conversely, $y_0 \in P_A(x)+y$. Then $y_0 - y \in P_A(x)$. So

$$\begin{aligned}\bigwedge\{t \succ \theta_E; N_c(x-(y_0-y),t) \geq \alpha\} &= \bigwedge_{a\in A} \wedge\{t \succ \theta_E; N_c(x-a,t) \geq \alpha\}\\ &= \bigwedge_{a\in A} \wedge\{t \succ \theta_E; N_c(x+y-y-a,t) \geq \alpha\}\\ &= \bigwedge_{a\in A} \wedge\{t \succ \theta_E; N_c(x+y-(y+a),t) \geq \alpha\}\\ &= \bigwedge_{z\in A+y} \wedge\{t \succ \theta_E; N_c(x+y-z,t) \geq \alpha\}\end{aligned}$$

where $z = a+y$. Then $y_0 \in P_{A+y}(x+y)$. Thus $P_A(x)+y \subset P_{A+y}(x+y)$. It follows that $P_A(x)+y = P_{A+y}(x+y)$.

(ii) Let $y_0 \in P_{\lambda A}(\lambda x)$. Then

$$\begin{aligned}\bigwedge\{t \succ \theta_E; N_c(\lambda x - y_0,t) \geq \alpha\} &= \bigwedge_{z\in\lambda A} \wedge\{t \succ \theta_E; N_c(\lambda x - z,t) \geq \alpha\}\\ &= \bigwedge_{a\in A}\bigwedge\{t \succ \theta_E; N_c(\lambda x-\lambda a,t) \geq \alpha\}\\ &= \bigwedge_{a\in A}\bigwedge\{t \succ \theta_E; N_c(x-a,\frac{t}{|\lambda|}) \geq \alpha\}\\ &=|\lambda| \bigwedge_{a\in A}\bigwedge\{t \succ \theta_E; N_c(x-a,t) \geq \alpha\}\end{aligned}$$

where $z = \lambda a$ for some $a \in A$. Thus

$$\bigwedge\{t \succ \theta_E; N_c(\lambda(x-\frac{y_0}{\lambda}),t) \geq \alpha\} = |\lambda| \bigwedge_{a\in A}\bigwedge\{t \succ \theta_E; N_c(x-a,t) \geq \alpha\}.$$

This means that

$$\bigwedge\{t \succ \theta_E; N_c((x-\frac{y_0}{\lambda}),\frac{t}{|\lambda|}) \geq \alpha\} = |\lambda| \bigwedge_{a\in A}\bigwedge\{t \succ \theta_E; N_c(x-a,t) \geq \alpha\}.$$

Then

$$|\lambda|\bigwedge\{t \succ \theta_E; N_c((x-\frac{y_0}{\lambda}),t) \geq \alpha\} = |\lambda| \bigwedge_{a\in A}\bigwedge\{t \succ \theta_E; N_c(x-a,t) \geq \alpha\}$$

and

$$\bigwedge\{t \succ \theta_E; N_c((x-\frac{y_0}{\lambda}),t) \geq \alpha\} = \bigwedge_{a\in A}\bigwedge\{t \succ \theta_E; N_c(x-a,t) \geq \alpha\}.$$

So $\frac{y_0}{\lambda} \in P_A(x)$. Hence, $y_0 \in \lambda P_A(x)$. Thus $P_{\lambda A}(\lambda x) \subset \lambda P_A(x)$.

Conversely, let $w_0 \in \lambda P_A(x)$. Then $w_0 = \lambda y_0$, where $y_0 \in P_A(x)$. Since $y_0 \in P_A(x)$, we have

$$\bigwedge\{t \succ \theta_E; N_c(x-y_0,t) \geq \alpha\} = \bigwedge_{a\in A}\bigwedge\{t \succ \theta_E; N_c(x-a,t) \geq \alpha\}.$$

This means that

$$\bigwedge\{t \succ \theta_E; N_c(x-\frac{w_0}{\lambda},t) \geq \alpha\} = \bigwedge_{a\in A}\bigwedge\{t \succ \theta_E; N_c(\frac{\lambda x-\lambda a}{\lambda},t) \geq \alpha\}.$$

Thus,

$$\bigwedge\{t \succ \theta_E; N_c(\frac{\lambda x-w_0}{\lambda},t) \geq \alpha\} = \bigwedge_{a\in A}\bigwedge\{t \succ \theta_E; N_c(\frac{\lambda x-\lambda a}{\lambda},t) \geq \alpha\}.$$

Then,

$$\bigwedge\{t \succ \theta_E; N_c(\lambda x - w_0, |\lambda| t) \geq \alpha\} = \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(\lambda x - \lambda a, |\lambda| t) \geq \alpha\}.$$

This means that

$$\begin{aligned}\bigwedge\{t \succ \theta_E; N_c(\lambda x - w_0, t) \geq \alpha\} &= \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(\lambda x - \lambda a, t) \geq \alpha\} \\ &= \bigwedge_{z\in \lambda A} \bigwedge\{t \succ \theta_E; N_c(\lambda x - z, t) \geq \alpha\}\end{aligned}$$

where $z = \lambda a$, for some $a \in A$. So $w_0 \in P_{\lambda A}(\lambda x)$. Thus $\lambda P_A(x) \subset P_{\lambda A}(\lambda x)$. It follows that $P_{\lambda A}(\lambda x) = \lambda P_A(x)$. □

Definition 3.6. Let $(X, N_c, *)$ be a fuzzy cone normed linear space. A subset T of X is said to be convex if $\lambda x + (1-\lambda)y \in T$ whenever $x, y \in T$ and $0 < \lambda < 1$.

Theorem 3.7. Let $(X, N_c, *)$ be a FCNLS and $A \subset X$ is a convex set. If $*$ satisfies $a * b \geq b$ then for $x_0 \in X$, $P_A(x_0)$ is a convex set.

Proof . Let $x, y \in P_A(x_0)$. So $x, y \in A$. Then $\lambda x + (1-\lambda)y \in A, 0 < \lambda < 1$, since A is a convex set. Now,

$$\bigwedge\{t \succ \theta_E; N_c(x_0 - (\lambda x + (1-\lambda)y), t) \geq \alpha\} \succeq \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(x_0 - a, t) \geq \alpha\} \tag{3.1}$$

since $\lambda x + (1-\lambda)y \in A$. By the assumption, $N_c((y-x)\lambda, t) * N_c(x_0 - y, t) \geq N_c(x_0 - y, t)$ and

$$\{t \succ \theta_E; N_c(x_0 - y, t) \geq \alpha\} \subset \{t \succ \theta_E; N_c((y-x)\lambda, t) * N_c(x_0 - y, t) \geq \alpha\}.$$

Then

$$\begin{aligned}\bigwedge\{t \succ \theta_E; N_c(x_0 - y, t) \geq \alpha\} &\succeq \bigwedge\{t \succ \theta_E; N_c((y-x)\lambda, t) * N_c(x_0 - y, t) \geq \alpha\} \\ &\succeq \bigwedge\{t \succ \theta_E; N_c(x_0 - (\lambda x + (1-\lambda)y), t) \geq \alpha\}\end{aligned}$$

this means that

$$\bigwedge\{t \succ \theta_E; N_c(x_0 - (\lambda x + (1-\lambda)y), t) \geq \alpha\} \preceq \bigwedge\{t \succ \theta_E; N_c(x_0 - y, t) \geq \alpha\}. \tag{3.2}$$

Since $y \in P_A(x_0)$,

$$\bigwedge\{t \succ \theta_E; N_c(x_0 - y, t) \geq \alpha\} = \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(x_0 - a, t) \geq \alpha\}.$$

Using (3.2),

$$\bigwedge\{t \succ \theta_E; N_c(x_0 - (\lambda x + (1-\lambda)y), t) \geq \alpha\} \preceq \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(x_0 - a, t) \geq \alpha\}. \tag{3.3}$$

Using (3.1) and (3.3), imply that

$$\bigwedge\{t \succ \theta_E; N_c(x_0 - (\lambda x + (1-\lambda)y), t) \geq \alpha\} = \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(x_0 - a, t) \geq \alpha\}.$$

Thus, $\lambda x + (1-\lambda)y \in P_A(x_0)$. So, $P_A(x_0)$ is a convex set. □

Remark 3.8. Let $(X, N_c, *)$ be a FCNLS and $A \subset X$. Then for $w_0 \in A$, $A \cap Q_A(w_0) = \{w_0\}$.

Proof . Let $x \in A \cap Q_A(w_0)$. Then $x \in A$ and $x \in Q_A(w_0)$. So, $x \in A$ and $\bigwedge\{t \succ \theta_E; N_c(x - w_0, t) \geq \alpha\} = \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(x - a, t) \geq \alpha\}$. Since $x \in A$, $\bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(x - a, t) \geq \alpha\} = \theta_E$. It follows that $\bigwedge\{t \succ \theta_E; N_c(x - w_0, t) \geq \alpha\} = \theta_E$. Thus, for all $t \succ \theta_E, \alpha \in (0, 1), N_c(x - w_0, t) \geq \alpha$. Hence, $N_c(x - w_0, t) = 1$. It follows that $x - w_0 = \theta_X$. So we have $x = w_0$. □

Corollary 3.9. Let $(X, N_c, *)$ be a FCNLS and $A \subset X$. Then for $w_0 \in A$, $P_A(w_0) \cap Q_A(w_0) = \{w_0\}$. It follows from $P_A(w_0) \cap Q_A(w_0) \subset A \cap Q_A(w_0)$.

Theorem 3.10. Let $(X, N_c, *)$ be a FCNLS and $A \subset X$ be a subspace. Then for $w_0 \in A$, $Q_A(w_0) = w_0 + Q_A(\theta_X)$.

Proof . Let $x \in w_0 + Q_A(\theta_X)$. Then $x - w_0 \in Q_A(\theta_X)$. So

$$\begin{aligned}\bigwedge\{t \succ \theta_E; N_c(x - w_0 - \theta_X, t) \geq \alpha\} &= \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(x - w_0 - a, t) \geq \alpha\} \\ &= \bigwedge_{a^{'}\in A} \bigwedge\{t \succ \theta_E; N_c(x - a^{'}, t) \geq \alpha\},\end{aligned}$$

where $a^{'} = w_0 + a$. Thus, $x \in Q_A(w_0)$.

Conversely, let $x \in Q_A(w_0)$. Then

$$\begin{aligned}\bigwedge\{t \succ \theta_E; N_c(x - w_0, t) \geq \alpha\} &= \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(x - a, t) \geq \alpha\} \\ &= \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(x - w_0 + w_0 - a, t) \geq \alpha\} \\ &= \bigwedge_{a\in A} \bigwedge\{t \succ \theta_E; N_c(x - w_0 - (a - w_0), t) \geq \alpha\} \\ &= \bigwedge_{a^{'}\in A} \bigwedge\{t \succ \theta_E; N_c(x - w_0 - a^{'}, t) \geq \alpha\},\end{aligned}$$

where $a^{'} = a - w_0 \in A$ (since A is a subspace). Hence, $x - w_0 \in Q_A(\theta_X)$. This implies that $x \in w_0 + Q_A(\theta_X)$. Hence $Q_A(w_0) = w_0 + Q_A(\theta_X)$. □

Theorem 3.11. Let $(X, N_c, *)$ be a FCNLS and $M \subset X$ be a l-fuzzy closed subspace of X. Define $N^* : X/M \times E \longrightarrow [0, 1]$ by

$$N^*(x + M, s) = \begin{cases} \bigvee \alpha \in (0, 1), & \bigwedge_{m\in M} \bigwedge\{t \succ \theta_E; N_c(x - m, t) \geq \alpha\} \preceq s \\ 0, & (x + M, s) = (M, \theta_E). \end{cases}$$

Then N^* is a fuzzy cone norm on X/M. We call $(X/M, N^*, *)$ a quotient fuzzy cone normed linear space.

Proof . (i) N^* is well defined. For, $x + M = y + M$, $x - m_1 = y - m_2$ for some $m_1, m_2 \in M$. Thus,

$$\bigwedge_{m\in M} \bigwedge\{t \succ \theta_E; N_c(x - m, t) \geq \alpha\} = \bigwedge_{m\in M} \bigwedge\{t \succ \theta_E; N_c(y - m, t) \geq \alpha\}.$$

Then we have $N^*(x + M, s) = N^*(y + M, s)$, for all $s \succ \theta_E$.

(ii) $N^*(x + M, s) = 0$, for all $s \preceq \theta_E$ (by the definition of N^*).

(iii) $N^*(x + M, s) = 1$, for all $s \succ \theta_E$. Then

$$\bigwedge_{m\in M} \bigwedge\{t \succ \theta_E; N_c(x - m, t) \geq \alpha\} \preceq s$$

for all $s \succ \theta_E$. So

$$\bigwedge_{m\in M} \bigwedge\{t \succ \theta_E; N_c(x - m, t) \geq \alpha\} = \theta_E$$

for all $\alpha \in (0, 1)$. Then there is a sequence $\{g_n\}$ in M such that

$$\bigwedge\{t \succ \theta_E; N_c(x - g_n, t) \geq \alpha\} = \theta_E$$

as $n \to \infty$, for all $\alpha \in (0, 1)$. Then $\{g_n\}$ is l-fuzzy convergent to x. Thus, $x \in M$, since M is l-fuzzy closed. Hence, $x + M = M = 0 + M$.

Conversely, let $x + M = 0 + M = M$. Then $x \in M$ and

$$\bigwedge_{m\in M} \bigwedge\{t \succ \theta_E; N_c(x - m, t) \geq \alpha\} = \theta_E \preceq s$$

for all $s \succ \theta_E$ and $\alpha \in (0,1)$. Thus,
$$N^*(x+M,s)=1 \ \ \forall s \succ \theta_E.$$

(iv) We have to show $N^*(\lambda(x+M),s) = N^*(\lambda x+M,s) = N^*(x+M,\frac{s}{|\lambda|}),\ 0 \neq \lambda \in K$. Since for all $\alpha \in (0,1)$, $\bigwedge_{m\in M}\bigwedge\{t \succ \theta_E; N_c(\lambda x - m,t) \geq \alpha\} \preceq s$, we have $\bigwedge_{m\in M}\bigwedge\{t \succ \theta_E; N_c(x-\frac{m}{\lambda},\frac{t}{|\lambda|}) \geq \alpha\} \preceq s$. Then $\bigwedge_{m'\in M}\bigwedge\{t \succ \theta_E; N_c(x-m',\frac{t}{|\lambda|}) \geq \alpha\} \preceq s$, since M is a subspace of X, $m'=\frac{m}{\lambda} \in M$. Then

$$\bigwedge_{m'\in M}\bigwedge\{t \succ \theta_E; N_c(x-m',t) \geq \alpha\} \preceq \frac{s}{|\lambda|}.$$

Hence $N^*(\lambda(x+M),s) = N^*(x+M,\frac{s}{|\lambda|})$.

(v) Now we have to show that

$$N^*((x+M)+(y+M)),s+t) \geq N^*(x+M,s) * N^*(y+M,t), \forall s,t \in E.$$

If this is not true, then $\exists s_0, t_0 \in E$, such that

$$N^*((x+M)+(y+M)),s_0+t_0) < N^*(x+M,s_0) * N^*(y+M,t_0).$$

Choose α_0 such that

$$N^*((x+M)+(y+M)),s_0+t_0) < \alpha_0 < N^*(x+M,s_0) * N^*(y+M,t_0).$$

Since $*$ is upper semi-continuous, there exist $\alpha_1,\alpha_2 \in (0,1)$ such that $\alpha_1 * \alpha_2 > \alpha_0$ and $N^*(x+M,s_0) > \alpha_1$, $N^*(y+M,t_0) > \alpha_2$. Since $N^*(x+M,s_0) > \alpha_1$,

$$\bigwedge_{m\in M}\bigwedge\{t \succ \theta_E; N_c(x-m,t) \geq \alpha_1\} \preceq s_0.$$

Similarly, $\bigwedge_{m\in M}\bigwedge\{t \succ \theta_E; N_c(y-m,t) \geq \alpha_2\} \preceq t_0$. Thus

$$\bigwedge_{m\in M}\bigwedge\{t \succ \theta_E; N_c(x-m,t) \geq \alpha_1\} + \bigwedge_{m\in M}\bigwedge\{t \succ \theta_E; N_c(y-m,t) \geq \alpha_2\} \preceq s_0+t_0.$$

Then for some $m_1, m_2 \in M$,

$$\bigwedge\{t \succ \theta_E; N_c(x-m_1,t) \geq \alpha_1\} + \bigwedge\{t \succ \theta_E; N_c(y-m_2,t) \geq \alpha_2\} \preceq s_0+t_0.$$

Again,

$$\bigwedge\{t \succ \theta_E; N_c(x+y-(m_1+m_2),t) \geq \alpha_1 * \alpha_2 > \alpha_0\} \preceq \bigwedge\{t \succ \theta_E; N_c(x-m_1,t) \geq \alpha_1\} + \bigwedge\{t \succ \theta_E; N_c(y-m_2,t) \geq \alpha_2\}.$$

So, $\bigwedge\{t \succ \theta_E; N_c(x+y-m,t) > \alpha_0\} \preceq s_0+t_0$, where $m = m_1+m_2 \in M$. Then $\bigwedge_{m\in M}\bigwedge\{t \succ \theta_E; N_c(x+y-m,t) > \alpha_0\} \preceq s_0+t_0$. So, $N^*(x+y+M),s_0+t_0) \geq \alpha_0$, which is a contradiction. Hence for all $s,t \in E$,

$$N^*((x+M)+(y+M)),s+t) \geq N^*(x+M,s) * N^*(y+M,t).$$

□

Definition 3.12. A sequence $\{x_n+M\}$ in X/M is said to be convergent to $x+M$ if $\lim_{n\to\infty} N^*(x_n+M-(x+M)),s) = 1$ for all $s \succ \theta_E$.

Definition 3.13. A sequence $\{x_n+M\}$ in X/M is said to be a Cauchy sequence to if $\lim_{n\to\infty} N^*(x_n+M-(x_{n+p}+M)),s) = 1$, for all $s \succ \theta_E,\ p=1,2,3,\ldots$.

Theorem 3.14. If $x_n \longrightarrow x$ in $(X,N_c,*)$, then $x_n+M \longrightarrow x+M$ in $(X/M,N^*,*)$.

Proof . Let $x_n \longrightarrow x$ in $(X, N_c, *)$. Thus $\lim_{n\to\infty} N_c(x_n - x, t) = 1$, for all $t \succ \theta_E$. Then there exists a positive integer N, such that $N_c(x_n - x, t) > \alpha$, for all $t \succ \theta_E, \alpha \in (0,1), n \geq N$. Then

$$\bigwedge\{t \succ \theta_E; N_c(x_n - x, t) > \alpha\} = \theta_E$$

for all $\alpha \in (0,1),\ n \geq N$. Again, $N^*(x_n + M - (x+M), s) = \{\vee\alpha \in (0,1),\ \bigwedge_{m\in M}\bigwedge\{t \succ \theta_E; N_c(x_n - x - m, t) \geq \alpha\} \preceq s\}$. Thus for all $n \geq N$ and $\alpha \in (0,1)$,

$$\bigwedge_{m\in M}\bigwedge\{t \succ \theta_E; N_c(x_n - x - m, t) \geq \alpha\} = \theta_E \preceq s$$

for all $s \succ \theta_E$ (putting $m = \theta_X$). Hence $N^*(x_n + M - (x+M), s) = 1$ for all $s \succ \theta_E, n \geq N$. Then

$$\lim_{n\to\infty} N^*(x_n + M - (x+M), s) = 1.$$

Thus, $\{x_n + M\}$ converges to $x + M$. □

Theorem 3.15. Let M be a l-fuzzy closed subspace of X. If $(X, N_c, *)$ is complete FCNLS, then $(X/M, N^*, *)$ is also complete.

Proof . Let $\{x_n + M\}$ is a Cauchy sequence in X/M. Then there exists $\epsilon_n > 0$ such that $\epsilon_n \to 0$ and

$$N^*(x_n + M - (x_{n+1} + M)), s) \geq 1 - \epsilon_n\ \forall s \succ \theta_E. \tag{3.4}$$

Fix $y_1 \in M$, then $N^*(x_1 + y_1 - x_2 + M)), s) \geq 1 - \epsilon_1\ \ \forall s \succ \theta_E$. By Remark 2.10, there exists $\beta \in (0,1)$, such that

$$\beta \geq N^*(x_1 + y_1 - x_2 + M)), s) * 1 - \epsilon_1$$

and

$$\bigwedge_{m\in M}\bigwedge\{t \succ \theta_E; N_c(x_1 + y_1 - x_2 - m, t) \geq \beta\} \preceq s\ \ \forall s \succ \theta_E.$$

Then

$$\bigwedge_{m\in M}\bigwedge\{t \succ \theta_E; N_c(x_1 + y_1 - x_2 - m, t) \geq N^*(x_1 + y_1 - x_2 + M)), s) * 1 - \epsilon_1 \geq 1 - \epsilon_1 * 1 - \epsilon_1\} \preceq s$$

for all $s \succ \theta_E$. Then there exists $y_2 \in M$, such that

$$\bigwedge\{t \succ \theta_E; N_c(x_1 + y_1 - x_2 - y_2, t) \geq 1 - \epsilon_1 * 1 - \epsilon_1\} \preceq s$$

for all $s \succ \theta_E$. Again for $y_2 \in M$, we have $N^*(x_2 + y_2 - x_3 + M)), s) \geq 1 - \epsilon_2$ for all $s \succ \theta_E$ from (3.4). Similarly, we get

$$\wedge\{t \succ \theta_E; N_c(x_2 + y_2 - x_3 - y_3, t) \geq 1 - \epsilon_2 * 1 - \epsilon_2\} \preceq s$$

for all $s \succ \theta_E$. Proceeding in similar way, we get

$$\bigwedge\{t \succ \theta_E; N_c(x_n + y_n - (x_{n+1} + y_{n+1}), t) \geq 1 - \epsilon_n * 1 - \epsilon_n\} \preceq s$$

for all $s \succ \theta_E$. Then

$$\bigwedge\{t \succ \theta_E; N_c(x_n + y_n - (x_{n+1} + y_{n+1}), t) \geq 1 - \epsilon_n * 1 - \epsilon_n\} = \theta_E.$$

Thus for each $t \succ \theta_E$, $\ N_c(x_n + y_n - (x_{n+1} + y_{n+1}), t) \geq 1 - \epsilon_n * 1 - \epsilon_n$. This implies that

$$\lim_{n\to\infty} N_c(x_n + y_n - (x_{n+1} + y_{n+1}), t) = 1$$

and $\{x_n + y_n\}$ is a Cauchy sequence in (X, N_c).

Since $(X, N_c, *)$ is complete, there exists $x \in X$ such that $\{x_n + y_n\}$ converges to x. Since $y_n \in M$, for each n, $x_n + y_n + M = x_n + M$. Again, $x_n + y_n \longrightarrow x$ as $n \longrightarrow \infty$. Then $x_n + y_n + M \longrightarrow x + M$ as $n \longrightarrow \infty$. Hence $x_n + M \longrightarrow x + M$ as $n \longrightarrow \infty$. It follows that $(X/M, N^*, *)$ is complete. □

Theorem 3.16. Let M be a c-proximal subspace of $(X, N_c, *)$ and $W \supseteq M$ be a subspace of X and $k \in X$. If $w_0 \in P_W(k)$ then $w_0 + M \in P_{W/M}(k + M)$.

Proof . Since $w_0 \in P_W(k)$,

$$\bigwedge\{t \succ \theta_E; N_c(k - w_0, t) \geq \alpha\} = \bigwedge_{w \in W} \bigwedge\{t \succ \theta_E; N_c(k - w, t) \geq \alpha\}. \tag{3.5}$$

If $w_0 + M \notin P_{W/M}(k + M)$, then there exists $w^{'} \in W$ such that by the definition of N^*,

$$\bigwedge\{t \succ \theta_E; N^*(k - w_0 + M, t) \geq \alpha\} \succ \bigwedge\{t \succ \theta_E; N^*(k - w^{'} + M, t) \geq \alpha\} = \bigwedge_{m \in M} \bigwedge\{t \succ \theta_E; N_c(k - w^{'} - m, t) \geq \alpha\}. \tag{3.6}$$

Again, by the definition of N^*

$$\begin{aligned}\bigwedge\{t \succ \theta_E; N^*(k - w_0 + M, t) \geq \alpha\} &= \bigwedge_{m \in M} \bigwedge\{t \succ \theta_E; N_c(k - w_0 - m, t) \geq \alpha\} \\ &\preceq \bigwedge\{t \succ \theta_E; N_c(k - w_0, t) \geq \alpha\} \end{aligned} \tag{3.7}$$

for $m = \theta_X$. Now,

$$\begin{aligned}\bigwedge_{m \in M} \bigwedge\{t \succ \theta_E; N_c(k - w^{'} - m, t) \geq \alpha\} &\prec \bigwedge\{t \succ \theta_E; N^*(k - w_0 + M, t) \geq \alpha\} \quad \text{using (3.6)} \\ & preceq \bigwedge\{t \succ \theta_E; N_c(k - w_0, t) \geq \alpha\} \quad \text{using (3.7)} \\ &= \bigwedge_{w \in W} \bigwedge\{t \succ \theta_E; N_c(k - w, t) \geq \alpha\} \quad \text{using (3.5)}\end{aligned}$$

Thus,

$$\bigwedge_{m \in M} \bigwedge\{t \succ \theta_E; N_c(k - w^{'} - m, t) \geq \alpha\} \prec \bigwedge_{w \in W} \bigwedge\{t \succ \theta_E; N_c(k - w, t) \geq \alpha\}. \tag{3.8}$$

Since $M \subset W$,

$$\bigwedge_{m \in M} \bigwedge\{t \succ \theta_E; N_c(k - w^{'} - m, t) \geq \alpha\} \succeq \bigwedge_{w \in W} \bigwedge\{t \succ \theta_E; N_c(k - w^{'} - w, t) \geq \alpha\} \tag{3.9}$$

$$= \bigwedge_{w \in W} \bigwedge\{t \succ \theta_E; N_c(k - w, t) \geq \alpha\} \tag{3.10}$$

because $w^{'} + w \in W$. Using (3.8) and (3.9), we get

$$\bigwedge_{w \in W} \bigwedge\{t \succ \theta_E; N_c(k - w, t) \geq \alpha\} \prec \bigwedge_{w \in W} \bigwedge\{t \succ \theta_E; N_c(k - w, t) \geq \alpha\}$$

which is a contradiction. Hence $w_0 + M \in P_{W/M}(k + M)$. □

Corollary 3.17. Let M be a c-proximal subspace of $(X, N_c, *)$ and $W \supseteq M$ be a subspace of X. If W is a c-proximal subspace then W/M is c-proximal subspace of X/M.

Theorem 3.18. Let M be a c-proximal subspace of $(X, N_c, *)$ and $W \supseteq M$ be a subspace of X and $k \in X$. If $w_0 + M \in P_{W/M}(k + M)$ and $m_0 \in P_M(k - w_0)$, then $w_0 + m_0 \in P_W(k)$.

Proof . Since $w_0 + M \in P_{W/M}(k + M)$,

$$\begin{aligned}\bigwedge\{t \succ \theta_E; N^*(k - w_0 + M, t) \geq \alpha\} &= \bigwedge_{w + M \in W/M} \bigwedge\{t \succ \theta_E; N^*(k - w + M, t) \geq \alpha\} \\ &= \bigwedge_{w \in W} \bigwedge\{t \succ \theta_E; N^*(k - w + M, t) \geq \alpha\}.\end{aligned}$$

Using the definition of N^*,

$$\bigwedge_{m\in M}\bigwedge\{t\succ\theta_E; N_c(k-w_0-m,t)\geq\alpha\} = \bigwedge_{w\in W}\bigwedge_{m\in M}\wedge\{t\succ\theta_E; N_c(k-w-m,t)\geq\alpha\}$$
$$= \bigwedge_{w'\in W}\bigwedge\{t\succ\theta_E; N_c(k-w',t)\geq\alpha\}. \tag{3.11}$$

Since W is a subspace and $M\subset W, w+m=w'\in W$. Again, $m_0\in P_M(k-w_0)$ implies that

$$\bigwedge\{t\succ\theta_E; N_c(k-w_0-m_0,t)\geq\alpha\} = \bigwedge_{m\in M}\bigwedge\{t\succ\theta_E; N_c(k-w_0-m,t)\geq\alpha\}$$
$$= \bigwedge_{w'\in W}\bigwedge\{t\succ\theta_E; N_c(k-w',t)\geq\alpha\} \quad \text{(using (3.11))}.$$

This means that

$$\bigwedge\{t\succ\theta_E; N_c(k-(w_0+m_0),t)\geq\alpha\} = \bigwedge_{w'\in W}\bigwedge\{t\succ\theta_E; N_c(k-w',t)\geq\alpha\}.$$

Hence, $w_0+m_0\in P_W(k)$. □

Example 3.19. Consider a fuzzy cone normed linear space $(X,N_c,*)$ as in Example 3.3. Take $X=R^2$ and $\|(a,b)\|=\sup\{|a|,|b|\},\ (a,b)\in R^2$. Then $M=\{(m,0),m\in R\}$ is c-proximal subspace of R^2 and take $W=M$. Then $(1,0)\in P_W((0,1))$. Now for $\alpha\in(0,1)$,

$$\bigwedge_{w'\in W/M}\bigwedge\{s\succ\theta_E; N^*((0,1)-w'+M,s)\geq\alpha\}$$
$$= \bigwedge_{(w,0)\in W/M}\bigwedge\{s\succ\theta_E; N^*((0,1)-(w,0)+M,s)\geq\alpha\} \quad \text{since } M=W$$
$$= \bigwedge_{(w,0)\in W/M}\bigwedge\{s\succ\theta_E; N^*((-w,1)+M,s)\geq\alpha\}$$
$$= \bigwedge_{(w,0)\in W/M}\bigwedge_{(m,0)\in M}\bigwedge\{t\succ\theta_E; N_c((-w,1)-(m,0),t)\geq\alpha\} \quad \text{by the definition of } N^*$$
$$= \bigwedge_{(w,0)\in W/M}\bigwedge_{(m,0)\in M}\bigwedge\{t\succ\theta_E; N_c((-w-m,1),t)\geq\alpha\}$$
$$= \bigwedge_{(w,0)\in W/M}\bigwedge_{(m,0)\in M}\bigwedge\{(t_1,0); \frac{t_1}{t_1+\|(-w-m,1)\|}\geq\alpha\} \quad \text{where } \ t=(t_1,0)\in P \text{ for some } t_1>0$$
$$= \bigwedge_{(w,0)\in W/M}\bigwedge_{(m,0)\in M}\bigwedge\{(t_1,0); t_1\geq\frac{\alpha}{1-\alpha}\|(-w-m,1)\|\}$$
$$= \bigwedge_{(w,0)\in W/M}\bigwedge_{(m,0)\in M}\{(\frac{\alpha}{1-\alpha}\|(-w-m,1)\|,0)\}$$
$$= \bigwedge_{(w,0)\in W/M}\bigwedge_{(m,0)\in M}\{(\frac{\alpha}{1-\alpha}\sup\{|w+m|,1\},0)\}$$
$$= (\frac{\alpha}{1-\alpha},0).$$

Again,

$$\begin{aligned}\bigwedge\{s\succ\theta_E;N^*((0,1)-(1,0)+M,s)\geq\alpha\} &= \bigwedge\{s\succ\theta_E;N^*((-1,1)+M,s)\geq\alpha\}\\ &= \bigwedge_{(m,0)\in M}\bigwedge\{t\succ\theta_E;N_c((-1,1)-(m,0),t)\geq\alpha\} \quad \text{by the definition of } N^*\\ &= \bigwedge_{(m,0)\in M}\bigwedge\{t\succ\theta_E;N_c((-1-m,1),t)\geq\alpha\}\\ &= \bigwedge_{(m,0)\in M}\bigwedge\{(t_1,0);t_1\geq\frac{\alpha}{1-\alpha}\|(-1-m,1)\|\}\\ &= \bigwedge_{(m,0)\in M}\{(\frac{\alpha}{1-\alpha}\sup\{|1+m|,1\},0)\}\\ &= (\frac{\alpha}{1-\alpha},0).\end{aligned}$$

Thus $(1,0)+M=M\in P_{W/M}((0,1)+M)$. Also $(2,0)+M\in P_{W/M}((0,1)+M)$ but $(2,0)\notin P_W((0,1))$. Now consider $P_M((0,1)-(2,0))=P_M((-2,1))$. Then from Example 3.4, for any $m\in R$ satisfying $|-2-m|\leq 1$, $(m,0)\in P_M((-2,1))$. Since $|m+2|\leq 1$, $(2,0)+(m,0)=(m+2,0)\in P_W((0,1))$.

Corollary 3.20. Let M be a c-proximal subspace of $(X,N_c,*)$ and $W\supseteq M$ be a subspace of X. If W/M is c-proximal in X/M, then W is c-proximal in X.

Theorem 3.21. Let W and M be a subspace of FCNLS $(X,N_c,*)$. If M is c-proximal in X and W is c-Chebyshev in X, then W/M is c-Chebyshev in X/M.

Proof . Suppose W/M is not c-Chebyshev in X/M. Then $k+M, k\in X$ has two distinct best c-approximation such as $w+M$ and $w^{'}+M$ in W/M, i.e, $w+M\in P_{W/M}(k+M)$ and $w^{'}+M\in P_{W/M}(k+M)$. Thus $w-w^{'}$ is not in M. Since M is c-proximal in X, there exists a best c-approximation m and $m^{'}$ to $k-w$ and $k-w^{'}$ from M respectively, i.e, $m\in P_M(k-w)$ and $m^{'}\in P_M(k-w^{'})$. By the Theorem 3.16, $w+m\in P_W(k)$ and $w^{'}+m^{'}\in P_W(k)$. Since W is c-Chebyshev in X, $w+m=w^{'}+m^{'}$. Thus $w-w^{'}=m^{'}-m\in M$, which is a contradiction. Hence W/M is c-Chebyshev in X/M. □

References

[1] T. Bag and S.K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), no. 3, 687–705.

[2] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, Ann. Fuzzy Math. Inf. **6** (2013), no 2, 271–283.

[3] S.C. Cheng and J.N. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), no. 5, 429–436.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.

[5] C. Felbin, *Finite dimensional fuzzy normed linear spaces*, Fuzzy Sets Syst. **48** (1992), 239–248.

[6] M. Goudarzi and S.M. Vaezpour, *Best simultaneous approximatıon in fuzzy normed spaces*, Iran. J. Fuzzy Syst. **7** (2010), no. 3, 87–96.

[7] O. Kaleva and S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets Syst. **12** (1984), 215–229.

[8] A.K. Katsaras, Fuzzy topological vector spaces, Fuzzy Sets Syst. **12** (1984), 143–154.

[9] George J. Klir and B. O. Yuan, Fuzzy Sets and Fuzzy Logic, Printice-Hall of India Private 353 Limited, New Delhi, 1997.

[10] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.

[11] H. Long-Guang, Z. Xian, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. **332** (2007) 1468-1476.

[12] R. Saadati and S.M. Vaezpour, *Some results on fuzzy Banach spaces*, J. Appl. Math. Comput. **17** (2005), 475-484.

[13] P. Tamang and T. Bag, *Some results on finite dimensional fuzzy cone normed linear space*, Ann. Fuzzy Math. Inf. **13** (2017), no. 1, 123–134.

[14] P. Tamang and T. Bag, *Some fixed point results in fuzzy cone normed linear space*, J. Egypt. Math. Soc. **27** (2019), 46.

[15] S.M. Vaezpour and F. Karimi, *t-best approximation in fuzzy normed spaces*, Iran. J. Fuzzy Syst. **5** (2008), No. 2, 93–99.

[16] L.A. Zadeh, *Fuzzy sets*, Inf. Control **8** (1965), 338–353.