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New approaches for solving Caputo time-fractional nonlinear system of equations describing the unsteady flow of a polytropic gas

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Abstract

The main object of this manuscript is to achieve the solutions of a time-fractional nonlinear system of equations describing the unsteady flow of a polytropic gas using two different approaches based on the combination of new general integral transform in the sense of Caputo fractional derivative and homotopy perturbation method and variational iteration method, respectively. The solutions are obtained in the form of rapidly convergent infinite series with easily computable terms. Numerical results reveal that the proposed approaches are very effective and simple to obtain approximate and analytical solutions for nonlinear systems of fractional partial differential equations.

Keywords: Systems of nonlinear time-fractional partial differential equations, Caputo fractional derivative, new general integral transform, homotopy perturbation method, variational iteration method
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1 Introduction

Recently, the theory and applications of fractional calculus have expanded greatly during the 19th and 20th centuries, and many researchers have given definitions for fractional derivatives and integrals to formulate many nonlinear problems that occur in engineering and applied sciences. However, mathematical modeling of real-world problems often leads to systems of nonlinear fractional partial differential equations, including acoustics and thermal systems, rheology and mechanical systems, signal processing and systems identification, control and robotics systems, nonlinear biology systems and other areas of applications. In order to develop engineering and applied sciences, it is necessary to study analytical and numerical methods to solve all available problems. Most systems of nonlinear fractional partial differential equations do not have explicit solutions expressible in finite terms; even if a solution can be found, it is often too complicated to clearly display the main characteristics of the solution. Due to these difficulties, one of the most time-consuming and difficult tasks appear among the researchers of nonlinear fractional problems [1, 2, 3, 8, 12, 14, 16, 17, 18, 19, 21].

Several mathematical methods have been developed to obtain exact and approximate analytical solutions. Among these methods, the homotopy perturbation method [7] and the variational iteration method [22] are the most clear

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methods of solution of systems of nonlinear fractional partial differential equations, because they provide immediate and visible symbolic terms of analytical solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization.

The polytropic gas in astrophysics is defined by [4]

$$w = k\rho^{1+\frac{1}{n}}, \tag{1.1}$$

where

$$\rho = \frac{\theta}{\varkappa}, \tag{1.2}$$

is the energy density, $\varkappa$ is the container volume, θ is the total energy of the gas, n is the polytropic index, and k is a constant. Degenerate adiabatic gas and electron gas are two instances of such gases. The study of polytropic gases plays a vital role in cosmology and astrophysics [11] and these gases can behave like dark energy [15].

This manuscript aims to apply two advanced approaches, called homotopy perturbation transform method (HPTM) and variational iteration transform method (VITM), for solving time-fractional system of nonlinear equations of unsteady flow of a polytropic gas in two dimensions of the following form

$$\begin{aligned} &D_t^\alpha u + u_x u + v u_y + \frac{w_x}{\rho} = 0, \\ &D_t^\alpha v + u v_x + v v_y + \frac{w_y}{\rho} = 0, \\ &D_t^\alpha \rho + u\rho_x + v\rho_y + \rho u_x + \rho v_y = 0, \\ &D_t^\alpha w + u w_x + v w_y + \tau u_x + \tau v_y = 0, \end{aligned} \tag{1.3}$$

under the initial conditions

$$\begin{aligned} u(x,y,0) &= a(x,y), \\ v(x,y,0) &= b(x,y), \\ \rho(x,y,0) &= c(x,y), \\ w(x,y,0) &= d(x,y), \end{aligned} \tag{1.4}$$

where D_t^α is the Caputo time-fractional derivative operator of order α with $0 < \alpha \leq 1$, $u = u(x,y,t)$ and $v = v(x,y,t)$ are the velocity components, $\rho = \rho(x,y,t)$ is the density, $w = w(x,y,t)$ is the pressure and τ is the ratio of the specific heat and it represents the adiabatic index.

Both approaches are new numerical-analytical techniques for dealing with both linear and non-linear problems, which enables us to obtain analytical and approximate solutions in convergent series by combining the new general integral transform in the sense of the Caputo fractional derivative and homotopy perturbation method and variational iteration method, respectively, without requiring constrained assumptions.

This manuscript is structured in the following way. In section 2, we present some basic definitions and mathematical preliminaries of the fractional calculus and the new general integral transform, helping us to understand the main results of this manuscript. Section 3 is devoted to the analysis of two numerical schemes which are the HPTM and the VITM to solve the nonlinear system (1.3) under the initial conditions (1.4) and we also demonstrate the applicability of the proposed numerical schemes by considering a numerical example in section 4. Finally we conclude our work in section 5.

2 Definition and preliminaries

This section presents the basic definitions and properties of fractional calculus theory and the new general integral transform, which are used further in this manuscript.

Definition 2.1. [10] Let $u \in L^1(0,T), T > 0$. The Riemann-Liouville fractional integral of order $\alpha > 0$, is defined by

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)d\tau, t > 0, \tag{2.1}$$

$$I^0 u(t) = u(t), \tag{2.2}$$

where $\Gamma(.)$ is the Euler gamma function, defined as follows

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt, z \in \mathbb{C} \text{ with } Re(z) > 0. \tag{2.3}$$

Definition 2.2. [10] Let $u^{(m)} \in L^1(0,T), T > 0$. The Caputo fractional derivative of order α, is defined by

$$D^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} u^{(m)}(\tau)d\tau, t > 0, \tag{2.4}$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$.

Definition 2.3. [10] For n to be the smallest integer that exceed α, the Caputo time-fractional derivative of order α, is defined as

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} u^{(m)}(x,\tau)d\tau, t > 0, \tag{2.5}$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$.

Definition 2.4. [10] The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^i}{\Gamma(i\alpha+\beta)}, \alpha, \beta > 0, z \in \mathbb{C}. \tag{2.6}$$

If $\beta = 1$, this function is denoted by $E_\alpha(.)$ and if $\alpha = \beta = 1$ this function represents e^z.

Definition 2.5. [6] Let $u(t)$ be a integrable function defined for $t \geq 0$, $p(s) \neq 0$ and $q(s)$ are positive real functions, we define the general integral transform $\mathcal{T}\ (s)$ of $u(t)$ by the formula

$$T[u(t)] = \mathcal{T}(s) = p(s) \int_0^{+\infty} u(t) \exp[-q(s)t]\, dt, \tag{2.7}$$

provided the integral exists for some $q(s)$.

Some basic properties of the new general integral transform are given as follows.

Property 2.1. The new general integral transform is also a linear operator

$$T[au_1(t) + bu_2(t)] = aT[u_1(t)] + bT[u_2(t)], a, b \in \mathbb{R}. \tag{2.8}$$

Property 2.2. If $u(t)$ is n^{th} differentiable and $p(s)$ and $q(s)$ are positive real functions, then

$$T\left[u^{(n)}(t)\right] = q^n(s)\mathcal{T}(s) - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s)u^{(k)}(0). \tag{2.9}$$

Property 2.3. (Convolution) Let $u_1(t)$ and $u_2(t)$ have new general integral transform $\mathcal{T}_1(s)$ and $\mathcal{T}_2(s)$. Then the new general integral transform of the convolution of u_1 and u_2 is

$$T[(u_1 * u_2)(t)] = T\left[\int_0^{+\infty} u_1(t)u_2(t-\tau)d\tau\right] = \frac{1}{p(s)}\mathcal{T}_1(s)\mathcal{T}_2(s). \tag{2.10}$$

Property 2.4. Some special new general integral transform

$$\begin{aligned}
T(1) &= \frac{p(s)}{q(s)}, \\
T(t) &= \frac{p(s)}{q^2(s)}, \\
T\left[\frac{t^n}{n!}\right] &= \frac{p(s)}{q^{n+1}(s)}, n = 0, 1, 2, ... \\
T[t^\alpha] &= \frac{p(s)}{q^{\alpha+1}(s)}\Gamma(\alpha+1), \alpha > 0.
\end{aligned} \tag{2.11}$$

Theorem 2.6. [9, Theorem 4.2] If $n \in \mathbb{Z}^+$ where $m-1 < \alpha \leq m$ and $\mathcal{T}(s)$ be the new general integral transform of the function $u(t)$, then, the new general integral transform of the Caputo fractional derivative of order $\alpha > 0$, is

$$T\left[D^{\alpha}u(t)\right] = q^{\alpha}(s)\mathcal{T}(s) - p(s)\sum_{k=0}^{m-1} q^{\alpha-1-k}(s)u^{(k)}(0). \tag{2.12}$$

3 Main result

This section gives our main result related to the HPTM and VITM to solve the nonlinear Caputo time-fractional partial differential equations.

3.1 Analysis of the HPTM

Theorem 3.1. Consider the nonlinear time-fractional partial differential equation in two dimensions

$$D_t^{\alpha}u(x,y,t) + Lu(x,y,t) + Nu(x,y,t) = f(x,y,t), \tag{3.1}$$

under the initial conditions

$$u^{(k)}(x,y,0) = u_k(x,y), k = 0,1,...,m-1, \tag{3.2}$$

where D^{α} is the Caputo fractional derivative operator of order α with $m-1 < \alpha \leq m$, L and N are a linear and nonlinear operators, respectively, and f is the source term.

Then, the exact solution of equations (3.1)-(3.2) using the HPTM, can be defined as follows

$$u(x,y,t) = \lim_{\varphi \to 1} \sum_{n=0}^{+\infty} \varphi^n u_n(x,y,t) = \sum_{n=0}^{+\infty} u_n(x,y,t), \tag{3.3}$$

where

$$\sum_{n=0}^{+\infty} \varphi^n u_n(x,y,t) = G(x,y,t) - T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[L\sum_{n=0}^{+\infty}\varphi^n u_n + \sum_{n=0}^{+\infty}\varphi^n H_n(u)\right]\right), \tag{3.4}$$

$T\left[.\right]$ is the new general integral transform and $H_n(u)$ is He's polynomials.

Proof . Taking the new general integral transform on both sides of equation (3.1) and using the Theorem 2.6, we obtain

$$T\left[u(x,y,t)\right] = \sum_{k=0}^{n-1}\frac{p(s)}{q^{k+1}(s)}u^{(k)}(x,y,0) - \frac{1}{q^{\alpha}(s)}T\left[Lu(x,y,t) + Nu(x,y,t)\right] + T\left[f(x,y,t)\right]. \tag{3.5}$$

Implementing the inverse new general integral transform on both sides of equation (3.4), we obtain

$$u(x,y,t) = G(x,y,t) - T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[Lu(x,y,t) + Nu(x,y,t)\right]\right), \tag{3.6}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we use the homotopy perturbation method

$$u(x,y,t) = \sum_{n=0}^{+\infty}\varphi^n u_n(x,y,t), \tag{3.7}$$

and the nonlinear term can be decomposed as

$$Nu(x,y,t) = \sum_{n=0}^{+\infty}\varphi^n H_n(u), \tag{3.8}$$

where $H_n(u)$ shows He's polynomials [5] and is determined with the help of the formula

$$H_n(u_0,u_1,...,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial\varphi^n}N\left[\sum_{i=0}^{+\infty}\varphi^i u_i\right]_{\varphi=0}, n \geq 0. \tag{3.9}$$

Substituting equations (3.7) and (3.8) in equation (3.6), we obtain

$$\sum_{n=0}^{+\infty}\varphi^n u_n(x,y,t)=G(x,y,t)-T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[L\sum_{n=0}^{+\infty}\varphi^n u_n(x,y,t)+\sum_{n=0}^{+\infty}\varphi^n H_n(u)\right]\right). \tag{3.10}$$

Comparing the coefficient of like powers of φ, the following approximations are obtained

$$\begin{aligned}
\varphi^0 &: \quad u_0(x,y,t)=G(x,y,t),\\
\varphi^1 &: \quad u_1(x,y,t)=-T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[Lu_0(x,y,t)+H_0(u)\right]\right),\\
\varphi^2 &: \quad u_2(x,y,t)=-T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[Lu_1(x,y,t)+H_1(u)\right]\right),\\
&\quad\vdots\\
\varphi^n &: \quad u_n(x,y,t)=-T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[Lu_{n-1}(x,y,t)+H_{n-1}(u)\right]\right).
\end{aligned} \tag{3.11}$$

Then, the exact solution of equations (3.1)-(3.2) can be defined as follows

$$u(x,y,t)=\lim_{\varphi\to 1}\sum_{n=0}^{+\infty}\varphi^n u_n(x,y,t)=\sum_{n=0}^{+\infty}u_n(x,y,t). \tag{3.12}$$

The proof is complete. □

3.2 Analysis of the convergence

Theorem 3.2. Let $\mathcal{H}$ be the Hilbert space. Then, the obtained solution $\sum_{n=0}^{+\infty} u_n(x,y,t)$ will be convergent to the exact solution $u(x,y,t)$ of equation (3.1), if there exists μ, $0<\mu<1$ such that

$$\|u_n(x,y,t)\|\le\mu\|u_{n-1}(x,y,t)\|,\forall n\in\mathbb{N}. \tag{3.13}$$

Proof . We make a sequence of $\sum_{n=0}^{+\infty} u_n(x,y,t)$

$$\begin{aligned}
S_0(x,y,t) &= u_0(x,y,t),\\
S_1(x,y,t) &= u_0(x,y,t)+u_1(x,y,t),\\
S_2(x,y,t) &= u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t),\\
&\ \ \vdots\\
S_n(x,y,t) &= u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+...+u_n(x,y,t).
\end{aligned} \tag{3.14}$$

Now, we must show that $S_n(x,y,t)$ forms a Cauchy sequence. Consider the following

$$\begin{aligned}
\|S_{n+1}(x,y,t)-S_n(x,y,t)\| &\le \|u_{n+1}(x,y,t)\|\le\mu\|u_n(x,y,t)\|\\
&\le \mu^2\|u_{n-1}(x,y,t)\|\le...\le\mu^{n+1}\|u(x,y,t)\|.
\end{aligned} \tag{3.15}$$

For every $n,m\in\mathbb{N}$, $n\ge m$, we obtain

$$\begin{aligned}
\|S_n(x,y,t)-S_m(x,y,t)\| &= \|S_{m+1}(x,y,t)-S_m(x,y,t)+S_{m+2}(x,y,t)-S_{m+1}(x,y,t)\\
&\quad +...+S_n(x,y,t)-S_{n-1}(x,y,t)\|\\
&\le \|_{m+1}(x,y,t)-S_m(x,y,t)\|+\|S_{m+2}(x,y,t)-S_{m+1}(x,y,t)\|\\
&\quad +...+\|S_n(x,y,t)-S_{n-1}(x,y,t)\|\\
&\le \mu^{m+1}\|u_0(x,y,t)\|+\mu^{m+2}\|u_0(x,y,t)\|+...+\mu^n\|u_0(x,y,t)\|\\
&= \mu^{m+1}\left(1+\mu+...+\mu^{n-m-1}\right)\|u_0(x,y,t)\|\\
&\le \mu^{m+1}\left(\frac{1-\mu^{n-m}}{1-\mu}\right)\|u_0(x,y,t)\|.
\end{aligned} \tag{3.16}$$

Since $0 < \mu < 1$ and $\|u_0(x,y,t)\|$ is bounded, we obtain

$$\lim_{n,m\longrightarrow\infty} \|S_n(x,y,t) - S_m(x,y,t)\| = 0. \tag{3.17}$$

Thus, the sequences $S_n(x,y,t)$ forms a Cauchy sequence in $\mathcal{H}$. That is, the following sequence $\sum\limits_{n=0}^{+\infty} u_n(x,y,t)$ is a convergent sequences with the limits

$$\lim_{m\to\infty} \sum_{n=0}^{m} u_n(x,y,t) = u(x,y,t). \tag{3.18}$$

for $u_n(x,y,t) \in \mathcal{H}$.

The proof is complete. □

Theorem 3.3. Let $\sum\limits_{k=0}^{m} u_k(x,y,t)$ be finite and $u(x,y,t)$ be the obtained series result. Therefore, the maximum absolute truncation error is estimated to be

$$\left\| u(x,y,t) - \sum_{k=0}^{m} u_k(x,y,t) \right\| \leq \frac{\mu^{m+1}}{1-\mu} \|u_0(x,y,t)\|. \tag{3.19}$$

Proof . Since $\sum\limits_{k=0}^{m} u_k(x,y,t)$ is finite, this implies that $\sum\limits_{k=0}^{m} u_k(x,y,t) < \infty$.

Consider

$$\begin{aligned} \left\| u(x,y,t) - \sum_{k=0}^{m} u_k(x,y,t) \right\| &= \left\| \sum_{k=m+1}^{\infty} u_k(x,y,t) \right\| \\ &= \sum_{k=m+1}^{\infty} \|u_k(x,y,t)\| \\ &\leq \sum_{k=m+1}^{\infty} \mu^k \|u_0(x,y,t)\| \\ &\leq \mu^{m+1}(1+\mu+\mu^2+...) \|u_0(x,y,t)\| \\ &\leq \frac{\mu^{m+1}}{1-\mu} \|u_0(x,y,t)\|. \end{aligned} \tag{3.20}$$

The proof is complete. □

3.3 Analysis of the VITM

Theorem 3.4. Consider the nonlinear time-fractional partial differential equation (3.1) under the initial conditions (3.2). Then, the exact solution of equations (3.1)-(3.2) using the VITM, is given as a limit of the successive approximations $u_n(x,y,t), n = 0,1,2,...$, in other words

$$u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t), \tag{3.21}$$

where

$$u_n(x,y,t) = \sum_{k=0}^{n-1} u_k(x,y) \frac{t^k}{k!} - T^{-1} \left(\frac{1}{q^\alpha(s)} T\left[Lu_{n-1}(x,y,t) + Nu_{n-1}(x,y,t) - f(x,y,t) \right] \right), \tag{3.22}$$

and $T[.]$ is the new general integral transform.

Proof . According to the variational iteration transform [23], the correction functional of equation (3.1), is given as

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^t \lambda(t-\tau) \left[\begin{array}{c} D_t^\alpha u_n(x,y,\tau) + Lu_n(x,y,\tau) \\ +Nu_n(x,y,\tau) - f(x,y,\tau) \end{array} \right] d\tau, \tag{3.23}$$

where $\lambda(t-\tau)$ is a general lagrange multiplier, the subscript $n \geq 0$ denotes the n^{th} approximation.

Taking the new general integral transform and using convolution property in equation (3.4), we have

$$\begin{aligned} T\left[u_{n+1}(x,y,t)\right] &= T\left[u_n(x,y,t)\right] + T\left[\int_0^t \lambda(t-\tau)\left[\begin{array}{c} D_t^{\alpha}u_n(x,y,\tau)+Lu_n(x,y,\tau) \\ +Nu_n(x,y,\tau)-f(x,y,\tau)\end{array}\right]d\tau\right] \\ &= T\left[u_n(x,y,t)\right] + \frac{1}{p(s)}T\left[\lambda(t)\right]T\left[\begin{array}{c} D_t^{\alpha}u_n(x,y,t)+Lu_n(x,y,t) \\ +Nu_n(x,y,t)-f(x,y,t)\end{array}\right]. \end{aligned} \tag{3.24}$$

Using the Theorem 2.6 and initial conditions (3.3), equation (3.5), becomes

$$\begin{aligned} T\left[u_{n+1}(x,y,t)\right] &= T\left[u_n(x,y,t)\right] + \frac{1}{p(s)}T\left[\lambda(t)\right]\Bigg(q^{\alpha}(s)T\left[u_n(x,y,t)\right] - p(s)\sum_{k=0}^{m-1}q^{\alpha-1-k}(s)u_k(x,y) \\ &\quad +T\left[Lu_n(x,y,t)+Nu_n(x,y,t)-f(x,y,t)\right]\Bigg). \end{aligned} \tag{3.25}$$

The optimal value of λ can be identified by making the equation (3.6) stationary with respect to $u_n(x,y,t)$

$$\begin{aligned} \delta\left(T\left[u_{n+1}(x,y,t)\right]\right) &= \delta\left(T\left[u_n(x,y,t)\right]\right) \\ &\quad +\frac{1}{p(s)}\delta\Bigg(T\left[\lambda(t)\right]\left(q^{\alpha}(s)T\left[u_n(x,y,t)\right] - p(s)\sum_{k=0}^{m-1}q^{\alpha-1-k}(s)u_k(x,y)\right. \\ &\quad \left.+T\left[Lu_n(x,y,t)+Nu_n(x,y,t)-f(x,y,t)\right]\right)\Bigg). \end{aligned} \tag{3.26}$$

Considering $T\left[Lu_n(x,y,t)+Nu_n(x,y,t)\right]$ as restricted variation, i.e.,

$$\delta\left(T\left[Lu_n(x,y,t)+Nu_n(x,y,t)\right]\right)=0, \tag{3.27}$$

we have

$$1+\frac{q^{\alpha}(s)}{p(s)}T\left[\lambda(t)\right]=0, \tag{3.28}$$

which implies that

$$T\left[\lambda(t)\right] = -\frac{p(s)}{q^{\alpha}(s)}. \tag{3.29}$$

Using (3.9) in equation (3.6) and taking the inverse new general integral transform, we attain a new correction functional

$$u_{n+1}(x,y,t) = \sum_{k=0}^{m-1}u_k(x,y)\frac{t^k}{k!} - T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[Lu_n(x,y,t)+Nu_n(x,y,t)-f(x,y,t)\right]\right). \tag{3.30}$$

The initial value $u_0(x,y,t)$ can be find as

$$u_0(x,y,t) = \sum_{k=0}^{m-1}u_k(x,y)\frac{t^k}{k!}. \tag{3.31}$$

The successive approximations rapidly converge to the exact solution of equation (3.1) as $n\to\infty$, that is

$$u(x,y,t) = \lim_{n\to\infty}u_n(x,y,t). \tag{3.32}$$

The proof is complete. □

4 Application of methods and results

This section provides an application of Caputo time-fractional system of equations describing the unsteady flow of a polytropic gas to assess the applicability, accuracy and efficiency of the HPTM and VITM.

Example 4.1. Consider the Caputo time-fractional system of nonlinear equations of unsteady flow of a polytropic gas in two dimensions [20]

$$\begin{aligned}
&D_t^{\alpha} u + u_x u + v u_y + \frac{w_x}{\rho} = 0,\\
&D_t^{\alpha} v + u v_x + v v_y + \frac{w_y}{\rho} = 0,\\
&D_t^{\alpha} \rho + u \rho_x + v \rho_y + \rho u_x + \rho v_y = 0,\\
&D_t^{\alpha} w + u w_x + v w_y + \tau u_x + \tau v_y = 0,
\end{aligned} \tag{4.1}$$

under the initial conditions

$$\begin{aligned}
u(x,y,0) &= e^{x+y},\\
v(x,y,0) &= -1 - e^{x+y},\\
\rho(x,y,0) &= e^{x+y},\\
w(x,y,0) &= c.
\end{aligned} \tag{4.2}$$

The exact solution, when $\alpha = 1$, is [13]

$$\begin{aligned}
u(x,y,t) &= e^{x+y+t},\\
v(x,y,t) &= -1 - e^{x+y+t},\\
\rho(x,y,t) &= e^{x+y+t},\\
w(x,y,t) &= c,
\end{aligned} \tag{4.3}$$

where c is a real constant.

Case 1. We solve the system (4.1) under the initial conditions (4.2) using the HPTM.

According to the HPTM algorithm proposed in Section 3, we get

$$\begin{aligned}
\sum_{n=0}^{+\infty} \varphi^n u_n(x,y,t) &= e^{x+y} - T^{-1}\left(\frac{1}{q^{\alpha}(s)} T\left[\sum_{n=0}^{+\infty}\varphi^n A_n + \sum_{n=0}^{+\infty}\varphi^n B_n + \sum_{n=0}^{+\infty}\varphi^n C_n\right]\right),\\
\sum_{n=0}^{+\infty} \varphi^n v_n(x,y,t) &= -1 - e^{x+y} - T^{-1}\left(\frac{1}{q^{\alpha}(s)} T\left[\sum_{n=0}^{+\infty}\varphi^n D_n + \sum_{n=0}^{+\infty}\varphi^n E_n + \sum_{n=0}^{+\infty}\varphi^n F_n\right]\right),\\
\sum_{n=0}^{+\infty} \varphi^n \rho_n(x,y,t) &= e^{x+y} - T^{-1}\left(\frac{1}{q^{\alpha}(s)} T\left[\sum_{n=0}^{+\infty}\varphi^n G_n + \sum_{n=0}^{+\infty}\varphi^n H_n + \sum_{n=0}^{+\infty}\varphi^n I_n + \sum_{n=0}^{+\infty}\varphi^n J_n\right]\right),\\
\sum_{n=0}^{+\infty} \varphi^n w_n(x,y,t) &= e^{x+y} - T^{-1}\left(\frac{1}{q^{\alpha}(s)} T\left[\sum_{n=0}^{+\infty}\varphi^n K_n + \sum_{n=0}^{+\infty}\varphi^n M_n + \tau\sum_{n=0}^{+\infty}\varphi^n u_{nx} + \tau\sum_{n=0}^{+\infty}\varphi^n v_{ny}\right]\right),
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
u_x u &= \sum_{n=0}^{+\infty}\varphi^n A_n,\, v u_y = \sum_{n=0}^{+\infty}\varphi^n B_n,\, \frac{w_x}{\rho} = \sum_{n=0}^{+\infty}\varphi^n C_n,\, u v_x \sum_{n=0}^{+\infty}\varphi^n D_n,\\
v v_y &= \sum_{n=0}^{+\infty}\varphi^n E,\, \frac{w_y}{\rho} = \sum_{n=0}^{+\infty}\varphi^n F_n,\, u\rho_x = \sum_{n=0}^{+\infty}\varphi^n G_n,\, v\rho_y = \sum_{n=0}^{+\infty}\varphi^n H_n,\\
\rho u_x &= \sum_{n=0}^{+\infty}\varphi^n I_n,\, \rho v_y = \sum_{n=0}^{+\infty}\varphi^n J_n,\, u w_x = \sum_{n=0}^{+\infty}\varphi^n K_n,\, v w_y = \sum_{n=0}^{+\infty}\varphi^n M_n,
\end{aligned} \tag{4.5}$$

are He's polynomials and can be calculated using the formula (3.7). On comparing the coefficient of the like power of

φ, on both sides in (4.4), we get

$$\begin{aligned} \varphi^0 \quad &: \quad u_0(x,y,t) = e^{x+y}, \\ &: \quad v_0(x,y,t) = -1 - e^{x+y}, \\ &: \quad \rho_0(x,y,t) = e^{x+y}, \\ &: \quad w_0(x,y,t) = e^{x+y}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \varphi^1 \quad &: \quad u_1(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[A_0+B_0+C_0\right]\right) = e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ &: \quad v_1(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[D_0+E_0+F_0\right]\right) = -e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ &: \quad \rho_1(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[G_0+H_0+I_0+J_0\right]\right) = e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ &: \quad w_1(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[K_0+M_0+\tau u_{0x}+\tau v_{0y}\right]\right) = 0, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \varphi^2 \quad &: \quad u_2(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[A_1+B_1+C_1\right]\right) = e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ &: \quad v_2(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[D_1+E_1+F_1\right]\right) = -e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ &: \quad \rho_2(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[G_1+H_1+I_1+J_1\right]\right) = e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ &: \quad w_2(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[K_1+M_1+\tau u_{1x}+\tau v_{1y}\right]\right) = 0, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \varphi^3 \quad &: \quad u_3(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[A_2+B_2+C_2\right]\right) = e^{x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ &: \quad v_3(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[D_2+E_2+F_2\right]\right) = -e^{x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ &: \quad \rho_3(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[G_2+H_2+I_2+J_2\right]\right) = e^{x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ &: \quad w_3(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[K_2+M_2+\tau u_{1x}+\tau v_{1y}\right]\right) = 0. \end{aligned} \tag{4.9}$$

We continue to get

$$\begin{aligned} \varphi^{n+1} \quad &: \quad u_{n+1}(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[A_n+B_n+C_n\right]\right) = e^{x+y}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ &: \quad v_{n+1}(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[D_n+E_n+F_n\right]\right) = -e^{x+y}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ &: \quad \rho_{n+1}(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[G_n+H_n+I_n+J_n\right]\right) = e^{x+y}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ &: \quad w_{n+1}(x,y,t) = -T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[K_n+M_n+\tau u_{nx}+\tau v_{ny}\right]\right) = 0, \end{aligned} \tag{4.10}$$

where $n \geq 0$. Finally, the series solution of the unknown functions $u(x,y,t)$, $v(x,y,t)$, $\rho(x,y,t)$ and $w(x,y,t)$ of system

(4.1) under (4.2) are given by

$$
\begin{aligned}
u(x,y,t) &= \sum_{n=0}^{+\infty} u_n(x,y,t) \\
&= e^{x+y}\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}+\ldots\right) \\
&= e^{x+y}\sum_{n=0}^{+\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = e^{x+y}E_{\alpha}(t^{\alpha}), \\
v(x,y,t) &= \sum_{n=0}^{+\infty} v_n(x,y,t) \\
&= -1-e^{x+y}\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}+\ldots\right) \\
&= -1-e^{x+y}\sum_{n=0}^{+\infty}\frac{t^{n\beta}}{\Gamma(n\beta+1)} = -1-e^{x+y}E_{\alpha}(t^{\alpha}), \qquad (4.11)\\
\varphi(x,y,t) &= \sum_{n=0}^{+\infty} \varphi_n(x,y,t) \\
&= e^{x+y}\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}+\ldots\right) \\
&= e^{x+y}\sum_{n=0}^{+\infty}\frac{t^{n\gamma}}{\Gamma(n\gamma+1)} = e^{x+y}E_{\alpha}(t^{\alpha}), \\
w(x,y,t) &= \sum_{n=0}^{+\infty} w_n(x,y,t) \\
&= c+0+0+0+\ldots = c,
\end{aligned}
$$

where $E_{\alpha}(t^{\alpha})$ is the Mittag-Leffler functions and c is real constant. When $\alpha = 1$, the solutions (4.11) become

$$
\begin{aligned}
u(x,y,t) &= e^{x+y}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots+\frac{t^n}{n!}+\ldots\right) = e^{x+y+t}, \\
v(x,y,t) &= -1-e^{x+y}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots+\frac{t^n}{n!}+\ldots\right) = -1-e^{x+y+t}, \qquad (4.12)\\
\varphi(x,y,t) &= e^{x+y}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots+\frac{t^n}{n!}+\ldots\right) = e^{x+y+t}, \\
w(x,y,t) &= c,
\end{aligned}
$$

which is an exact solution of nonlinear system of equations of unsteady flow of a polytropic gas in two dimensions that is given in (4.3).

Case 2. We solve the system (4.1) under the initial conditions (4.2) using the VITM. According to the VITM algorithm proposed in Section 3, we get the following iteration formulas

$$
\begin{aligned}
u_{n+1}(x,y,t) &= e^{x+y} - T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[u_{nx}u_n + v_n u_{ny} + \frac{w_{nx}}{\rho_n}\right]\right), \\
v_{n+1}(x,y,t) &= -1-e^{x+y} - T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[u_n v_{nx} + v_n v_{ny} + \frac{w_{ny}}{\rho_n}\right]\right), \qquad (4.13)\\
\rho_{n+1}(x,y,t) &= e^{x+y} - T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[u_n\rho_{nx} + v_n\rho_{ny} + \rho_n u_{nx} + \rho_n v_{ny}\right]\right), \\
w_{n+1}(x,y,t) &= c - T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[u_n w_{nx} + v_n w_{ny} + \tau u_{nx} + \tau v_{ny}\right]\right),
\end{aligned}
$$

where the initial iterations of $u_0(x,y,t), v_0(x,y,t), \rho_0(x,y,t)$ and $w_0(x,y,t)$ are given as

$$\begin{aligned} u_0(x,y,t) &= e^{x+y}, \\ v_0(x,y,t) &= -1-e^{x+y}, \\ \rho_0(x,y,t) &= e^{x+y}, \\ w_0(x,y,t) &= c. \end{aligned} \tag{4.14}$$

Then, we have the following iterations by (4.13) and (4.14).

$$\begin{aligned} u_1(x,y,t) &= e^{x+y} - T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[u_{0x}u_0 + v_0u_{0y} + \frac{w_{0x}}{\rho_0}\right]\right) \\ &= e^{x+y} + e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ v_1(x,y,t) &= -1-e^{x+y} - T^{-1}\left(\frac{1}{q^\beta(s)}T\left[u_0v_{0x} + v_0v_{0y} + \frac{w_{0y}}{\rho_0}\right]\right) \\ &= -1-e^{x+y} - e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ \rho_1(x,y,t) &= e^{x+y} - T^{-1}\left(\frac{1}{q^\gamma(s)}T\left[u_0\rho_{0x} + v_0\rho_{0y} + \rho_0u_{0x} + \rho_0v_{0y}\right]\right) \\ &= e^{x+y} + e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ w_1(x,y,t) &= c - T^{-1}\left(\frac{1}{q^\eta(s)}T\left[u_0w_{0x} + v_0w_{0y} + \tau u_{0x} + \tau v_{0y}\right]\right) = c, \end{aligned} \tag{4.15}$$

$$\begin{aligned} u_2(x,y,t) &= e^{x+y} - T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[u_{1x}u_1 + v_1u_{1y} + \frac{w_{1x}}{\rho_1}\right]\right) \\ &= e^{x+y} + e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)} + e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ v_2(x,y,t) &= -1-e^{x+y} - T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[u_1v_{1x} + v_1v_{1y} + \frac{w_{1y}}{\rho_1}\right]\right) \\ &= -1-e^{x+y} - e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)} - e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ \rho_2(x,y,t) &= e^{x+y} - T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[u_1\rho_{1x} + v_1\rho_{1y} + \rho_1u_{1x} + \rho_1v_{1y}\right]\right) \\ &= e^{x+y} + e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)} + e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ w_2(x,y,t) &= c - T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[u_1w_{1x} + v_1w_{1y} + \tau u_{1x} + \tau v_{1y}\right]\right) = c, \\ u_3(x,y,t) &= e^{x+y} - T^{-1}\left(\frac{1}{q^\alpha(s)}T\left[u_{2x}u_2 + v_2u_{2y} + \frac{w_{2x}}{\rho_2}\right]\right) \\ &= e^{x+y} + e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)} + e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{aligned} \tag{4.16}$$

$$\begin{aligned}
v_3(x,y,t) &= -1-e^{x+y}-T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[u_2v_{2x}+v_2v_{2y}+\frac{w_{2y}}{\rho_2}\right]\right)\\
&= -1-e^{x+y}-e^{x+y}\frac{t^{\alpha}}{\Gamma(\alpha+1)}-e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-e^{x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \qquad (4.17)\\
\rho_3(x,y,t) &= e^{x+y}-T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[u_2\rho_{2x}+v_2\rho_{2y}+\rho_2u_{2x}+\rho_2v_{2y}\right]\right)\\
&= e^{x+y}+e^{x+y}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+e^{x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
w_3(x,y,t) &= c-T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[u_2w_{2x}+v_2w_{2y}+\tau u_{2x}+\tau v_{2y}\right]\right)=c.
\end{aligned}$$

We continue to get

$$\begin{aligned}
u_{n+1}(x,y,t) &= e^{x+y}-T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[u_{nx}u_n+v_nu_{ny}+\frac{w_{nx}}{\rho_n}\right]\right)\\
&= e^{x+y}+e^{x+y}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+e^{x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...+e^{x+y}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},
\end{aligned}$$

$$\begin{aligned}
v_{n+1}(x,y,t) &= -1-e^{x+y}-T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[u_nv_{nx}+v_nv_{ny}+\frac{w_{ny}}{\rho_n}\right]\right)\\
&= -1-e^{x+y}-e^{x+y}\frac{t^{\alpha}}{\Gamma(\alpha+1)}-e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-e^{x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}-...-e^{x+y}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \qquad (4.18)\\
\rho_{n+1}(x,y,t) &= e^{x+y}-T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[u_n\rho_{nx}+v_n\rho_{ny}+\rho_nu_{nx}+\rho_nv_{ny}\right]\right)\\
&= e^{x+y}+e^{x+y}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+e^{x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...+e^{x+y}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},\\
w_{n+1}(x,y,t) &= c-T^{-1}\left(\frac{1}{q^{\alpha}(s)}T\left[u_nw_{nx}+v_nw_{ny}+\tau u_{nx}+\tau v_{ny}\right]\right)=c.
\end{aligned}$$

Finally, the series solution of the unknown functions $u(x,y,t)$, $v(x,y,t)$, $\rho(x,y,t)$ and $w(x,y,t)$ of system (4.1) under (4.2) are given by

$$\begin{aligned}
u(x,y,t) &= \lim_{n\to\infty}u_n(x,y,t)\\
&= \lim_{n\to\infty}e^{x+y}\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right)\\
&= \lim_{n\to\infty}e^{x+y}\sum_{n=0}^{n}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}=e^{x+y}E_{\alpha}(t^{\alpha}),\\
v(x,y,t) &= \lim_{n\to\infty}v_n(x,y,t)\\
&= \lim_{n\to\infty}\left(-1-e^{x+y}\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right)\right)\\
&= \lim_{n\to\infty}\left(-1-e^{x+y}\sum_{n=0}^{n}\frac{t^{n\beta}}{\Gamma(n\beta+1)}\right)=-1-e^{x+y}E_{\alpha}(t^{\alpha}), \qquad (4.19)\\
\varphi(x,y,t) &= \lim_{n\to\infty}\varphi_n(x,y,t)\\
&= \lim_{n\to\infty}e^{x+y}\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right)\\
&= \lim_{n\to\infty}e^{x+y}\sum_{n=0}^{+\infty}\frac{t^{n\gamma}}{\Gamma(n\gamma+1)}=e^{x+y}E_{\alpha}(t^{\alpha}),\\
w(x,y,t) &= \lim_{n\to\infty}w_n(x,y,t)=c+0+0+0+...=c,
\end{aligned}$$

where $E_\alpha(t^\alpha)$ is the Mittag-Leffler functions and c is real constant. When $\alpha = 1$, the solutions (4.19) become

$$\begin{aligned} u(x,y,t) &= e^{x+y}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...+\frac{t^n}{n!}+...\right) = e^{x+y+t}, \\ v(x,y,t) &= -1-e^{x+y}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...+\frac{t^n}{n!}+...\right) = -1-e^{x+y+t}, \\ \varphi(x,y,t) &= e^{x+y}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...+\frac{t^n}{n!}+...\right) = e^{x+y+t}, \\ w(x,y,t) &= c, \end{aligned} \tag{4.20}$$

which is an exact solution of nonlinear system of equations of unsteady flow of a polytropic gas in two dimensions that is given in (4.3).

5 Conclusion

In this manuscript, homotopy perturbation transform method and variational iteration transform method for obtaining the solutions of Caputo time-fractional nonlinear system of equations describing the unsteady flow of a polytropic gas are implemented. By using these approaches, solutions are calculated in form of a convergent series with easily computable components since where we can arrive at the exact solution after few iterations. The advantage of the proposed approaches is that these approaches avoid linearity and unrealistic assumptions, and provide an efficient numerical solution. Hence, we can deduce that these approaches can be applied to a wide range of nonlinear systems of fractional partial differential equations.

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