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On the value distribution of the differential polynomial $\phi f^n f^{(k)} - 1$

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Abstract

In the paper, we study the value distribution of the differential polynomial $\phi f^n f^{(k)} - 1$, where $f(z)$ is a transcendental meromorphic function, $\phi(z)(\not\equiv 0)$ is a small function of $f(z)$ and $n(> 2), k(\geq 1)$ are integers. We prove an inequality which will give an upper bound for the characteristic function $T(r, f)$ in terms of reduced counting function only.

Keywords: Value distribution, Meromorphic function, Differential polynomial
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1 Introduction

In this paper by meromorphic function we shall always mean meromorphic function in the complex plane $\mathbb{C}$. We shall use standard notations of the Nevanlinna theory of meromorphic functions as explained in [3, 7, 13, 14]. We denote by $T(r, f)$ the Nevanlinna characteristic function of a nonconstant meromorphic function $f(z)$ and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ for all r possibly outside a set of finite logarithmic measure. A meromorphic function $\phi(z)$ is said to be a small function of $f(z)$, if $T(r, \phi) = S(r, f)$.

In this research work the following definitions will be needed.

Definition 1.1. [14] Let $f(z)$ be a nonconstant meromorphic function and p be a positive integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N_{p)}(r, \frac{1}{f-a})$ the counting function of those zeros of $f(z) - a$ whose multiplicities are not greater than p and by $\overline{N}_{p)}(r, \frac{1}{f-a})$ the corresponding reduced counting function. We denote by $N_{(p+1}(r, \frac{1}{f-a})$ the counting function of those zeros of $f(z) - a$ whose multiplicities are greater than p and by $\overline{N}_{(p+1}(r, \frac{1}{f-a})$ the corresponding reduced counting function. We denote by $N_p(r, \frac{1}{f-a})$ the counting function of those zeros of $f(z) - a$ whose multiplicities are exactly p.

Definition 1.2. [14] Suppose that $f(z)$ is a nonconstant meromorphic function in the complex plane $\mathbb{C}$, and $\alpha(z)$ is a small function of $f(z)$. Let $n_0, n_1, \cdots, n_k$ be nonnegative integers. We denote by $M(f) = \alpha f^{n_0}(f')^{n_1} \cdots (f^{(k)})^{n_k}$ the differential monomial in f and by $n = \sum_{j=0}^{k} n_j$ the degree of $M(f)$. Also let $M_1(f), M_2(f), \cdots, M_k(f)$ be differential

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monomials in f of degree $m_1, m_2, \cdots, m_k$ respectively. The summation $P(f) = \sum_{j=1}^{k} M_j(f)$ is said to be differential polynomial in f and $m = max\{m_1, m_2, \cdots, m_k\}$, the degree of $P(f)$.

2 Preliminaries

A lot of research works have been done in the field of value distribution of differential polynomials of meromorphic functions by many mathematicians from different part of the world (See [4, 5, 9, 11, 12, 15, 16]). In 1979, E. Mues [8] first proved a qualitative result in this topic. The result is as follows:

Theorem 2.1. Let $f(z)$ be a transcendental meromorphic function in the complex plane. Then $f^2f' - 1$ has infinitely many zeros.

In 1992, Q.D. Zhang [15] proved a quantitative result of Theorem 2.1, which is as follows:

Theorem 2.2. Let $f(z)$ be a transcendental meromorphic function in the complex plane. Then

$$T(r, f) \leq 6N\left(r, \frac{1}{f^2f' - 1}\right) + S(r, f).$$

In 2011, J.F. Xu, H.X. Yi and Z.L. Zhang [12] improved Theorem 2.2 by estimating the reduced counting function. Their result is as follows:

Theorem 2.3. Let $f(z)$ be a transcendental meromorphic function in the complex plane. Then

$$T(r, f) \leq M\overline{N}\left(r, \frac{1}{f^2f^{(k)} - 1}\right) + S(r, f),$$

where M is 6 if $k = 1$ or $k \geq 3$ and M is 10 if $k = 2$.

In 1992, Q.D. Zhang [16] also studied the value distribution in case of small functions involved in differential equation and got the following result:

Theorem 2.4. Let $f(z)$ be a transcendental meromorphic function in the complex plane and $\phi(z)(\not\equiv 0)$ be a small function of $f(z)$. Then

$$T(r, f) \leq 6N\left(r, \frac{1}{\phi f^2f' - 1}\right) + S(r, f).$$

In 2016, J.F. Xu and H.X. Yi [10] improved Theorem 2.4 by considering reduced counting function and proved the following result:

Theorem 2.5. Let $f(z)$ be a transcendental meromorphic function in the complex plane and $\phi(z)(\not\equiv 0)$ be a small function of $f(z)$. Then

$$T(r, f) \leq 6\overline{N}\left(r, \frac{1}{\phi f^2f' - 1}\right) + S(r, f).$$

In 2018, H. Karmakar and P. Sahoo [6] proved following result which certainly improves Theorem 2.3.

Theorem 2.6. Let $f(z)$ be a transcendental meromorphic function and $n(\geq 2)$, $k(\geq 1)$ be integers. Then

$$T(r, f) \leq \frac{6}{2n-3}\overline{N}\left(r, \frac{1}{f^nf^{(k)} - 1}\right) + S(r, f).$$

Now it is natural to ask the following question:

Question 2.1. What will be the result if we replace $f^n f^{(k)} - 1$ by $\phi f^n f^{(k)} - 1$ in Theorem 2.6 where $\phi(z)(\not\equiv 0)$ is a small function of $f(z)$?

Recently, G. Biswas and P. Sahoo [1] gave answer to the above question for $n = 2$. They proved the following result:

Theorem 2.7. Let $f(z)$ be a transcendental meromorphic function in the complex plane, $k(\geq 2)$ be an integer and $\phi(z)(\not\equiv 0)$ be a small function of $f(z)$ such that the set of zeros and poles of $f(z)$ and that of $\phi(z)$ are disjoint and $\phi(z)$ has no zero of multiplicity 2. Then

$$T(r,f) \leq 6\overline{N}\left(r, \frac{1}{\phi f^2 f^{(k)} - 1}\right) + S(r,f).$$

3 Main Result

In this paper we investigate to find out possible answer for the question 2.1 for $n > 2$ and obtain the following result:

Theorem 3.1. Let $f(z)$ be a transcendental meromorphic function in the complex plane, $n(> 2)$, $k(\geq 1)$ be any integers and $\phi(z)(\not\equiv 0)$ be a small function of $f(z)$. If the sets $A = \{z : f(z) = 0 \text{ or } \infty\}$ and $B = \{z : \phi(z) = 0 \text{ or } \infty\}$ are disjoint and $\phi(z)$ has no zero of order n then

$$T(r,f) \leq \frac{6}{2n-3}\overline{N}\left(r, \frac{1}{\phi f^n f^{(k)} - 1}\right) + S(r,f).$$

Remark 3.1. Theorem 3.1 is a direct extension of Theorem 2.6 for $(n > 2)$ as it proves that the result remains unaffected if we involve a small function as coefficient.

4 Lemmas

Suppose that $f(z)$ is a transcendental meromorphic function and $\phi(z)(\not\equiv 0)$ is a small function of $f(z)$. Let us define $g(z) = \phi(z)f^n(z)f^{(k)}(z) - 1$ and $h(z) = \frac{g'(z)}{f^{n-1}(z)}$ where $n \geq 2$, $k \geq 1$ are integers. Also let

$$\begin{aligned} F(z) = {} & a_1\left(\frac{g'(z)}{g(z)}\right)^2 + a_2\left(\frac{g'(z)}{g(z)}\right)' + a_3\frac{g'(z)}{g(z)}\cdot\frac{h'(z)}{h(z)} + a_4\left(\frac{h'(z)}{h(z)}\right)^2 + a_5\left(\frac{h'(z)}{h(z)}\right)' \\ & + a_6\frac{g'(z)}{g(z)}\cdot\frac{\phi'(z)}{\phi(z)} + a_7\frac{h'(z)}{h(z)}\cdot\frac{\phi'(z)}{\phi(z)} + a_8\left(\frac{\phi'(z)}{\phi(z)}\right)^2 + a_9\left(\frac{\phi'(z)}{\phi(z)}\right)', \end{aligned} \tag{4.1}$$

where for $k = 1$,
$a_1 = 2(4n^2 + 8n + 7)$, $a_2 = 2(n+2)(4n^2 - 1)$,
$a_3 = -2(n+2)(2n^2 + 3n + 4)$, $a_4 = (n+1)(n+2)^2$,
$a_5 = -(n+2)^2(2n-1)$, $a_6 = 2(n+1)(n+2)(2n-5)$,
$a_7 = 3(n+2)^2$, $a_8 = -(n+2)^2(4n^2 - 5n - 12)$,
$a_9 = -(n+2)^2(4n^2 - 4n - 11)$
and for $k \geq 2$,
$a_1 = \{(n-1)k + (3n-1)\}\{(n-1)k^3 - 3(n^3 - 2n + 1)k^2 - 3(6n^3 - 3n + 1)k - (27n^3 - 4n + 1)\}$,
$a_2 = (n + k + 1)\{(n-1)k + (3n-1)\}^2\{(n-1)k^2 - (3n^2 - 5n + 2)k - (9n^2 - 4n + 1)\}$,
$a_3 = -2n(n + k + 1)\{(n-1)k + (3n-1)\}\{(n-1)k^2 - (3n^2 - 5n + 2)k - (9n^2 - 4n + 1)\}$,
$a_4 = n^2(n-1)(k+1)(n+k+1)^2\{(n-1)k + (3n-1)\}$,
$a_5 = -n(n-1)(k+1)(n+k+1)^2\{(n-1)k + (3n-1)\}^2$,
$a_6 = 2(n-1)\{(n-1)k^2 - (3n^2 - 5n + 2)k - (9n^2 - 4n + 1)\}\{(n-1)k^2 + (n^2 + 3n - 2)k + (3n^2 + 2n - 1)\}$,
$a_7 = -2n(n-1)^2(k+1)(n+k+1)^2\{(n-1)k + (3n-1)\}$,
$a_8 = (n-1)^3(k+1)(n+k+1)^2\{(n-1)k + (3n-1)\}$,
$a_9 = (n-1)^2(k+1)\{(n-1)k^2 + (n^2 + 3n - 2)k + (3n^2 + 2n - 1)\}^2$.

Lemma 4.1. [2] Suppose that $f(z)$ is a transcendental meromorphic function and $f^nP(f) = Q(f)$, where $P(f)$ and $Q(f)$ are differential polynomials in $f(z)$ with functions of small proximity related to $f(z)$ as the coefficient and the degree of $Q(z)$ is at most n. Then $m(r, P(f)) = S(r, f)$.

Lemma 4.2. [6] For two integers $n(> 2)$, $k(> 2)$, if

$$\begin{aligned} f(x) &= (n-1)\Big[\big\{(k+1)n^4 + 2(k^2+5k+10)n^3 + (k+1)^2(k+2)n^2 - (k+1)^2(2k+5)n \\ &+ (k+1)^3\big\}x^2 + (n+k+1)(k+1)\big\{(k+1)n^3 + (k^2+4k+9)n^2 - (2k^2+7k+5)n \\ &+ (k+1)^2\big\}x - n(n+k+1)^2(k+1)\big\{(n-1)k + (2n-1)\big\}\Big], \end{aligned}$$

then $f(x) = 0$ has no solution in $\mathbb{Z}_+$.

Lemma 4.3. Let $f(z)$, $\phi(z)(\not\equiv 0)$ and $g(z)$ be defined as in the beginning of the section. Then $g(z)$ is not equivalently constant.

Proof . Suppose $\phi(z)f^n(z)f^{(k)}(z) \equiv C$(a constant). Obviously $C \neq 0$. Hence we have

$$\frac{1}{f^{n+1}} = \frac{\phi}{C} \cdot \frac{f^{(k)}}{f} \; and \; \frac{1}{f^n f^{(k)}} = \frac{\phi}{C}.$$

Therefore

$$m\left(r, \frac{1}{f^{n+1}}\right) = m\left(r, \frac{\phi}{C} \cdot \frac{f^{(k)}}{f}\right).$$

i.e.,

$$(n+1)m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{\phi}{C}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + O(1) = S(r, f).$$

Also

$$N\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{f^n f^{(k)}}\right) = N\left(r, \frac{\phi}{C}\right) = S(r, f).$$

Therefore

$$T(r, f) = S(r, f),$$

a contradiction. Thus $\phi(z)f^n(z)f^{(k)}(z)$ is not equivalently constant and hence $g(z)$ is not equivalently constant. This completes the proof of *Lemma 4.3*. □

Lemma 4.4. Let $f(z)$, $\phi(z)(\not\equiv 0)$ and $g(z)$ be defined as in the beginning of the section. Then

$$\begin{aligned} (n+1)T(r,f) \leq \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + N_{k)}\left(r,\frac{1}{f}\right) + k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right) \\ + \overline{N}\left(r,\frac{1}{g}\right) - N_0\left(r,\frac{1}{g'}\right) + S(r,f) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \Big\{N(r,f) - \overline{N}(r,f)\Big\} &+ \left\{N\left(r,\frac{1}{f}\right) - \overline{N}\left(r,\frac{1}{f}\right)\right\} + \left\{N_{(k+1}\left(r,\frac{1}{f}\right) - k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\right\} \\ &+ (n-2)N\left(r,\frac{1}{f}\right) + m(r,f) + n\, m\left(r,\frac{1}{f}\right) \\ &\leq \overline{N}\left(r,\frac{1}{g}\right) - N_0\left(r,\frac{1}{g'}\right) + S(r,f), \end{aligned} \tag{4.3}$$

where $N_0\left(r, \frac{1}{g'}\right)$ denotes the counting function of those zeros of g' which are not zero of f or g.

Proof .

By Lemma 4.3, we have g is not equivalently constant. Therefore we can write

$$\frac{1}{f^{n+1}} = \frac{\phi f^n f^{(k)}}{f^{n+1}} - \frac{g'}{f^{n+1}} \cdot \frac{g}{g'}.$$

Then

$$\begin{aligned}
(n+1)m\left(r, \frac{1}{f}\right) &\leq m\left(r, \frac{\phi f^{(k)}}{f}\right) + m\left(r, \frac{g'}{f^{n+1}}\right) + m\left(r, \frac{g}{g'}\right) + O(1) \\
&\leq m\left(r, \frac{g}{g'}\right) + S(r, f) \\
&\leq T\left(r, \frac{g}{g'}\right) - N\left(r, \frac{g}{g'}\right) + S(r, f) \\
&= N\left(r, \frac{g'}{g}\right) - N\left(r, \frac{g}{g'}\right) + S(r, f) \\
&\leq \overline{N}(r, g) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g'}\right) + S(r, f) \\
&\leq \overline{N}(r, f) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g'}\right) + S(r, f).
\end{aligned}$$

Therefore

$$\begin{aligned}
(n+1)T(r, f) &= (n+1)m\left(r, \frac{1}{f}\right) + (n+1)N\left(r, \frac{1}{f}\right) + O(1) \\
&\leq (n+1)N\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g'}\right) + S(r, f). \qquad (4.4)
\end{aligned}$$

Let

$$N\left(r, \frac{1}{g'}\right) = N_{000}\left(r, \frac{1}{g'}\right) + N_{00}\left(r, \frac{1}{g'}\right) + N_0\left(r, \frac{1}{g'}\right) + S(r, f),$$

where $N_{000}\left(r, \frac{1}{g'}\right)$ denotes the counting function of those zeros of g' which comes from the zeros of g and $N_{00}\left(r, \frac{1}{g'}\right)$ denotes the counting function of those zeros of g' which comes from the zeros of f. Therefore

$$N\left(r, \frac{1}{g}\right) - N_{000}\left(r, \frac{1}{g'}\right) = \overline{N}\left(r, \frac{1}{g}\right).$$

Let z_0 be a zero of $f(z)$ with multiplicity p and pole of $\phi(z)$ with multiplicity q. Let us observe the following cases:
Case 1: Let $p \leq k$. If $q < np$, then z_0 is a zero of $g'(z)$ with multiplicity at least $(np - q - 1)$. If $q \geq np$, then z_0 is not a zero of $g'(z)$. Hence the zeros of $g'(z)$ come from those zeros of $f(z)$ with multiplicities not greater than k which are poles of $\phi(z)$ with multiplicities less than np.
Case 2: Let $p \geq k+1$. If $q < (n+1)p - k$, then z_0 is zero of $g'(z)$ with multiplicity at least $(n+1)p - k - q - 1$. If $q \geq (n+1)p$, then z_0 is not a zero of $g'(z)$. Hence the zeros of $g'(z)$ come from the zeros of $f(z)$ with multiplicities greater than k and which are poles of $\phi(z)$ with multiplicities less than $(n+1)p - k$.
Therefore

$$\begin{aligned}
N_{00}\left(r, \frac{1}{g'}\right) &\geq nN_{k)}\left(r, \frac{1}{f}\right) - \overline{N}_{k)}\left(r, \frac{1}{f}\right) + (n+1)N_{(k+1}\left(r, \frac{1}{f}\right) \\
&- (k+1)\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) - \big((n+1)p - k - 1\big)\overline{N}(r, \phi) \\
&= nN\left(r, \frac{1}{f}\right) + N_{(k+1}\left(r, \frac{1}{f}\right) - k\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) - \overline{N}\left(r, \frac{1}{f}\right) + S(r, f).
\end{aligned}$$

Therefore from (4.4) we get

$$\begin{aligned}(n+1)T(r,f) &\leq (n+1)N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{g}\right)-nN\left(r,\frac{1}{f}\right)\\ &- N_{(k+1}\left(r,\frac{1}{f}\right)+k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)-N_0\left(r,\frac{1}{g'}\right)+S(r,f)\\ &= \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+N_{k)}\left(r,\frac{1}{f}\right)+k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\\ &- N_0\left(r,\frac{1}{g'}\right)+S(r,f),\end{aligned}$$

which is (4.2). Also

$$\begin{aligned}(n+1)T(r,f) &= T(r,f)+n\,T\left(r,\frac{1}{f}\right)+O(1)\\ &= N(r,f)+m(r,f)+(n-2)N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)+N_{k)}\left(r,\frac{1}{f}\right)\\ &+ N_{(k+1}\left(r,\frac{1}{f}\right)+n\,m\left(r,\frac{1}{f}\right)+S(r,f).\end{aligned}$$

Therefore

$$\begin{aligned}\left\{N(r,f)-\overline{N}(r,f)\right\} &+ \left\{N\left(r,\frac{1}{f}\right)-\overline{N}\left(r,\frac{1}{f}\right)\right\}+\left\{N_{(k+1}\left(r,\frac{1}{f}\right)-k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\right\}\\ &+ (n-2)N\left(r,\frac{1}{f}\right)+m(r,f)+n\,m\left(r,\frac{1}{f}\right)\\ &\leq \overline{N}\left(r,\frac{1}{g}\right)-N_0\left(r,\frac{1}{g'}\right)+S(r,f),\end{aligned}$$

which is (4.3). This completes the proof of Lemma 4.4. □

Lemma 4.5. Let $f(z)$, $\phi(z)(\not\equiv 0)$, $g(z)$, $h(z)$, $F(z)$, a_i 's$(i=1,2,\cdots,9)$, n and k be defined as in the beginning of the section. If the sets $A=\{z:f(z)=0\ or\ \infty\}$ and $B=\{z:\phi(z)=0\ or\ \infty\}$ are disjoint, then the simple poles of $f(z)$ are zeros of $F(z)$.

Proof .

Let z_0 be a simple pole of f. Since $\phi(z_0)\neq 0,\infty$, in some neighbourhood of z_0, we write

$$f(z)=\frac{a}{z-z_0}\Big[1+b_0(z-z_0)+b_1(z-z_0)^2+b_2(z-z_0)^3+O\big((z-z_0)^4\big)\Big] \tag{4.5}$$

and

$$\phi(z)=b\Big[1+c_1(z-z_0)+c_2(z-z_0)^2+c_3(z-z_0)^3+O\big((z-z_0)^4\big)\Big], \tag{4.6}$$

where $a(\neq 0)$, $b(\neq 0)$, b_0, b_1, b_2, c_1, c_2 and c_3 are constants. From (4.5) and (4.6) we get

$$f'(z)=\frac{a}{(z-z_0)^2}\Big[-1+b_1(z-z_0)^2+2b_2(z-z_0)^3+O\big((z-z_0)^4\big)\Big];$$

$$f^{(k)}(z)=\frac{(-1)^k ak!}{(z-z_0)^{k+1}}\Big[1+(-1)^k b_k(z-z_0)^{k+1}+O\big((z-z_0)^{k+2}\big)\Big];$$

$$\begin{aligned} f^n(z) &= \frac{a^n}{(z-z_0)^n}\Big[1 + nb_0(z-z_0) + \frac{1}{2}\Big\{n(n-1)b_0^2 + 2nb_1\Big\}(z-z_0)^2 \\ &+ O\big((z-z_0)^3\big)\Big]; \end{aligned}$$

$$\frac{\phi'(z)}{\phi(z)} = \Big[c_1 + \big(2c_2 - c_1^2\big)(z-z_0) + O\big((z-z_0)^2\big)\Big]; \tag{4.7}$$

$$\left(\frac{\phi'(z)}{\phi(z)}\right)^2 = \Big[c_1^2 + \big(4c_1c_2 - 2c_1^3\big)(z-z_0) + O\big((z-z_0)^2\big)\Big] \tag{4.8}$$

and

$$\left(\frac{\phi'(z)}{\phi(z)}\right)' = \Big[\big(2c_2 - c_1^2\big) + O\big((z-z_0)\big)\Big]. \tag{4.9}$$

Now we discuss the following two cases.
Case 1: Let $k = 1$. Then

$$\begin{aligned} g(z) &= \phi(z)f^n(z)f'(z) - 1 = \frac{-a^{n+1}b}{(z-z_0)^{n+2}}\Big[1 + \big(nb_0 + c_1\big)(z-z_0) \\ &+ \frac{1}{2}\Big\{n(n-1)b_0^2 + 2(n-1)b_1 + 2nb_0c_1 + 2c_2\Big\}(z-z_0)^2 + O\big((z-z_0)^3\big)\Big] \end{aligned}$$

and

$$\begin{aligned} h(z) &= \frac{g'(z)}{f^{n-1}(z)} = \frac{a^2b}{(z-z_0)^4}\Big[(n+2) + \big\{2b_0 + (n+1)c_1\big\}(z-z_0) \\ &+ \big\{b_0c_1 - 2(n-1)b_1 + nc_2\big\}(z-z_0)^2 + O\big((z-z_0)^3\big)\Big]. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \frac{g'(z)}{g(z)} &= \frac{-1}{z-z_0}\Big[(n+2) - \big(nb_0 + c_1\big)(z-z_0) + \big\{nb_0^2 - 2(n-1)b_1 \\ &+ c_1^2 - 2c_2\big\}(z-z_0)^2 + O\big((z-z_0)^3\big)\Big]; \end{aligned} \tag{4.10}$$

$$\begin{aligned} \left(\frac{g'(z)}{g(z)}\right)^2 &= \frac{1}{(z-z_0)^2}\Big[(n+2)^2 - 2(n+2)\big(nb_0 + c_1\big)(z-z_0) \\ &+ \big\{n(3n+4)b_0^2 - 4(n-1)(n+2))b_1 + 2nb_0c_1 + (2n+5)c_1^2 \\ &- 4(n+2)c_2\big\}(z-z_0)^2 + O\big((z-z_0)^3\big)\Big]; \end{aligned} \tag{4.11}$$

$$\begin{aligned} \left(\frac{g'(z)}{g(z)}\right)' &= \frac{1}{(z-z_0)^2}\Big[(n+2) - \big\{nb_0^2 - 2(n-1)b_1 + c_1^2 - 2c_2\big\}(z-z_0)^2 \\ &+ O\big((z-z_0)^3\big)\Big]; \end{aligned} \tag{4.12}$$

$$\begin{aligned} \frac{h'(z)}{h(z)} &= \frac{-1}{z-z_0}\Big[4 - \frac{2b_0 + (n+1)c_1}{n+2}(z-z_0) + \Big\{\frac{4b_0^2 + 2nb_0c_1 + (n+1)^2c_1^2}{(n+2)^2} \\ &+ 2\,\frac{2(n-1)b_1 - nc_2}{n+2}\Big\}(z-z_0)^2 + O\big((z-z_0)^3\big)\Big]; \end{aligned} \tag{4.13}$$

$$\left(\frac{h'(z)}{h(z)}\right)^2 = \frac{1}{(z-z_0)^2}\Bigg[16-8\,\frac{2b_0+(n+1)c_1}{n+2}(z-z_0)+\Bigg\{16\,\frac{2(n-1)b_1-nc_2}{n+2} + \frac{36b_0^2+4(5n+1)b_0c_1+9(n+1)^2c_1^2}{(n+2)^2}\Bigg\}(z-z_0)^2+O\big((z-z_0)^3\big)\Bigg] \tag{4.14}$$

and

$$\left(\frac{h'(z)}{h(z)}\right)' = \frac{1}{(z-z_0)^2}\Bigg[4-\Bigg\{\frac{4b_0^2+2nb_0c_1+(n+1)^2c_1^2}{(n+2)^2}+2\,\frac{2(n-1)b_1-nc_2}{n+2}\Bigg\}(z-z_0)^2 + O\big((z-z_0)^3\big)\Bigg]. \tag{4.15}$$

From (4.7), (4.10) and (4.13) we get

$$\frac{g'(z)}{g(z)}\cdot\frac{h'(z)}{h(z)} = \frac{1}{(z-z_0)^2}\Big[4(n+2)-\big\{2(2n+1)b_0+(n+5)c_1\big\}(z-z_0) + \big\{2(2n+1)b_0^2-4(n-1)b_1+(n+5)c_1^2-2(n+4)c_2 + (n+1)b_0c_1\big\}(z-z_0)^2+O\big((z-z_0)^3\big)\Big]; \tag{4.16}$$

$$\frac{g'(z)}{g(z)}\cdot\frac{\phi'(z)}{\phi(z)} = \frac{-1}{z-z_0}\Big[(n+2)c_1-\big\{(n+3)c_1^2-2(n+2)c_2+nb_0c_1\big\}(z-z_0) + O\big((z-z_0)^2\big)\Big] \tag{4.17}$$

and

$$\frac{h'(z)}{h(z)}\cdot\frac{\phi'(z)}{\phi(z)} = \frac{-1}{z-z_0}\Bigg[4c_1-\Bigg\{\frac{2b_0c_1+(5n+9)c_1^2}{n+2}-8c_2\Bigg\}(z-z_0)+O\big((z-z_0)^2\big)\Bigg]. \tag{4.18}$$

Now substituting values from (4.8), (4.9), (4.11), (4.12) and (4.14) - (4.18) in the expression (4.1) we get $F(z) = O((z-z_0))$, which shows that z_0 is a zero of $F(z)$.

Case 2: Let $k \geq 2$. Then

$$g(z) = \phi(z)f^n(z)f^{(k)}(z)-1 = \frac{(-1)^k k! a^{n+1} b}{(z-z_0)^{n+k+1}}\Bigg[1+\big(nb_0+c_1\big)(z-z_0) + \frac{1}{2}\Big\{n(n-1)b_0^2+2nb_1+2nb_0c_1+2c_2\Big\}(z-z_0)^2+O\big((z-z_0)^3\big)\Bigg]$$

and

$$h(z) = \frac{g'(z)}{f^{n-1}(z)} = \frac{(-1)^{k+1}k!a^2b}{(z-z_0)^{k+3}}\Big[(n+k+1)+\big\{(k+1)b_0+(n+k)c_1\big\}(z-z_0) + \big\{kb_0c_1-(n-k-1)b_1+(n+k-1)c_2\big\}(z-z_0)^2+O\big((z-z_0)^3\big)\Big].$$

Therefore

$$\frac{g'(z)}{g(z)} = \frac{-1}{z-z_0}\Big[(n+k+1)-\big(nb_0+c_1\big)(z-z_0)+\big\{nb_0^2-2nb_1 + c_1^2-2c_2\big\}(z-z_0)^2+O\big((z-z_0)^3\big)\Big]; \tag{4.19}$$

$$\begin{aligned}\left(\frac{g'(z)}{g(z)}\right)^2 &= \frac{1}{(z-z_0)^2}\Big[(n+k+1)^2 - 2(n+k+1)\big(nb_0+c_1\big)(z-z_0)\\ &+ \big\{n(3n+2k+2)b_0^2 - 4n(n+k+1)b_1 + 2nb_0c_1 + (2n+2k+3)c_1^2\\ &- 4(n+k+1)c_2\big\}(z-z_0)^2 + O\big((z-z_0)^3\big)\Big];\end{aligned} \tag{4.20}$$

$$\begin{aligned}\left(\frac{g'(z)}{g(z)}\right)' &= \frac{1}{(z-z_0)^2}\Big[(n+k+1) - \big\{nb_0^2 - 2nb_1 + c_1^2 - 2c_2\big\}(z-z_0)^2\\ &+ O\big((z-z_0)^3\big)\Big];\end{aligned} \tag{4.21}$$

$$\begin{aligned}\frac{h'(z)}{h(z)} &= \frac{-1}{z-z_0}\Big[(k+3) - \frac{(k+1)b_0+(n+k)c_1}{n+k+1}(z-z_0)\\ &+ \left\{\frac{(k+1)^2b_0^2 + 2nb_0c_1 + (n+k)^2c_1^2}{(n+k+1)^2} + 2\,\frac{(n-k-1))b_1 - (n+k-1)c_2}{n+k+1}\right\}(z-z_0)^2\\ &+ O\big((z-z_0)^3\big)\Big];\end{aligned} \tag{4.22}$$

$$\begin{aligned}\left(\frac{h'(z)}{h(z)}\right)^2 &= \frac{1}{(z-z_0)^2}\Big[(k+3)^2 - 2(k+3)\frac{(k+1)b_0+(n+k)c_1}{n+k+1}(z-z_0)\\ &+ \left\{\frac{(k+1)^2(2k+7)b_0^2 + (14n+6nk+2k^2+2k)b_0c_1 + (n+k)^2(2k+7)c_1^2}{(n+k+1)^2}\right.\\ &+ \left.4(k+3)\frac{(n-k-1)b_1 - (n+k-1)c_2}{n+k+1}\right\}(z-z_0)^2 + O\big((z-z_0)^3\big)\Big]\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}\left(\frac{h'(z)}{h(z)}\right)' &= \frac{1}{(z-z_0)^2}\Big[(k+3) - \left\{\frac{(k+1)^2b_0^2 + 2nb_0c_1 + (n+k)^2c_1^2}{(n+k+1)^2}\right.\\ &+ \left.2\,\frac{(n-k-1)b_1 - (n+k-1)c_2}{n+k+1}\right\}(z-z_0)^2 + O\big((z-z_0)^3\big)\Big].\end{aligned} \tag{4.24}$$

From (4.7), (4.19) and (4.22) we get

$$\begin{aligned}\frac{g'(z)}{g(z)}\cdot\frac{h'(z)}{h(z)} &= \frac{1}{(z-z_0)^2}\Big[(n+k+1)(k+3) - \big\{(nk+k+3n+1)b_0\\ &+ (n+2k+3)c_1\big\}(z-z_0) + \big\{(nk+k+3n+1)b_0^2\\ &- 2(nk+k+2n+1)b_1 + (n+2k+3)c_1^2 - 2(n+2k+2)c_2\\ &+ (n+1)b_0c_1\big\}(z-z_0)^2 + O\big((z-z_0)^3\big)\Big];\end{aligned} \tag{4.25}$$

$$\begin{aligned}\frac{g'(z)}{g(z)}\cdot\frac{\phi'(z)}{\phi(z)} &= \frac{-1}{z-z_0}\Big[(n+k+1)c_1 + \big\{2(n+k+1)c_2\\ &- (n+k+2)c_1^2 - nb_0c_1\big\}(z-z_0) + O\big((z-z_0)^2\big)\Big]\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}\frac{h'(z)}{h(z)}\cdot\frac{\phi'(z)}{\phi(z)} &= \frac{-1}{z-z_0}\Big[(k+3)c_1 + \left\{2(k+3)c_2 - \frac{(k+1)b_0c_1}{n+k+1}\right.\\ &- \left.\frac{((n+k+1)(k+3)+(n+k))\,c_1^2}{n+k+1}\right\}(z-z_0) + O\big((z-z_0)^2\big)\Big].\end{aligned} \tag{4.27}$$

Now substituting values from (4.8), (4.9), (4.20), (4.21) and (4.23) - (4.27) in the expression (4.1) we get $F(z) = O\big((z-z_0)\big)$ i.e., z_0 is a zero of $F(z)$. This proves that the simple poles of $f(z)$ are zeros of $F(z)$. This completes the proof of *Lemma 4.5.* □

Lemma 4.6. Let $f(z)$, $\phi(z)(\not\equiv 0)$, $g(z)$, $h(z)$, $F(z)$, a_i's $(i = 1, 2, \cdots, 9)$, $n(> 2)$ and k be defined as in the beginning of this section. If the sets $A = \{z : f(z) = 0 \ or \ \infty\}$ and $B = \{z : \phi(z) = 0 \ or \ \infty\}$ are disjoint and $\phi(z)$ has no zero of multiplicity n, then $F(z) \not\equiv 0$.

Proof . If possible, we assume that $F(z) \equiv 0$. Under this hypothesis we shall show that
i) $g(z)$ has no zero,
ii) $\phi(z)$ has no zero and pole,
iii) $h(z)$ has no zero,
iv) $f(z)$ has no multiple zero.

Suppose that z_1 is a zero of $g(z)$ of multiplicity $l_1(\geq 1)$. Then it is clear that $f(z_1) \neq 0, \infty$, $\phi(z_1) \neq 0, \infty$ and z_1 is a zero of $h(z)$ with multiplicity $(l_1 - 1)$. Then from Laurent series expansion of $F(z)$ we get the coefficient of $(z - z_1)^{-2}$ as

$$A(l_1) = (a_1 + a_3 + a_4)l_1^2 - (a_2 + a_3 + 2a_4 + a_5)l_1 + (a_4 + a_5).$$

For $k = 1$, putting values of a_i's $(i = 1, 2, \cdots, 9)$ we get

$$A(l_1) = -\big\{(n+1)(3n^2 - 2n - 2)l_1^2 + (n+2)(4n^2 - 3n - 4)l_1 + (n+2)^2(n-2)\big\}.$$

Obviously $A(l_1)$ does not vanish for any positive integral values of l_1. Thus z_1 is a pole of $F(z)$, which contradicts our hypothesis.
For $k \geq 2$, putting the values of a_i's $(i = 1, 2, \cdots, 9)$ we get

$$\begin{aligned} A(l_1) &= (n-1)\big\{(n-1)k + (3n-1)\big\}\Big[\big\{(k+1)n^4 + 2(k^2+5k+10)n^3 + (k+1)^2(k+2)n^2 \\ &- (k+1)^2(2k+5)n + (k+1)^3\big\}l_1^2 + (n+k+1)(k+1)\big\{(k+1)n^3 + (k^2+4k+9)n^2 \\ &- (2k^2+7k+5)n + (k+1)^2\big\}l_1 - n(n+k+1)^2(k+1)\big\{(n-1)k + (2n-1)\big\}\Big]. \end{aligned}$$

By Lemma 4.2 we get that $A(l_1)$ does not vanish for any positive integral values of l_1. Therefore z_1 is a pole of $F(z)$, which is contradictory to our hypothesis. Thus z_1 is not a zero of $g(z)$. Therefore $g(z)$ has no zero.

Now let z_2 be a zero of $\phi(z)$ of multiplicity $l_2(\geq 1)$. Then it is a zero of $h(z)$ of multiplicity $(l_2 - 1)$ but not a zero of $g(z)$. Therefore from Laurent series expansion of $F(z)$ we get the coefficient of $(z - z_2)^{-2}$ as

$$B(l_2) = (a_4 + a_7 + a_8)l_2^2 - (2a_4 + a_5 + a_7 + a_9)l_2 + (a_4 + a_5).$$

For $k = 1$, putting values of a_i's $(i = 1, 2, \cdots, 9)$ we get

$$B(l_2) = -(n+2)^2\big\{2(2n^2 - 3n - 8)l_2^2 - (4n^2 - 4n - 17)l_2 + (n-2)\big\}.$$

$B(l_2) = 0$ gives

$$l_2 = \frac{(4n^2 - 4n - 17) \pm \sqrt{16n^4 - 48n^3 - 64n^2 + 152n + 161}}{4(2n^2 - 3n - 8)}.$$

Now let $d = 16n^4 - 48n^3 - 64n^2 + 152n + 161$ and $M = (4n^2 - 6n - 13)$. Then we see that $M^2 < d < (M+1)^2$ for $n > 2$. Thus d is not a perfect square for $n > 2$. Therefore $B(l_2)$ does not vanish for any positive integer value of l_2. Then z_2 is pole of $F(z)$, a contradiction.

For $k \geq 2$, putting the values of a_i's $(i = 1, 2, \cdots, 9)$ we get

$$B(l_2) = (k+1)(n-1)(n+k+1)^2\big\{(n-1)k + (3n-1)\big\}\Big[l_2^2 + (k+1)(n-1)l_2 - n\big\{(n-1)k + (2n-1)\big\}\Big].$$

$B(l_2) = 0$ gives $l_2 = n, -\big\{(n-1)k + (2n-1)\big\}$. Since $\phi(z)$ has no zero of multiplicity n, $B(l_2)$ does not vanish for any positive integral values of l_2. Then z_2 is a pole of $F(z)$, a contradiction. Therefore z_2 is not zero of $\phi(z)$ and hence $\phi(z)$ has no zero.

Let z_3 be a pole of $\phi(z)$ of multiplicity $l_3(\geq 1)$. Then it is a pole of $g(z)$ of multiplicity l_3 and a pole of $h(z)$ of multiplicity (l_3+1). Therefore from Laurent series expansion of $F(z)$ we get the coefficient of $(z-z_3)^{-2}$ as

$$C(l_3) = (a_1+a_3+a_4+a_6+a_7+a_8)l_3^2 + (a_2+a_3+2a_4+a_5+a_7+a_9)l_3 + (a_4+a_5).$$

For $k=1$, putting values of a_i's $(i=1,2,\cdots,9)$ we get

$$C(l_3) = -\big\{(4n^4+10n^3-20n^2-62n-42)l_3^2 + (4n^4+8n^3-19n^2-62n-48)l_3 + (n+2)^2(n-2)\big\}.$$

Now $(4n^4+10n^3-20n^2-62n-42)$, $(4n^4+8n^3-19n^2-62n-48)$ and $(n+2)^2(n-2)$ are positve for all values of $n(>2)$. Therefore $C(l_3)$ does not vanish for any positive integral values of l_3. Then z_3 is a pole of $F(z)$, a contradiction. Therefore z_3 is not a pole of $\phi(z)$.

For $k(\geq 2)$, putting the values of a_i's $(i=1,2,\cdots,9)$ we get

$$\begin{aligned} C(l_3) &= -n(n-1)(k+1)\Big[\big\{3(n-1)^2k^2+(17n^2-23n+6)k+(24n^2-17n+3)\big\}l_3^2 \\ &+ 2\big\{2(n-1)^2k^3+(2n^3+9n^2-17n+6)k^2+(11n^3+11n^2-22n+6)k \\ &+ (15n^3+4n^2-9n+2)\big\}l_3 + (n+k+1)^2\big\{(n-1)k+(3n-1)\big\}\big\{(n-1)k+(2n-1)\big\}\Big] \end{aligned}$$

$C(l_3)=0$ gives $l_3 = -(n+k+1)$ or $-\frac{(n-1)k^2+(n^2+2n-2)k+(2n^2+n-1)}{3(n-1)k+8n-3}$. Clearly $C(l_3)$ does not vanish for any positive integral values of l_3. Then z_3 is a pole of $F(z)$, a contradiction. Therefore $\phi(z)$ has no pole.

Let z_4 be a zero of $h(z)$ of multiplicity l_4. Then z_4 may be a zero of $\phi(z)$ of multiplicity (l_4+1). But we already have shown that $\phi(z)$ has no zero. Also z_4 is not a zero or pole of $g(z)$. Then from Laurent series expansion of $F(z)$ we get the coefficient of $(z-z_4)^{-2}$ as

$$D(l_4) = a_4 l_4^2 - a_5 l_4.$$

Clearly $D(l_4)$ does not vanish for any positive values of l_4. Then z_4 is a pole of $F(z)$, a contradiction. Hence $h(z)$ has no zero.

Now since multiple zeros of $f(z)$ are also zero of $h(z)$, $f(z)$ has no multiple zero.

Set

$$\begin{aligned} \psi(z) &= \frac{h(z)}{g(z)} = \frac{g'(z)}{g(z)}\cdot\frac{1}{f^{n-1}(z)} \\ &= \frac{\phi(z)\big\{f(z)f^{(k+1)}(z)+nf'(z)f^{(k)}(z)\big\}+\phi'(z)f(z)f^{(k)}(z)}{\phi(z)f^n(z)f^{(k)}(z)-1}. \end{aligned} \tag{4.28}$$

Therefore

$$\frac{g'(z)}{g(z)} = \psi(z)f^{n-1}(z),\ h(z)=\psi(z)g(z)$$

and

$$\frac{h'(z)}{h(z)} = \frac{g'(z)}{g(z)} + \frac{\psi'(z)}{\psi(z)} = \psi(z)f^{n-1}(z) + \frac{\psi'(z)}{\psi(z)}.$$

Substituting these values in the expression (4.1) we get

$$\begin{aligned} &(a_1+a_3+a_4)\psi^2 f^{2n-2} + \left\{\Big(a_2+a_3+2a_4+a_5\Big)\frac{\psi'}{\psi} + \Big(a_6+a_7\Big)\frac{\phi'}{\phi}\right\}\psi f^{n-1} \\ &\qquad +(n-1)\big(a_2+a_5\big)\psi f^{n-2}f' + \left\{a_4\left(\frac{\psi'}{\psi}\right)^2 + a_5\left(\frac{\psi'}{\psi}\right)' \right. \\ &\qquad\qquad \left. +a_7\frac{\psi'}{\psi}\cdot\frac{\phi'}{\phi} + a_8\left(\frac{\phi'}{\phi}\right)^2 + a_9\left(\frac{\phi'}{\phi}\right)'\right\} \equiv 0. \end{aligned} \tag{4.29}$$

From this we have

$$f' = \frac{l_{1,1}}{\psi f^{n-2}} + l_{1,2}f + l_{1,3}f^n\psi, \tag{4.30}$$

where $l_{1,1}, l_{1,2}, l_{1,3}$ are differential polynomial of $\frac{\psi'}{\psi}$ and $\frac{\phi'}{\phi}$.
We observe that since $g(z)$ has no zero, $\phi(z)$ has no pole and poles of $f(z)$ can not be pole of $\psi(z)$, it (i.e., $\psi(z)$) is an entire function. Also since $f(z)$ has no multiple zero, zeros of $f(z)$ can not be a zero or a pole of $\psi(z)$ and $\phi(z)$. Also simple zeros of $f(z)$ are not zero of $l_{1,1}$.
Let z_5 be a zero of $f(z)$. Then for $n > 2$, from (4.30) we get z_5 is pole of $f'(z)$, which is a contradiction. Therefore $f(z)$ has no zero. Thus

$$N\left(r, \frac{1}{f}\right) = 0.$$

From (4.3) of Lemma 4.4 we get

$$m\left(r, \frac{1}{f}\right) = S(r, f).$$

Therefore

$$T(r, f) = T\left(r, \frac{1}{f}\right) + O(1) = N\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f}\right) + O(1) = S(r, f).$$

This is a contradiction. Therefore $F(z) \not\equiv 0$ for $n > 2$. This completes the proof of the lemma.
□

5 Proof of the Theorem

Proof . By Lemma 4.5 and Lemma 4.6 we have seen that the simple poles of $f(z)$ are zeros of $F(z)$ and $F(z) \not\equiv 0$. Now we have $g(z) = \phi(z)f^n(z)f^{(k)}(z) - 1$ and

$$h(z) = \frac{g'(z)}{f^{n-1}(z)} = \phi(z)\{f(z)f^{(k+1)}(z) + nf'(z)f^{(k)}(z)\} + \phi'(z)f(z)f^{(k)}(z).$$

Let

$$\beta(z) = h(z) - \phi(z)f^n(z)f^{(k)}(z)\frac{h(z)}{g(z)}. \tag{5.1}$$

Therefore

$$\beta f^{n-1} = -\frac{g'}{g} \ \textit{or}\ \beta = -\frac{g'}{g}\cdot\frac{1}{f^{n-1}} \ \textit{or}\ \beta = -\frac{h}{g} \ \textit{or}\ h = -\beta g. \tag{5.2}$$

Now we consider the poles of $\beta^2 F$. From Lemma 4.5 we observe that the poles of $F(z)$ are of multiplicities at most 2 and come from the zeros and poles of $g(z)$ or $h(z)$ or $\phi(z)$ except the zeros of $\phi(z)$ with multiplicity n. From (5.2) we can see that the poles of $\beta(z)$ are zeros of $g(z)$ or poles of $h(z)$. Now poles of $g(z)$ and $h(z)$ come from the poles of $\phi(z)$ and $f(z)$. But we see that a pole of $f(z)$ of order $s(\geq 2)$ is a zero of $\beta(z)$ of order $(n-1)s - 1 \geq 1$. Therefore poles of $f(z)$ can not be pole of $\beta^2 F$. Also from (5.2) we can see that zeros of $h(z)$ comes from zeros of $\beta(z)$ and $g(z)$. Now zeros of $\beta(z)$ come from multiple zeros of $f(z)$ and $\phi(z)$. But multiple zeros of $f(z)$ and $\phi(z)$ are pole of $F(z)$ of order at most 2 and zero of $\beta^2(z)$ of order at least 2. Therefore multiple zeros of $f(z)$ and $\phi(z)$ can not be pole of $\beta^2 F$. Then poles of $\beta^2 F$ comes only from zeros of $g(z)$, poles of $\phi(z)$ and simple zeros of $\phi(z)$.

Let z_7 be a zero of $g(z)$ of multiplicity t. Then z_7 is not a zero of $f(z)$ or $\phi(z)$. Therefore z_7 is a zero of $g'(z)$ and $h(z)$ with multiplicity $(t-1)$ and hence a simple pole of $\beta(z)$. Also we remember that the zeros of $h(z)$ can be pole of $F(z)$ of order at most 2. Therefore z_7 is a pole of $\beta^2 F$ of order at most 4. Therefore

$$N(r, \beta^2 F) \leq 4\overline{N}\left(r, \frac{1}{g}\right) + N(r, \phi) + 2\overline{N}\left(r, \frac{1}{\phi}\right) = 4\overline{N}\left(r, \frac{1}{g}\right) + S(r, f). \tag{5.3}$$

Now from the expression (4.1) of $F(z)$ we get $m(r,F) = S(r,f)$. Also using Lemma 4.1 we get from (5.2) that $m(r,\beta^2) = S(r,f)$. Thus $m(r,\beta^2 F) = S(r,f)$. Therefore

$$T(r,\beta^2 F) \leq 4\overline{N}\left(r,\frac{1}{g}\right) + S(r,f). \tag{5.4}$$

Now zeros of $f(z)$ of order $\mu(\geq k+1)$ are zero of $\beta(z)$ of order at least $(2\mu - k - 1)$. Also zeros of $f(z)$ are not zero of $g(z)$ but zero of $h(z)$ of order $(2\mu - k - 1)$ and then pole of $F(z)$ of order 2. Therefore zeros of $\beta^2 F$ are of multiplicity at least $(4\mu - 2k - 4)$. Also simple poles of $f(z)$ are zero of $\beta^2 F$. Therefore

$$\begin{aligned} & N_1(r,f) + 4N_{(k+1}\left(r,\frac{1}{f}\right) - 2(k+2)\overline{N}_{(k+1}\left(r,\frac{1}{f}\right) \\ \leq\ & N\left(r,\frac{1}{\beta^2 F}\right) \leq T\left(r,\frac{1}{\beta^2 F}\right) \leq 4\overline{N}\left(r,\frac{1}{g}\right) + S(r,f). \end{aligned}$$

Combining this inequality with twice of (4.2) of Lemma 4.4 we get

$$\begin{aligned} & 2(n+1)T(r,f) - 2\overline{N}(r,f) - 2\overline{N}\left(r,\frac{1}{f}\right) - 2N_{k)}\left(r,\frac{1}{f}\right) - 2k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right) \\ & + N_1(r,f) + 4N_{(k+1}\left(r,\frac{1}{f}\right) - 2(k+2)\overline{N}_{(k+1}\left(r,\frac{1}{f}\right) \leq 6\overline{N}\left(r,\frac{1}{g}\right) + S(r,f). \end{aligned} \tag{5.5}$$

Now

$$\begin{aligned} (2n+2)T(r,f) &= (2n-3)T(r,f) + T(r,f) + 4T\left(r,\frac{1}{f}\right) \\ &\geq (2n-3)T(r,f) + N(r,f) + 4N\left(r,\frac{1}{f}\right). \end{aligned} \tag{5.6}$$

From (5.5) and (5.6) we get

$$\begin{aligned} (2n-3)T(r,f) &+ \left\{N(r,f) + N_1(r,f) - 2\overline{N}(r,f)\right\} + \left\{4N\left(r,\frac{1}{f}\right) - 2\overline{N}\left(r,\frac{1}{f}\right)\right. \\ &- \left. 2N_{k)}\left(r,\frac{1}{f}\right) + 4N_{(k+1}\left(r,\frac{1}{f}\right) - 4(k+1)\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\right\} \\ &\leq 6\overline{N}\left(r,\frac{1}{g}\right) + S(r,f). \end{aligned} \tag{5.7}$$

Now

$$\begin{aligned} N(r,f) + N_1(r,f) - 2\overline{N}(r,f) &\geq N_1(r,f) + N_{(2}(r,f) + N_1(r,f) - 2\overline{N}_1(r,f) - 2\overline{N}_{(2}(r,f) \\ &= N_{(2}(r,f) - 2\overline{N}_{(2}(r,f) \geq 0 \end{aligned}$$

and

$$\begin{aligned} 4N\left(r,\frac{1}{f}\right) &- 2\overline{N}\left(r,\frac{1}{f}\right) - 2N_{k)}\left(r,\frac{1}{f}\right) + 4N_{(k+1}\left(r,\frac{1}{f}\right) - 4(k+1)\overline{N}_{(k+1}\left(r,\frac{1}{f}\right) \\ &= 2\left\{N\left(r,\frac{1}{f}\right) - \overline{N}\left(r,\frac{1}{f}\right)\right\} + 2\left\{N\left(r,\frac{1}{f}\right) - N_{k)}\left(r,\frac{1}{f}\right)\right\} \\ &+ 4\left\{N_{(k+1}\left(r,\frac{1}{f}\right) - (k+1)\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\right\} \geq 0. \end{aligned}$$

Therefore from (5.7) we have

$$T(r,f) \leq \frac{6}{2n-3}\overline{N}\left(r,\frac{1}{g}\right) + S(r,f).$$

This completes the proof. □

6 Open Problems

Question 6.1. Is it possible to remove the condition '$\phi(z)$ has no zero of multiplicity n' in Theorem 3.1 ?

Question 6.2. Can the condition that 'the set of zeros and poles of $f(z)$ and that of $\phi(z)$ are disjoint' in Theorem 3.1 be removed ?

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