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Integral inequalities for functions satisfying certain convexity conditions

Serap Kemali[a], Sevda Sezer[b], Zeynep Eken[b], Sinem Sezer Evcan[b,*]

[a] *Akdeniz University, Vocational School of Technical Sciences, Antalya, Turkey*

[b] *Akdeniz University, Faculty of Education, Antalya, Turkey*

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Abstract

In this article, some integral equations are obtained and based on these integral equations, new integral inequalities are obtained for convex functions that satisfy certain convexity conditions.

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1 Introduction

Recently, many studies have been carried out on the abstractization of the concept of convexity [1, 2, 3, 4, 5, 12, 17, 18, 19, 20]. Some important applications of different convex function classes obtained as a result of these studies are also examined. One of the important applications is to obtain integral inequalities for functions that meet certain conditions. Studies in this direction have contributed greatly to the Inequality Theory. For example, Hermite-Hadamard inequality known for classical convex functions has been obtained in different abstract convex functions [9, 14, 15, 24]. Moreover, for some abstract convex classes, inequalities with fractional integral operators have also been found [10, 13, 21, 22, 25].

One way to achieve such inequalities is to take advantage of known equalities and inequalities. For example, based on the equality known as Lemma 1.1, some inequalities have been obtained using Hölder integral inequality and Power-mean integral inequality [11, 16]. In the literature, there are many papers obtained for different abstract convexity classes using the similar method [6, 8, 14, 15, 16].

Throughout the article, I and I^0 will denote a subinterval of $\mathbb{R}$ and interior of I, respectively.

Lemma 1.1. [11] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be n-times differentiable function on I^o for $n \in \mathbb{N}$ and let $f^{(n)} \in L[a, b]$, where $a, b \in I^o$ with $a < b$. We have the identity

$$\sum_{k=0}^{n-1} (-1)^k \left[\frac{b^{k+1} f^{(k)}(b) - a^{k+1} f^{(k)}(a)}{(k+1)!}\right] - \int_a^b f(x)dx = \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x)dx.$$

*Corresponding author
Email addresses: skemali@akdeniz.edu.tr (Serap Kemali), sevdasezer@akdeniz.edu.tr (Sevda Sezer), zeynepeken@akdeniz.edu.tr (Zeynep Eken), sinemsezer@akdeniz.edu.tr (Sinem Sezer Evcan)

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One of the many theorems proved with the help of this lemma is as follows.

Theorem 1.2. [11] For $n \in \mathbb{N}$; let $f : I \subset (0,\infty) \to \mathbb{R}$ be $n-$times differentiable function on I^o and let $a, b \in I^o$ with $a < b$. If $\left|f^{(n)}\right|^q$ for $q > 1$ is quasi-convex on $[a, b]$, then the following inequality holds;

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{b-a}\right)-\int_a^b f(x)dx\right| \le \frac{1}{n!}(b-a)M_{n,q}^{\frac{1}{q}}(f)L_{np}^n(a,b).$$

One of the inequalities used to achieve integral inequalities is given in [8].

In this article, general form of Lemma 1.1 is proved. For the special cases of the g function mentioned in Lemma 2.1.1, ways of obtaining different integral inequalities are shown.

Let us give some concepts to be used in the article.

Let I be an interval on $\mathbb{R}$ and let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a functions. f is said to be convex if the inequality

$$f(\lambda x + \mu y) \le \lambda f(x) + \mu f(y)$$

is valid whenever $x, y \in I$ and $\lambda, \mu \ge 0$ with $\lambda + \mu = 1$. Geometrically, f is convex if every chord joining two points on its graph lies or above the graph. If $-f : I \longrightarrow \mathbb{R}$ is convex, then $f : I \longrightarrow \mathbb{R}$ is said to be concave.

Let I be an interval on $\mathbb{R}$ and let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function. f is said to be quasi-convex if

$$f(\lambda x + \mu y) \le \max\{f(x), f(y)\}$$

whenever $x, y \in I$ and $\lambda, \mu \ge 0$ with $\lambda + \mu = 1$.

The arithmetic mean and logarithmic mean for $x, y \in \mathbb{R}^+$ are defined as follows:

$$\begin{aligned} A(x,y) &= \frac{x+y}{2}, \\ L(x,y) &= \begin{cases} x & , \ x = y \\ \frac{x-y}{\ln x - \ln y} & , \ x \ne y. \end{cases} \end{aligned}$$

Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$. Hölder's inequality for integrals states that

$$\int_a^b |f(x)g(x)|\, dx \le \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}}\left(\int_a^b |g(x)|^q\, dx\right)^{\frac{1}{q}} \tag{1.1}$$

with equality when $|g(x)| = c\,|f(x)|^{p-1}$.

If $p = q = 2$, then Inequality (1.1) becomes to Schwarz's inequality.

2 Main Results

2.1 Generalization of Lemma 1.1.

Lemma 2.1.1. Let $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ be $n-$times differentiable functions on I^o for $n \in \mathbb{N}$ and $f^{(n)}, g^{(n)}$ be integrable functions on $[a, b]$ where $a, b \in I^o$ with $a < b$, we have the equality

$$\int_a^b g(x)f^{(n)}(x)dx = \sum_{k=1}^{n}(-1)^{k+1}\left[g^{(k-1)}(b)f^{(n-k)}(b) - g^{(k-1)}(a)f^{(n-k)}(a)\right] - (-1)^{n-1}\int_a^b g^{(n)}(x)f(x)dx. \tag{2.1}$$

Proof . The induction method can be used to proof. Using integration by parts for $n = 1$, we obtain

$$\int_a^b g(x)f'(x)dx = g(b)f(b) - g(a)f(a) - \int_a^b g'(x)f(x)dx \Longrightarrow \int_a^b [g(x)f(x)]'\, dx = g(b)f(b) - g(a)f(a). \tag{2.2}$$

Equality (2.2) coincides with Equality (2.1) for $n = 1$. Suppose that Equality (2.1) holds for $n = t$. That is

$$\int_a^b g(x)f^{(t)}(x)dx = \sum_{k=1}^{t}(-1)^{k+1}\left[g^{(k-1)}(b)f^{(t-k)}(b) - g^{(k-1)}(a)f^{(t-k)}(a)\right] - (-1)^{t-1}\int_a^b g^{(t)}(x)f(x)dx.$$

Using the integration by parts, for $n = t+1$ we have

$$\begin{aligned}\int_a^b g(x)f^{(t+1)}(x)dx &= g(b)f^{(t)}(b) - g(a)f^{(t)}(a) - \int_a^b g'(x)f^{(t)}(x)dx \\ &= \sum_{k=0}^{t}(-1)^k\left[g^{(k)}(b)f^{(t-k)}(b) - g^{(k)}(a)f^{(t-k)}(a)\right] - (-1)^t\int_a^b g^{(t+1)}(x)f(x)dx \\ &= \sum_{k=1}^{t+1}(-1)^{k+1}\left[g^{(k-1)}(b)f^{(t-k+1)}(b) - g^{(k-1)}(a)f^{(t-k+1)}(a)\right] - (-1)^t\int_a^b g^{(t+1)}(x)f(x)dx.\end{aligned}$$

We proved the accuracy of Equality (2.1) for $t+1$ and the proof is complete. □

Remark 2.1.2. If the function g is taken specifically $g(x) = x^n$, Lemma 2.1.1 is obtained [11]. So, Lemma 1.1 in [11] is a special case of Lemma 2.1.1.

It is possible to achieve some inequalities in the Equation (2.1) by replacing $g(x)$ with different functions or by taking $f^{(n)}(x)$ from different function classes [11, 16].

We can achieve different results by changing the function $g(x)$. For example, let us take $g(x) = e^x$.

Corollary 2.1.3. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be $n-$times differentiable function on I^o for $n \in \mathbb{N}$ and $f^{(n)}$ be an integrable function on $[a, b]$ where $a, b \in I^o$ with $a < b$, we have the identity

$$\int_a^b e^x f^{(n)}(x)dx = \sum_{k=1}^{n}(-1)^{k+1}\left[e^b f^{(n-k)}(b) - e^a f^{(n-k)}(a)\right] - (-1)^{n-1}\int_a^b e^x f(x)dx.$$

2.2 Some applications of Corollary 2.1.3

Let us give some of the results obtained with the help of Corollary 2.1.3 which is one of the special cases of Lemma 2.1.1.

Theorem 2.2.1. Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be $n-$times differentiable function on I^o for $n \in \mathbb{N}$. If $f^{(n)}$ is an integrable function and $\left|f^{(n)}\right|^q$ for $q > 1$ is convex on $[a, b]$ where $a, b \in I^o$ with $a < b$, then the following inequality holds:

$$\begin{aligned}&\left|\sum_{k=1}^{n}(-1)^{k+1}\frac{\left[e^b f^{(n-k)}(b) - e^a f^{(n-k)}(a)\right]}{b-a} - \frac{(-1)^{n-1}}{b-a}\int_a^b e^x f(x)dx\right| \\ &\qquad\leq A^{\frac{1}{q}}\left(\left|f^{(n)}(b)\right|^q, \left|f^{(n)}(a)\right|^q\right) L^{\frac{1}{p}}\left(e^{bp}, e^{ap}\right).\end{aligned} \tag{2.3}$$

Proof . It is given that $\left|f^{(n)}\right|^q$ is convex for $q > 1$ on $[a, b]$. Using Corollary 2.1.3, the Hölder integral inequality and the following equality

$$\left|f^{(n)}(x)\right|^q = \left|f^{(n)}\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right)\right|^q \leq \frac{x-a}{b-a}\left|f^{(n)}(b)\right|^q + \frac{b-x}{b-a}\left|f^{(n)}(a)\right|^q,$$

we have,

$$\begin{aligned}\left|\sum_{k=1}^{n}(-1)^{k+1}\left[e^{b}f^{(n-k)}(b)\right.\right. &- \left.\left. e^{a}f^{(n-k)}(a)\right]-(-1)^{n-1}\int_{a}^{b}e^{x}f(x)dx\right| \\ &\leq \int_{a}^{b}e^{x}\left|f^{(n)}(x)\right|dx \\ &\leq \left(\int_{a}^{b}e^{px}dx\right)^{\frac{1}{p}}\left(\int_{a}^{b}\left|f^{(n)}(x)\right|^{q}dx\right)^{\frac{1}{q}} \\ &\leq \left(\int_{a}^{b}e^{px}dx\right)^{\frac{1}{p}}\left(\int_{a}^{b}\left(\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^{q}+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^{q}\right)dx\right)^{\frac{1}{q}} \\ &= \left(\frac{e^{bp}-e^{ap}}{p(b-a)}\right)^{\frac{1}{p}}(b-a)\left(\frac{\left|f^{(n)}(b)\right|^{q}+\left|f^{(n)}(a)\right|^{q}}{2}\right)^{\frac{1}{q}} \\ &= (b-a)\,L^{\frac{1}{p}}\left(e^{bp},e^{ap}\right)A^{\frac{1}{q}}\left(\left|f^{(n)}(b)\right|^{q},\left|f^{(n)}(a)\right|^{q}\right).\end{aligned}$$

If both sides of the inequality are multiplied by $\frac{1}{b-a}$, inequality (2.3) is obtained. □

Under the conditions of Theorem 2.2.1, the inequality (2.3) for $n=1$ is as follows,

$$\left|\frac{e^{b}f(b)-e^{a}f(a)}{b-a}-\frac{1}{b-a}\int_{a}^{b}e^{x}f(x)dx\right|\leq L^{\frac{1}{p}}\left(e^{bp},e^{ap}\right)A^{\frac{1}{q}}\left(\left|f'(b)\right|^{q},\left|f'(a)\right|^{q}\right).$$

Theorem 2.2.2. Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be $n-$times differentiable function on I^{o} for $n\in\mathbb{N}$, If $f^{(n)}$is an integrable function and $\left|f^{(n)}\right|^{q}$ for $q>1$ is concave on $[a,b]$, where $a,b\in I^{o}$ with $a<b$, then the following inequality holds;

$$\left|\sum_{k=1}^{n}(-1)^{k+1}\frac{\left[e^{b}f^{(n-k)}(b)-e^{a}f^{(n-k)}(a)\right]}{b-a}-\frac{(-1)^{n-1}}{b-a}\int_{a}^{b}e^{x}f(x)dx\right|\leq\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|L^{\frac{1}{p}}\left(e^{bp},e^{ap}\right). \qquad (2.4)$$

Proof . Using Hermite-Hadamard type inequality and concavity of $\left|f^{(n)}\right|^{q}$ for $q>1$ on $[a,b]$, we get

$$\int_{a}^{b}\left|f^{(n)}(x)\right|^{q}dx\leq(b-a)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^{q}.$$

Then, using Corollary 2.1.3 and Hölder integral inequality we have

$$\begin{aligned}\left|\sum_{k=1}^{n}(-1)^{k-1}\left[e^{b}f^{(n-k)}(b)-e^{a}f^{(n-k)}(a)\right]\right. &- \left.(-1)^{n-1}\int_{a}^{b}e^{x}f(x)dx\right| \\ &\leq \int_{a}^{b}e^{x}\left|f^{(n)}(x)\right|dx \\ &\leq \left(\int_{a}^{b}e^{px}dx\right)^{\frac{1}{p}}\left(\int_{a}^{b}\left|f^{(n)}(x)\right|^{q}dx\right)^{\frac{1}{q}} \\ &\leq \left(\frac{e^{bp}-e^{ap}}{p}\right)^{\frac{1}{p}}\left((b-a)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}} \\ &= \left(\frac{e^{bp}-e^{ap}}{p(b-a)}\right)^{\frac{1}{p}}(b-a)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right| \\ &= (b-a)\,L^{\frac{1}{p}}\left(e^{bp},e^{ap}\right)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|.\end{aligned}$$

□

Under the conditions of Theorem 2.2.2, the Inequality (2.3) for $n=1$ is as follows,

$$\left|\frac{e^{b}f(b)-e^{a}f(a)}{b-a}-\frac{1}{b-a}\int_{a}^{b}e^{x}f(x)dx\right|\leq L^{\frac{1}{p}}\left(e^{bp},e^{ap}\right)\left|f'\left(\frac{a+b}{2}\right)\right|.$$

Theorem 2.2.3. Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be $n-$times differentiable function on I^o for $n \in \mathbb{N}$ If $f^{(n)}$ is an integrable function and $\left|f^{(n)}\right|^q$ for $q > 1$ is quasi-convex on $[a, b]$ where $a, b \in I^o$ with $a < b$ and

$$M_{n,q}(f) = \max\left\{\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right\},$$

then the following inequality holds;

$$\left|\sum_{k=1}^{n}(-1)^{k+1}\left[\frac{e^b f^{(n-k)}(b) - e^a f^{(n-k)}(a)}{b-a}\right] - (-1)^{n-1}\frac{1}{b-a}\int_a^b e^x f(x)dx\right| \leq L^{\frac{1}{p}}\left(e^{bp}, e^{ap}\right) M_{n,q}^{\frac{1}{q}}(f). \tag{2.5}$$

Proof . Using Corollary 2.1.3 and Hölder integral inequality and

$$\left|f^{(n)}(x)\right|^q = \left|f^{(n)}\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right)\right|^q \leq M_{n,q}(f),$$

we have

$$\begin{aligned} \left|\sum_{k=1}^{n}(-1)^{k+1}\left[e^b f^{(n-k)}(b) - e^a f^{(n-k)}(a)\right] - (-1)^{n-1}\int_a^b e^x f(x)dx\right| &\leq \left(\int_a^b e^{px}dx\right)^{\frac{1}{p}}\left(\int_a^b \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\ &\leq \left(\frac{e^{bp}-e^{ap}}{p}\right)^{\frac{1}{p}}\left((b-a)M_{n,q}(f)\right)^{\frac{1}{q}} \\ &\leq (b-a)L^{\frac{1}{p}}\left(e^{bp}, e^{ap}\right) M_{n,q}^{\frac{1}{q}}(f). \end{aligned}$$

□

Under the conditions of Theorem 2.2.3 for $n = 1$, we have the following inequality,

$$\left|\frac{e^b f(b) - e^a f(a)}{b-a} - \frac{1}{b-a}\int_a^b e^x f(x)dx\right| \leq L^{\frac{1}{p}}\left(e^{bp}, e^{ap}\right) M_{1,q}^{\frac{1}{q}}(f).$$

Theorem 2.2.4. Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be $n-$times differentiable function on I^o for $n \in \mathbb{N}$. If $f^{(n)}$is an integrable function and $\left|f^{(n)}\right|^q$ for $q > 1$ is quasi-convex on $[a, b]$, where $a, b \in I^o$ with $a < b$ and

$$M_{n,q}(f) = \max\left\{\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right\},$$

then the following inequality holds;

$$\left|\sum_{k=1}^{n}(-1)^{k+1}\left[\frac{e^b f^{(n-k)}(b) - e^a f^{(n-k)}(a)}{e^b - e^a}\right] - (-1)^{n-1}\frac{1}{e^b - e^a}\int_a^b e^x f(x)dx\right| \leq M_{n,q}^{\frac{1}{q}}(f). \tag{2.6}$$

Proof . From Corollary 2.1.3 and Power-mean integral inequality, we obtain

$$\begin{aligned} \left|\sum_{k=1}^{n}(-1)^{k+1}\left[e^b f^{(n-k)}(b) - e^a f^{(n-k)}(a)\right] - (-1)^{n-1}\int_a^b e^x f(x)dx\right| &\leq \left(\int_a^b e^x dx\right)^{1-\frac{1}{q}}\left(\int_a^b e^x \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\ &\leq \left(e^b - e^a\right)^{1-\frac{1}{q}}\left(\int_a^b e^x M_{n,q}(f)dx\right)^{\frac{1}{q}} \\ &= \left(e^b - e^a\right) M_{n,q}^{\frac{1}{q}}(f). \end{aligned}$$

□

Under the conditions of Theorem 2.2.4 for $n = 1$, we have the following inequality

$$\left|\frac{e^b f(b) - e^a f(a)}{e^b - e^a} - \frac{1}{e^b - e^a}\int_a^b e^x f(x)dx\right| \leq M_{1,q}^{\frac{1}{q}}(f).$$

Theorem 2.2.5. Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be $n-$times differentiable function on I^o for $n \in \mathbb{N}$ and $a, b \in I^o$ with $a < b$. If $f^{(n)}$ is an integrable function and $\left|f^{(n)}\right|^q$ is quasi-convex on $[a,b]$ for $p, q > 1$ and

$$M_{n,q}(f) = \max\left\{\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right\},$$

then the following inequality holds;

$$\left|\sum_{k=1}^{n}(-1)^{k-1}\left[\frac{e^b f^{(n-k)}(b) - e^a f^{(n-k)}(a)}{b-a}\right] - (-1)^{n-1}\frac{1}{b-a}\int_a^b e^x f(x)dx\right| \le L^{\frac{1}{q}}\left(e^{bq}, e^{aq}\right) M_{n,q}^{\frac{1}{q}}(f). \tag{2.7}$$

Proof . Using Corollary 2.1.3 and the Hölder integral inequality we have the following inequality

$$\begin{aligned}
\left|\sum_{k=1}^{n}(-1)^{k-1}\left[e^b f^{(n-k)}(b) - e^a f^{(n-k)}(a)\right] - (-1)^{n-1}\int_a^b e^x f(x)dx\right| &\le \left(\int_a^b 1^p dx\right)^{\frac{1}{p}}\left(\int_a^b e^{xq}\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\le (b-a)^{\frac{1}{p}}\left(\int_a^b e^{xq} M_{n,q}(f)dx\right)^{\frac{1}{q}} \\
&= (b-a)M_{n,q}^{\frac{1}{q}}(f)\left(\frac{e^{bq}-e^{aq}}{q(b-a)}\right)^{\frac{1}{q}} \\
&= (b-a)L^{\frac{1}{q}}\left(e^{bq}, e^{aq}\right) M_{n,q}^{\frac{1}{q}}(f).
\end{aligned}$$

□

Under the conditions Theorem 2.2.5 for $n = 1$, we have the following inequality

$$\left|\frac{e^b f(b) - e^a f(a)}{b-a} - \frac{1}{b-a}\int_a^b e^x f(x)dx\right| \le L^{\frac{1}{q}}\left(e^{bq}, e^{aq}\right) M_{1,q}^{\frac{1}{q}}(f).$$

Corollary 2.2.6. Under the conditions of Theorem 2.2.3 and Theorem 2.2.5 for $n = 1$, we have the following inequality

$$\left|\frac{e^b f(b) - e^a f(a)}{b-a} - \frac{1}{b-a}\int_a^b e^x f(x)dx\right| \le M_{1,q}^{\frac{1}{q}}(f)\min\left\{L^{\frac{1}{p}}\left(e^{bp}, e^{ap}\right), L^{\frac{1}{q}}\left(e^{bq}, e^{aq}\right)\right\}. \tag{2.8}$$

Proof . The proof is clear from Theorem 2.2.3 and Theorem 2.2.5. □

3 Conclusion

As a result, integral inequalities related to convex functions are established. Then, some new results are obtained by choosing $n = 1$. Our results are in the integral form and discrete form of the sum of the function values together. Interested readers can find new results using the above mentioned Lemma 2.1.1 for other types of convexity or try to get better upper bounds for the left side of this lemma using different properties. Also by applying several convex functions to this inequalities demonstrated in the study can be found the relationships between some new special tools.

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