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A generalization to parametric metric spaces

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Abstract

Following the concept of parametric metric space introduced by N. Hussain et al., the definition of generalized parametric metric space is given. A decomposition theorem from a generalized metric into a family of crisp metrics is established. Another decomposition theorem from a family of the crisp metric into a generalized parametric metric identical to the first one under certain conditions is proved. Some basic results are studied and develop a Banach type fixed point theorem with an application to an integral equation.

Keywords: Parametric metric, decomposition theorem, metric, generalized parametric metric
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1 Introduction

In physical formulation, most of the non-linear equations transformed into a fixed point equation $F(x) = x$. To obtain results, existence and uniqueness of such fixed point equation are examined. These mappings basically satisfies some contraction or expansion conditions which are either the celebrated Banach type contraction principle or its variants on different types of generalized metric spaces.

Parametric metric space is one of those generalized metric spaces introduced by N. Hussain et al. in 2014 [1]. Some basic concepts such as notion of convergence of sequences, Cauchy sequence and some fixed-point theorems on complete parametric metric spaces and triangular intuitionistic fuzzy metric spaces are also studied by them.

Being inspired by N. Hussain et al., our aim in this paper to introduce a notion of generalized parametric metric by making some changes in the definition given by them. Analyzing the conditions of generalized parametric metric spaces we achieve a decomposition theorem of generalized parametric metric into a family of crisp metrics. There is another decomposition theorem which deduce a generalized parametric metric from the family of crisp metrics and we show that under certain conditions those two generalized parametric metrics are identical. Lastly we developed Banach type contraction principle in this new setting with an application to integral equation.

To develop the results of our manuscript we study some concerned papers for references(please see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]).

The organization of the paper is as follows: Section 2 contains some preliminary results. In Section 3, definition of generalized parametric metric, some examples are given, decomposition theorems and results related to notion of convergence are developed. Section 4 consists of a version of Banach type contraction principle in complete generalized parametric metric spaces and its application to integral equation.

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2 Preliminaries

First we recall some definitions and results which are used in this paper.

Definition 2.1. [4] Let X be a non-empty set. A pair (X, d) is called a metric space if there is a function $d : X \times X \to \mathbb{R}^+$, called metric which satisfies the following conditions:

(M1) $d(a, b) = 0$ if and only if $a = b$,
(M2) $d(a, b) = d(b, a)$, for all $a, b \in X$,
(M3) $d(a, b) \leq d(a, x) + d(b, x)$, for all $a, b, x \in X$.

Definition 2.2. [4] In a metric space (X, d), a sequence $\{x_n\} \subset X$ is said to

(i) converge to a point $x \in X$ if for any $\epsilon > 0, \exists m \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$, for all $n \geq m$,
(ii) be a Cauchy sequence if for any $\epsilon > 0, \exists r \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$, for all $n, m \geq r$.
(iii) (X, d) is said to be complete if every Cauchy sequence in X converges to some point in X.

Definition 2.3. [1] Let X be a non-empty set and $P : X \times X \times (0, \infty) \to \mathbb{R}^+$ be a function which satisfies the following conditions:

(i) $P(a, b, t) = 0, \ \forall\ t > 0$ if and only if $a = b$;
(ii) $P(a, b, t) = P(b, a, t)$, for all $a, b \in X$ and for all $t > 0$;
(iii) $P(a, b, t) \leq P(a, x, t) + P(b, x, t)$, for all $a, b, x \in X$ and for all $t > 0$.

Then the pair (X, P) is called a parametric metric space.

Definition 2.4. [1] Let (X, P) be a parametric metric space.

(i) Let $x \in X$ and $r > 0$, then the set $B(x, r) = \{y \in X : P(x, y, t) < r \text{ for all } t > 0\}$ is called an open ball with center at x and radius $r > 0$.
(ii) A sequence $\{x_n\} \subset X$ is said to converge to a point $x \in X$ if $\lim\limits_{n\to\infty} P(x_n, x, t) = 0$, for all $t > 0$.
(iii) If for any to sequences $\{x_n\}$ and $\{y_n\}$ in X converging to x and y respectively. If $\lim\limits_{n\to\infty} P(x_n, y_n, t) = P(x, y, t)$, for all $t > 0$, then P is said to be continuous in its two variables.
(iv) A sequence $\{x_n\}$ is said to be Cauchy if $\lim\limits_{n\to\infty} P(x_n, x_m, t) = 0$, for all $t > 0$.
(v) (X, P) is said to be complete if every Cauchy sequence converges in X.

3 Generalized parametric metric space

In this section, definition of generalized parametric space is given and some basic properties are studied. Two decomposition theorems are established.

Definition 3.1. Let $o : [0, \infty) \times [0, \infty) \to [0, \infty)$ be a binary operation which satisfies the following conditions $\forall \alpha,\ \beta,\ \gamma \in [0, \infty)$

(a) $\alpha\ o\ 0 = \alpha$
(b) $\alpha \leq \beta \implies \alpha o \gamma \leq \beta o \gamma$ (monotonicity)
(c) $\alpha o \gamma = \gamma o \alpha$ (commutativity)
(d) $\alpha o (\beta o \gamma) = (\alpha o \beta) o \gamma$ (associativity)

Following are the examples of such binary operation.

(i) $\alpha o \beta = \max\{\alpha, \beta\}$.
(ii) $\alpha o \beta = \alpha + \beta$.
(iii) $\alpha o \beta = (\alpha^n + \beta^n)^{\frac{1}{n}}$, for all $n \in \mathbb{N}$.

Definition 3.2. $'o\,'$ is said to be continuous if for any sequence $\{\alpha_n\},\ \{\beta_n\}$ in $[0, \infty)$ converging to α and β respectively, $\{\alpha_n o \beta_n\}$ converges to $\alpha o \beta$.

There are some additional axioms for $'o\,'$.
(e) $'o\,'$ is a continuous function.
(f) $\alpha < \beta$ and $\gamma < \delta \implies \alpha o \gamma < \beta o \delta$, for all $\alpha,\ \beta,\ \gamma,\ \delta \in [0,\infty)$ (strictly monotonicity).
(g) $\alpha o \alpha > \alpha$, for all $\alpha \in [0,\infty)$ (supper idempotency).

Definition 3.3. Let X be a non-empty set and $P : X \times X \times (0,\infty) \to [0,\infty)$ be a function which satisfies the following conditions:

(P1) $(P(a,b,t) = 0,\ \forall t > 0)$ if and only if $a = b$;
(P2) $P(a,b,t) = P(b,a,t)$, for all $t > 0$ and for all $a,b \in X$;
(P3) for $s,t > 0$ and for all $a,b,x \in X$, $P(a,b,s+t) \leq P(a,x,s)\ o\ P(b,x,t)$.

Then the function P is said to be generalized parametric metric and the triple (X,P,o) is called a generalized parametric metric space.

Example 3.4. Let us define a mapping $P : \mathbb{R} \times \mathbb{R} \times (0,\infty) \to [0,\infty)$ by

$$P(a,b,t) = \frac{|a-b|^p}{t},\ \text{for all } a,b \in \mathbb{R},\ \text{for all } t > 0 \text{ and } 0 < p < 1$$

and we consider the binary operation $'o\,'$ as $\alpha o \beta = \alpha + \beta$, for all $\alpha,\beta \in [0,\infty)$. We show that P is a generalized parametric metric on $\mathbb{R}$. For,

(i) Clearly $P(a,b,t) \geq 0$ and $P(a,b,t) = 0$, for all $t > 0$ if and only if $|a-b|^p = 0$ if and only if $a = b$. Thus (P1) holds.
(ii) (P2) holds trivially.
(iii) Let $a,b,x \in \mathbb{R}$ and $s,t > 0$. Then

$$\begin{aligned} P(a,b,s+t) &= \frac{|a-b|^p}{s+t} \\ &\leq \frac{|a-x|^p + |x-b|^p}{s+t} \\ &= \frac{|a-x|^p}{s+t} + \frac{|x-b|^p}{s+t} \\ &\leq \frac{|a-x|^p}{s} + \frac{|x-b|^p}{t} \\ &= P(a,x,s) + P(b,x,t) \end{aligned}$$

Therefore, (P3) holds.

Hence $(\mathbb{R},P,+)$ is a generalized parametric metric space.

Example 3.5. We consider a metric space (X,d) and define a non-negative real valued function P by

$$P(a,b,t) = \begin{cases} \frac{d(a,b)}{t} & if \quad 0 < t \leq 2d(a,b) \\ \frac{1}{2} & if \quad 2d(a,b) < t \leq 3d(a,b) \\ 2 - \frac{t}{2d(a,b)} & if \quad 3d(a,b) < t \leq 4d(a,b) \\ 0 & if \quad 4d(a,b) < t < \infty \end{cases}$$

for all $a,b \in X$.
We show that (X,P,o) is a generalized parametric metric space where the binary operation $'o\,'$ on $[0,\infty)$ is taken as $\alpha o \beta = \max\{\alpha,\beta\}$. For,

(i) $P(a,b,t) \geq 0$, for all $t > 0$ and $a,b \in X$. If $a = b$ then $d(a,b) = 0$. So, $P(a,b,t) = 0$, for all $t > 0$. Again, $P(a,b,t) = 0$, for all $t > 0$, implies that $t > 4d(a,b)$, for all $t > 0$. So, $d(a,b) = 0$. Hence, $a = b$. Thus (P1) holds.
(ii) (P2) holds trivially, since $d(a,b) = d(b,a)$, for all $a,b \in X$.
(iii) To prove (P3), we show that $P(a,b,s+t) \leq \max\{P(a,x,s),P(b,x,t)\}$, for all $a,b,x \in X$, for all $s,t > 0$.
Let $a,b,x \in X$. For all $s,t > 0$ we have the following cases.

(a) $s > 4d(a, x),\ t > 4d(x, b)$. Then $s + t > 4d(a, x) + 4d(x, b) \geq 4d(a, b)$ which implies $P(a, b, s + t) = 0$. Again
$$\max\{P(a, x, s), P(b, x, t)\} = 0.$$
(b) $s > 4d(a, x),\ 3d(x, b) < t \leq 4d(x, b)$. Then $s + t > 4d(a, x) + 3d(x, b) \geq 3d(a, b)$ and $P(a, x, s) = 0$, $P(b, x, t) = 2 - \frac{t}{2d(x,b)}$ implies $\max\{P(a, x, s), P(b, x, t)\} = 2 - \frac{t}{2d(x,b)}$. Again $P(a, b, s+t) \leq 2 - \frac{s+t}{2d(a,b)}$. Now,
$$\begin{aligned}\frac{s+t}{2d(a,b)} - \frac{t}{2d(x,b)} &= \frac{(s+t)d(b,x) - td(a,b)}{2d(a,b)d(b,x)}\\ &\geq \frac{sd(b,x) + td(x,b)) - td(a,x) - td(b,x)}{2d(a,b)d(b,x)}\\ &= \frac{sd(b,x) - td(a,x)}{2d(a,b)d(b,x)}\\ &\geq \frac{4d(a,x)d(b,x) - 4d(b,x)d(a,x)}{2d(a,b)d(b,x)} = 0\end{aligned}$$
implies $P(a, b, s + t) \leq P(b, x, t) = \max\{P(a, x, s), P(b, x, t)\}$.

(c) $s > 4d(a, x),\ 2d(x, b) < t \leq 3d(x, b)$. Then $P(b, x, t) = \frac{1}{2}$ and $\max\{P(a, x, s), P(b, x, t)\} = \frac{1}{2}$. Now, $s + t > 4d(a, x) + 2d(x, b) \geq 2d(a, b)$ implies $P(a, b, s + t) = \frac{1}{2} = \max\{P(a, x, s), P(b, x, t)\}$.

(d) $s > 4d(a, x),\ 0 < t \leq 2d(x, b)$. Then $P(b, x, t) = \frac{d(b,x)}{t}$ and $\max\{P(a, x, s), P(b, x, t)\} = \frac{d(b,x)}{t}$. Now, $s + t > 4d(a, x) > 0$ implies $P(a, b, s + t) \leq \frac{d(a,b)}{s+t}$. Then,
$$\begin{aligned}\frac{d(b,x)}{t} - \frac{d(a,b)}{s+t} &= \frac{(s+t)d(b,x) - td(a,b)}{t(s+t)}\\ &\geq \frac{(s+t)d(b,x) - t(d(a,x) + d(x,b))}{t(s+t)}\\ &= \frac{sd(b,x) - td(a,x)}{t(s+t)}\\ &\geq \frac{4d(a,x)d(b,x) - 2d(b,x)d(a,x)}{t(s+t)}\\ &= \frac{2d(a,x)d(b,x)}{t(s+t)} \geq 0\end{aligned}$$
which shows that $P(a, b, s + t) \leq P(b, x, t) = \max\{P(a, x, s), P(b, x, t)\}$. Similarly, we can verify the other cases.

Thus (P3): $P(a, b, s + t) \leq \max\{P(a, x, s), P(b, x, t)\}$, for all $a, b, x \in X$ and$s, t > 0$ holds.

Hence P is a generalized parametric metric on X.

***Remark* 3.6.** Concept of parametric metric space (Definition 2.3) and concept of generalized parametric metric space (Definition 3.3) are totally different. We justify it by the following two examples. In Example 3.7, we show that a generalized parametric metric may not be a parametric metric space and in the Example 3.8, we prove that a parametric space need not to be a generalized parametric metric space. In Definition 3.3 (generalized parametric metric space) we use the general binary operation 'o' instead of '+' in the sense to achieve decomposition theorems which play crucial role to develop more results in generalized parametric spaces.

Example 3.7. We consider two metrics d_1 and d_2 on $\mathbb{R}^2$ defined by $d_1(a, b) = \max\{|a_i - b_i| : i = 1, 2\}$ and $d_2(a, b) = \sum_{i=1}^{2} |a_i - b_i|$, for all $a = (a_1, a_2),\ b = (b_1, b_2) \in \mathbb{R}^2$. Then clearly, $d_1(a, b) \leq d_2(a, b)$, for all $a,\ b \in \mathbb{R}^2$. Next, we define a function $P : \mathbb{R}^2 \times \mathbb{R}^2 \times (0, \infty) \to [0, \infty)$ by
$$P(a,b,t) = \begin{cases} 100 & if \quad 0 < t \leq d_1(a,b) \\ 50 & if \quad d_1(a,b) < t \leq d_2(a,b) \\ 25 & if \quad d_2(a,b) < t \leq 2d_2(a,b) \\ 0 & if \quad 2d_2(a,b) < t < \infty \end{cases}$$
for all $a, b \in \mathbb{R}^2$. We claim that P is a generalized parametric metric on $\mathbb{R}^2$. The binary operation 'o' is taken as '+'. Clearly P satisfies (P1) and (P2). We only verify the condition (P3). For all $a, b, x \in \mathbb{R}^2$ and $s, t > 0$, we have the following cases:

(a) $s > 2d_2(a,x),\ t > 2d_2(x,b)$. Therefore, $s+t > 2d_2(a,x)+2d_2(x,b) \geq 2d_2(a,b)$ that is $s+t > 2d_2(a,b)$. Hence $P(a,b,s+t) = 0 = P(a,x,s) = P(b,x,t)$ and so $P(a,b,s+t) = P(a,x,s)+P(b,x,t)$.

(b) $s > 2d_2(a,x),\ d_2(x,b) < t \leq 2d_2(x,b)$. Then $P(a,x,s) = 0$ and $P(b,x,t) = 25$. Now, $s+t > 2d_2(a,x)+d_2(x,b) > d_2(a,x)+d_2(x,b) \geq d_2(a,b)$. So $s+t > d_2(a,b)$. Therefore, $P(a,b,s+t) \leq 25 = P(a,x,s)+P(b,x,t)$.

(c) $s > 2d_2(a,x),\ d_1(x,b) < t \leq d_2(x,b)$. Therefore $P(b,x,t) = 50$ and $P(a,x,s) = 0$. Now, $s+t > 2d_2(a,x) + d_1(x,b) \geq 2d_1(a,x)+d_1(x,b) > d_1(a,x)+d_1(x,b) \geq d_1(a,b)$. Thus, $s+t > d_1(a,b)$ which implies $P(a,b,s+t) \leq 50 = P(a,x,s)+P(b,x,t)$.

(d) $s > 2d_2(a,x),\ 0 < t \leq d_1(x,b)$. So $P(a,x,s) = 0$ and $P(b,x,t) = 100$. Now, $s+t > 2d_2(a,x) > 0$ and hence $P(a,b,s+t) \leq 100 = P(a,x,s)+P(b,x,t)$. Similarly, the other cases can also be verified.
Thus (P3): $P(a,b,s+t) \leq P(a,x,s)+P(b,x,t)$, for all $a,b,x \in \mathbb{R}^2$ and $s,t > 0$ holds.

Hence $(\mathbb{R}^2, P, +)$ is a generalized parametric metric space. It is not a parametric metric space. For, we take $a = (1,0),\ b = (0,\frac{1}{2}),\ c = (\frac{1}{4},\frac{1}{8})$, and $t = 1$. We have $d_1(a,b) = \max\{1,\frac{1}{2}\} = 1$. So, $0 < t \leq d_1(a,b)$ which implies $P(a,b,t) = 100$. Now, $d_2(a,c) = |1-\frac{1}{4}|+|0-\frac{1}{8}| = \frac{7}{8}$ and $2d_2(a,c) = \frac{14}{8}$. So, $d_2(a,c) < t \leq 2d_2(a,c)$ which implies $P(a,c,t) = 25$. Again, $d_2(b,c) = |0-\frac{1}{4}|+|\frac{1}{2}-\frac{1}{8}| = \frac{5}{8}$ and $2d_2(b,c) = \frac{10}{8}$. Therefore, $d_2(b,c) < t \leq 2d_2(b,c)$ which implies $P(b,c,t) = 25$. Thus, $P(a,c,t)+P(b,c,t) = 25+25 = 50 < 100 = P(a,b,t)$. So, P is not parametric metric on $\mathbb{R}^2$.

Example 3.8. We consider a metric space (X,d) and define a function $P : X \times X \times (0,\infty) \to [0,\infty)$ by $P(a,b,t) = e^t d(a,b)$, for all $a,b \in X$, for all $t > 0$. We show that P is a parametric metric on X. For,

(i) $P(a,b,t) \geq 0,\ a,b \in X$, for all $t > 0$ and $P(a,b,t) = 0$, for all $t > 0$ if and only if $d(a,b) = 0$ if and only if $a = b$.
(ii) $P(a,b,t) = P(b,a,t),\ a,b \in X$, for all $t > 0$, since d is a metric on X.
(iii) For any $a,b,c \in X$ and for all $t > 0$,

$$\begin{aligned} P(a,b,t) =& e^t d(a,b) \\ \leq& e^t[d(a,c)+d(b,c)] \\ =& P(a,c,t)+P(b,c,t) \end{aligned}$$

Therefore (X,P) is a parametric metric space. Next, in particular we take, $X = \mathbb{R}$ and d as the usual metric. Choose $a = 2,\ b = 4,\ c = 3,\ s = t = 1$. We have $P(a,b,s+t) = e^2 d(a,b) = 2e^2$, $P(a,c,s) = ed(a,c) = e,\ P(b,c,t) = ed(b,c) = e$. So, $P(a,b,s+t) = 2e^2 > 2e = P(a,c,s)+P(b,c,t)$. Hence P is not a generalized parametric metric on $\mathbb{R}$.

Proposition 3.9. If P is a generalized parametric metric on X then $P(a,b,\cdot)$ is non-increasing function, for all $a,b \in X$.

Proof . The proof follows from (P1), (P2) and (P3). □

Definition 3.10. Let (X,P,o) be a generalized parametric metric space. Define

$$d_\alpha(a,b) = \inf\{t > 0 : P(a,b,t) < \alpha\},\ \alpha \in (0,\infty) \text{ and } \forall a,b \in X \tag{3.1}$$

Lemma 3.11. If P is a generalized parametric metric on X and d_α is defined as in (3.1) then d_α is non-increasing function, for each $\alpha \in (0,\infty)$.

Proof . The proof is straightforward. □

Lemma 3.12. Let (X,P,o) be a generalized parametric metric space and d_α, for each $\alpha \in (0,\infty)$ is defined as in (3.1). Then $\forall a,b,c \in X$ and $\forall \alpha,\ \beta \in (0,\infty)$, $d_{\alpha o \beta}(a,b) \leq d_\alpha(a,c)+d_\beta(c,b)$.

Proof . Let $\alpha,\ \beta \in (0,\infty)$ and $a,b,c \in X$. Then

$$\begin{aligned} d_\alpha(a,c)+d_\beta(c,b) =& \inf\{t > 0 : P(a,c,t) < \alpha\} + \inf\{s > 0 : P(b,c,s) < \beta\} \\ =& \inf\{s+t > 0 : P(a,c,t) < \alpha, P(b,c,s) < \beta\} \end{aligned}$$

Now $P(a,c,t) < \alpha,\ P(b,c,s) < \beta$ and $P(a,b,s+t) \le P(a,c,t)oP(c,b,s)$ implies $P(a,b,s+t) < \alpha o\beta$.
Thus,

$$\begin{aligned} &\{s+t>0 : P(a,c,t) < \alpha,\ P(c,b,s) < \beta\} \subset \{s+t>0 : P(a,b,s+t) < \alpha o\beta\} \\ \implies &\inf\{s+t>0 : P(a,c,t) < \alpha,\ P(c,b,s) < \beta\} \ge \inf\{s+t>0 : P(a,b,s+t) < \alpha o\beta\} \\ \implies &d_{\alpha o\beta}(a,b) \le d_\alpha(a,c) + d_\beta(c,b) \end{aligned}$$

□

***Remark* 3.13.** In Lemma 3.12, if the binary operation $'o\,'$ is taken as $'\max'$, then $\alpha o\alpha = \alpha$. Hence in particular, we obtain $d_\alpha(a,b) \le d_\alpha(a,c) + d_\alpha(c,b), \forall\alpha \in (0,\infty)$.

Theorem 3.14. (1st Decomposition Theorem) Let (X,P,o) be a generalized parametric metric space and d_α, for each $\alpha \in (0,\infty)$ is defined as in (3.1). Then $\forall a,b,c \in X$ and $\forall\alpha,\beta \in (0,\infty)$,

(d1) $d_\alpha(a,b) \ge 0$;
(d2) $(d_\alpha(a,b) = 0,\ \forall\alpha \in (0,\infty)) \iff a = b$;
(d3) $d_\alpha(a,b) = d_\alpha(b,a)$;
(d4) $d_{\alpha o\beta}(a,b) \le d_\alpha(a,c) + d_\beta(c,b)$;
(d5) $d_\alpha(a,b)$ is non-increasing.

Proof . We only prove (d1)-(d3).

(d1) From definition of $d_\alpha,\ \alpha \in (0,\infty)$, it is clear that $d_\alpha(a,b) \ge 0,\ \forall a,b \in X$.
(d2) for all $a,b \in X$,

$$\begin{aligned} &d_\alpha(a,b) = 0,\ \forall\alpha \in (0,\infty) \\ \implies &\inf\{t>0 : P(a,b,t) < \alpha\} = 0,\ \forall\alpha \in (0,\infty) \\ \implies &P(a,b,t) < \alpha,\ \forall t>0,\ \forall\alpha \in (0,\infty) \\ \implies &P(a,b,t) = 0,\ \forall t>0 \\ \implies &a = b \end{aligned}$$

Again, $a = b \implies P(a,b,t) = 0,\ \forall t > 0$ and hence by definition $d_\alpha(a,b) = 0, \forall\alpha \in (0,\infty)$.
(d3) Since $P(a,b,t) = P(b,a,t)$, for all $t > 0$ and $a,b \in X$, $d_\alpha(a,b) = d_\alpha(b,a)$, for all $a,b \in X$ and $\alpha \in (0,\infty)$.

The proof of (d4) and (d5) follows from Lemma 3.11 and Lemma 3.12. □

Further assume that,

$$(P4)\quad P(a,b,t) < \alpha,\ \forall t > 0,\quad \text{for any}\quad \alpha \in (0,\infty) \quad\implies a = b.$$

***Remark* 3.15.** If P satisfies $(P4)$ then $(d6)\quad d_\alpha(a,b) = 0 \iff a = b,\ \forall\alpha \in (0,\infty)$ Hence if (X,P,o) is a generalized parametric metric space satisfying $(P4)$ then $(d6)$ holds.

***Remark* 3.16.** It is obvious that (d6) implies (d2) .

Definition 3.17. A family of mappings $\{d_\alpha : \alpha \in (0,\infty)\}$ satisfying (d1), (d3), (d4) and (d6) is said to be a quasi metric family and $\{(X,d_\alpha) : \alpha \in (0,\infty)\}$ is called a generating space of quasi metric family.

***Remark* 3.18.** If the binary operation $'o\,'$ is taken as $'\max'$, then in Definition 3.17 the family $\{(X,d_\alpha) : \alpha \in (0,\infty)\}$ is a family of metrics. We call $\{d_\alpha : \alpha \in (0,\infty)\}$ are the α-metrics induced by the generalized parametric metric P.

Example 3.19. We consider a metric space (X,d) and define a mapping $P : X \times X \times (0,\infty) \to [0,\infty)$ by

$$P(a,b,t) = \frac{d(a,b)}{t},\quad \forall a,b \in \mathbb{R}\ \text{ and } \forall t > 0$$

Let the binary operation $'o\,'$ as $\alpha o\beta = \max\{\alpha,\beta\},\ \forall\alpha,\beta \in [0,\infty)$. Now,

(i) $P(a,b,t) \geq 0$, for all $a,b \in X$, $t>0$, and $P(a,b,t)=0$, for all $t>0$, $d(a,b)=0$ if and only if $a=b$. Thus (P1) holds.

(ii) Since, $d(a,b)=d(b,a)$, for all $a,b \in X$, $P(a,b,t)=P(b,a,t)$, for all $a,b \in X$ and $t>0$. Thus (P2) holds.

(iii) Let $a,b,x \in X$ and $s,t>0$. Then $P(a,b,s+t)=\frac{d(a,b)}{s+t}$ and $\max\{P(a,x,s),P(b,x,t)\}=\max\{\frac{d(a,x)}{s},\frac{d(x,b)}{t}\}$. Now if $\frac{d(a,x)}{s} \geq \frac{d(x,b)}{t}$, then

$$\begin{aligned} P(a,x,s)-P(a,b,s+t) =& \frac{d(a,x)}{s}-\frac{d(a,b)}{s+t} \\ =& \frac{(s+t)d(a,x)-sd(a,b)}{s(s+t)} \\ =& \frac{s(d(a,x)-d(a,b))+td(a,x)}{s(s+t)} \\ \geq& \frac{-sd(b,x)+td(a,x)}{s(s+t)} \quad \text{(follows from metric inequality)} \\ \geq& 0. \end{aligned}$$

Similarly, if $\frac{d(a,x)}{s} \leq \frac{d(x,b)}{t}$, then $P(b,x,t)-P(a,b,s+t) \geq 0$. Thus for all $a,b,x \in X$ and $s,t>0$, $(P3)$: $P(a,b,s+t) \leq \max\{P(a,x,s),P(b,x,t)\}$ holds.

Hence $(X,P,\max)$ is a generalized parametric metric space. Again,

$$\begin{aligned} & P(a,b,t)<\alpha,\ \forall t>0,\ \text{for any } \alpha>0 \\ \Longrightarrow\ & t>\frac{d(a,b)}{\alpha}, \forall t>0,\ \text{for any } \alpha>0 \\ \Longrightarrow\ & d(a,b)=0 \\ \Longrightarrow\ & a=b. \end{aligned}$$

Therefore P satisfies the condition (P4). Hence, for each $\alpha>0$, the α-metrics on X are

$$d_\alpha(a,b)=\inf\{t>0: P(a,b,t)<\alpha\}=\frac{d(a,b)}{\alpha}, \quad \forall a,b \in X$$

Clearly $\{d_\alpha(a,b): \alpha \in (0,\infty)\}$ is an non-increasing family of α-metrics on X.

Definition 3.20. Let (X,P,o) be a generalized parametric metric space and $\{d_\alpha : \alpha \in (0,\infty)\}$ is defined in (3.1). Then d_α is said to be upper semi-continuous on $(0,\infty)$ if for any $\alpha_0 \in (0,\infty)$, there exist a neighborhood U of α_0 such that $d_{\alpha_0}(a,b)<t \implies d_\alpha(a,b)<t$, $\forall \alpha \in U$ and $\forall a,b \in X$.

Proposition 3.21. Let (X,P,o) be a generalized parametric metric space and $\{d_\alpha : \alpha \in (0,\infty)\}$ is defined as in (3.1). Assume further that,

$$(P5) \ \ \text{for all } a,b \in X, \ \ P(a,b,.) \text{ is continuous function of } t, \ \ \forall t>0$$

If P satisfies (P5), then d_α is upper semi-continuous on $(0,\infty)$.

Proof . Let us choose $\alpha_0, t_0 \in (0,\infty)$. Then for $a,b \in X$,

$$\begin{aligned} & d_{\alpha_0}(a,b)<t_0 \\ \Longrightarrow\ & P(a,b,t_0)<\alpha_0 \\ \Longrightarrow\ & \exists\ \epsilon>0 \text{ such that } P(a,b,t_0)<\alpha_0-\epsilon \\ \Longrightarrow\ & d_\alpha(a,b) \leq t_0,\ \forall \alpha \in (\alpha_0-\epsilon, \alpha_0+\epsilon) \end{aligned}$$

This completes the proof. □

Theorem 3.22. Let $(X,P,\ \max)$ be a generalized parametric metric space satisfying (P4) and $\{d_\alpha : \alpha \in (0,\infty)\}$ be the family of α-metrics induced by P. We define

$$P'(a,b,t)=\inf\{\alpha>0: d_\alpha(a,b) \leq t\}, \quad \text{for all } t>0 \text{ and } a,b \in X. \tag{3.2}$$

Then $(X,P',\max)$ is a generalized parametric metric space.

Proof .

(i) $P'(a,b,t) \geq 0, \quad \forall t > 0$ and $\forall a,b \in X$. Then,

$$\begin{aligned} & P'(a,b,t) = 0, \quad \forall t > 0 \\ \iff & \inf\{\alpha > 0 : d_\alpha(a,b) \leq t\} = 0, \quad \forall t > 0 \\ \iff & d_\alpha(a,b) \leq t, \quad \forall t > 0, \ \forall\ \alpha > 0 \\ \iff & d_\alpha(a,b) = 0, \quad \forall\ \alpha > 0 \\ \iff & a = b \end{aligned}$$

So (P1) holds.

(ii) Since $d_\alpha, \ \alpha \in (0,\infty)$ is a metric, for all $a,b \in X, \ d_\alpha(a,b) = d_\alpha(b,a)$ implies

$$P'(a,b,t) = P'(b,a,t),$$

for all $t > 0$. Thus (P2) holds.

(iii) Let $a,b,x \in X$ and $s,t > 0$. We have to show that $P'(a,b,s+t) \leq \max\{P'(a,x,s), P'(b,x,t)\}$. Let $P'(a,x,s) = \alpha_1$ and $P'(b,x,t) = \alpha_2$. If $\alpha_1 = 0 = \alpha_2$ then the result is obvious. Suppose that $\alpha_1 \leq \alpha_2$. Let us choose α where $0 < \alpha_1 \leq \alpha_2 < \alpha$. Then $\exists \alpha_0 < \alpha$ such that $d_{\alpha_0}(a,x) \leq s$ and $\exists \beta_0 < \alpha$ such that $d_{\beta_0}(b,x) \leq t$. Let $\gamma = \max\{\alpha_0, \beta_0\} < \alpha$. Therefore $d_\gamma(a,x) \leq d_{\alpha_0}(a,x) \leq s$ and $d_\gamma(b,x) \leq d_{\beta_0}(b,x) \leq t$. Now for all $a,b,x \in X$,

$$\begin{aligned} & d_\gamma(a,b) \leq d_\gamma(a,x) + d_\gamma(x,b) \leq s + t \\ \implies & P'(a,b,s+t) \leq \gamma < \alpha \\ \implies & P'(a,b,s+t) < \alpha. \end{aligned}$$

Since $0 < \alpha_1 \leq \alpha_2 < \alpha$ is arbitrary, from above it follows that

$$\begin{aligned} & P'(a,b,s+t) \leq \alpha_2 = \max\{\alpha_1, \alpha_2\} \\ \implies & P'(a,b,s+t) \leq \max\{P'(a,x,s), P'(b,x,t)\}. \end{aligned}$$

Similarly if $\alpha_2 \leq \alpha_1$, then also the result holds.

So (P3) holds. □

Proposition 3.23. The generalized parametric metric P', defined in (3.2) is a non-increasing function of t.

Proof . The proof is straightforward. □

Following is a example for the justification of the Theorem 3.22.

Example 3.24. The generalized parametric metric P, given in Example 3.19 induced a non-increasing family of α-metrics on X. We define P' as

$$\begin{aligned} P'(a,b,t) &= \inf\{\alpha > 0 : d_\alpha(a,b) \leq t\}, \quad \forall t > 0, \ \forall a,b \in X \\ &= \inf\{\alpha > 0 : \frac{d(a,b)}{\alpha} \leq t\}, \quad \forall t > 0, \ \forall a,b \in X \\ &= \inf\{\alpha > 0 : \frac{d(a,b)}{t} \leq \alpha\}, \quad \forall t > 0, \ \forall a,b \in X \\ &= \frac{d(a,b)}{t}, \quad \forall t > 0, \end{aligned}$$

for all $a,b \in X$. Then P' is a parametric metric which is shown in Example 3.19. Moreover P' is a non-increasing function of t.

Lemma 3.25. Let $(X, P, \ \max)$ be a generalized parametric metric space satisfying (P4) and for each $\alpha \in (0,\infty)$, d_α be the α-metric induced by P. Choose $a_0, b_0 \in X$ with $a_0 \neq b_0$. If

(i) P is non-increasing and satisfies (P5) then for $t_0 > 0, \ P(a_0,b_0,t_0) = \alpha_0 \in (0,\infty)$ implies $P(a_0,b_0,d_{\alpha_0}(a_0,b_0)) = \alpha_0$. Furthermore, for any $\alpha > 0, \ P(a_0,b_0,d_\alpha(a_0,b_0)) = \alpha$.

(ii) P satisfies (P5) and P is strictly decreasing function of t then for any $\alpha > 0$, $P(a_0, b_0, t) = \alpha$ if and only if $d_\alpha(a_0, b_0) = t$.

Proof .

(i) Since $P(a_0, b_0, t_0) = \alpha_0$, we have

$$d_{\alpha_0}(a_0, b_0) \geq t_0. \tag{3.3}$$

Now, since $P(a_0, b_0, .)$ is non-increasing, from (3.3) we have,

$$\alpha_0 = P(a_0, b_0, t_0) \geq P(a_0, b_0, d_{\alpha_0}(a_0, b_0)). \tag{3.4}$$

If $P(a_0, b_0, d_{\alpha_0}(a_0, b_0)) < \alpha_0$, then since $P(a_0, b_0, \cdot)$ is continuous and non-increasing, there exist $t' < d_{\alpha_0}(a_0, b_0)$ such that $P(a_0, b_0, t') < \alpha_0$. Then

$$t' < d_{\alpha_0}(a_0, b_0) = \inf\{s > 0 : P(a_0, b_0, s) < \alpha_0\} \leq t'.$$

Therefore a contradiction to our assumption. Hence,

$$P(a_0, b_0, d_{\alpha_0}(a_0, b_0)) = \alpha_0. \tag{3.5}$$

Since $P(a_0, b_0, \cdot)$ is continuous, for any $\alpha \in (0, \infty)$, there exists $t > 0$ such that $P(a_0, b_0, t) = \alpha$. Hence (3.5) gives

$$P(a_0, b_0, d_\alpha(a_0, b_0)) = \alpha. \tag{3.6}$$

(ii) Let $P(a_0, b_0, t) = \alpha$. Then $P(a_0, b_0, d_\alpha(a_0, b_0)) = \alpha$ (by (i)). Therefore,

$$\begin{aligned} &P(a_0, b_0, t) = \alpha = P(a_0, b_0, d_\alpha(a_0, b_0)) \\ \Longrightarrow\ &t = d_\alpha(a_0, b_0), \quad \text{(since, } P \text{ is strictly decreasing)} \end{aligned}$$

Again,

$$\begin{aligned} &d_\alpha(a_0, b_0) = t \\ \Longrightarrow\ &P(a_0, b_0, t) = P(a_0, b_0, d_\alpha(a_0, b_0)) = \alpha \text{ (by (i))} \\ \Longrightarrow\ &P(a_0, b_0, t) = \alpha \end{aligned}$$

This completes the proof. □

***Remark* 3.26.** (a) The Lemma 3.25 does not hold without the strictly decreasing property of the generalized parametric metric P. To justify we consider the generalized parametric metric of Example 3.5. Now,

$$\begin{aligned} &P(a, b, t) < \alpha, \quad \forall\ t > 0, \text{ for any } \alpha \in (0, \infty) \\ \Longrightarrow\ &P(a, b, t) = 0, \ \forall\ t > 0 \\ \Longrightarrow\ &t > 4d(a, b), \forall t > 0 \\ \Longrightarrow\ &4d(a, b) = 0 \\ \Longrightarrow\ &a = b \end{aligned}$$

Thus P satisfies the condition (P4). Moreover for all $a, b \in X$, $P(a, b, \cdot)$ is continuous and non-increasing function, $\forall\ t > 0$. But P is not strictly decreasing function of t. Now for each $\alpha > 0$, the α-metrics on X are

$$d_\alpha(a, b) = \inf\{t > 0 : P(a, b, t) < \alpha\} = \begin{cases} 2(2-\alpha)d(a, b) & if \quad 0 < \alpha < \frac{1}{2} \\ 3d(a, b) & if \quad \alpha = \frac{1}{2} \\ \frac{d(a,b)}{\alpha} & if \quad \alpha > \frac{1}{2}, \end{cases}$$

for all $a, b \in X$. Choose $a_0 \neq b_0$ and let $t_0 = \frac{5}{2}d(a_0, b_0)$. Then $P(a_0, b_0, t_0) = \frac{1}{2} = \alpha_0(say)$. But $d_{\alpha_0}(a_0, b_0) = 3d(a_0, b_0) \neq \frac{5}{2}d(a_0, b_0) = t_0$. Therefore, $P(a_0, b_0, t_0) = \alpha_0 \not\Rightarrow d_{\alpha_0}(a_0, b_0) = t_0$.

(b) Following example shows that the Lemma 3.25 does not hold without the continuity of P.

Example 3.27. Let (X, d) be a metric space and define a function P by

$$P(a,b,t) = \begin{cases} \frac{2d(a,b)}{t} & if \quad 0 < t \leq 2d(a,b) \\ \frac{1}{2} & if \quad 2d(a,b) < t \leq 3d(a,b) \\ 0 & if \quad 3d(a,b) < t < \infty \end{cases}$$

for all $a, b \in X$. We consider the binary operation $'o\,'$ as $\alpha o \beta = \max\{\alpha, \beta\}$. First we show that P is a generalized parametric metric on X. For,

(i) Clearly $P(a,b,t) \geq 0,\ \forall\ t > 0$ and $\forall\ a, b \in X$. If $a = b$ then $d(a,b) = 0$. Hence $P(a,b,t) = 0$, for all $t > 0$. Again, if $P(a,b,t) = 0,\ \forall t > 0$ then $t > 3d(a,b), \forall t > 0 \implies d(a,b) = 0 \implies a = b$. Thus (P1) holds.

(ii) (P2) holds trivially.

(iii) To prove (P3), we have to show that $P(a,b,s+t) \leq \max\{P(a,x,s), P(b,x,t)\}$, for all $a, b, x \in X$ and for all $s, t > 0$. Let $a, b, x \in X$ and $s, t > 0$. Then for all $s, t > 0$, we have the following cases.

(a) $0 < s \leq 2d(a,x),\quad 0 < t \leq 2d(b,x)$. Then $P(a,x,s) = \frac{2d(a,x)}{s},\ P(b,x,t) = \frac{2d(x,b)}{t}$ and $P(a,b,s+t) \leq \frac{2d(a,b)}{s+t}$ and hence the proof of the inequality $P(a,b,s+t) \leq \max\{P(a,x,s), P(b,x,t)\}$ follows from the Example 3.19.

(b) $0 < s \leq 2d(a,x),\quad 2d(b,x) < t \leq 3d(b,x)$. Then $P(b,x,t) = \frac{1}{2}$ and $s + t > 2d(b,x) > 0$ implies $P(a,b,s+t) \leq \frac{2d(a,b)}{s+t}$. Now, $\max\{P(a,x,s), P(b,x,t)\} = \max\{\frac{2d(a,x)}{s}, \frac{1}{2}\} = \frac{2d(a,x)}{s}$ and by the similar lines of proof of the case (d) of Example 3.5 it follows that $P(a,b,s+t) \leq P(a,x,s)$.

(c) $0 < s \leq 2d(a,x),\quad t > 3d(x,b)$. Therefore $P(b,x,t) = 0$ and the proof is similar as the above case (b).

The other cases can be verified similarly. Thus (P3): $P(a,b,s+t) \leq \max\{P(a,x,s), P(b,x,t)\}$, for all $a, b, x \in X$, and $s, t > 0$ holds.

Hence $(X, P, \max)$ is a generalized parametric metric space but for all $a, b \in X,\ P(a,b,\cdot)$ is discontinuous function of t. Again,

$$\begin{aligned} & P(a,b,t) < \alpha, \quad \forall\ t > 0, \text{ for any } \alpha \in (0,\infty) \\ \implies & P(a,b,t) = 0,\ \forall\ t > 0 \\ \implies & t > 3d(a,b), \forall t > 0 \\ \implies & d(a,b) = 0 \\ \implies & a = b. \end{aligned}$$

Thus (P4) holds. So we can construct the α-metrics on X. For all $a, b \in X$, the induced α-metrics are

$$d_\alpha(a,b) = \begin{cases} 3d(a,b) & if \quad 0 < \alpha \leq \frac{1}{2} \\ 2d(a,b) & if \quad \frac{1}{2} < \alpha \leq 1 \\ \frac{2d(a,b)}{\alpha} & if \quad \alpha > 1. \end{cases}$$

Choose $a_0 \neq b_0 \in X$ and let $t_0 = \frac{5}{2}d(a_0, b_0)$. Then $P(a_0, b_0, t_0) = \frac{1}{2} = \alpha_0(say)$. But $d_{\alpha_0}(a_0, b_0) = 3d(a_0, b_0) \neq \frac{5}{2}d(a_0, b_0) = t_0$. Therefore, $P(a_0, b_0, t_0) = \alpha_0 \not\Rightarrow d_{\alpha_0}(a_0, b_0) = t_0$.

Lemma 3.28. Let $(X, P,\ \max)$ be a generalized parametric metric space satisfying (P4), (P5) and $\{d_\alpha : \alpha \in (0,\infty)\}$ be the family of α-metrics on X induced by P. Then for any increasing or decreasing sequence $\{\alpha_n\}$ in $(0,\infty)$, $\alpha_n \to \alpha$ implies $d_{\alpha_n}(a,b) \to d_\alpha(a,b)$, for all $a, b \in X$.

Proof . If $a = b$, then $\alpha_n \to \alpha$ implies $d_{\alpha_n}(a,b) \to d_\alpha(a,b)$. Let $a \neq b$. First we assume $\{\alpha_n\}$ is an increasing sequence such that $\alpha_n \to \alpha$. Let $d_{\alpha_n}(a,b) = t_n$ and $d_\alpha(a,b) = t$. Then by Lemma 3.25,

$$P(a,b,t_n) = \alpha_n, \quad \text{for all } n \quad \text{and} \quad P(a,b,t) = \alpha. \tag{3.7}$$

Since $\{d_\alpha : \alpha \in (0,\infty)\}$ is a non-increasing family of metrics, $\{t_n\}$ is a non-increasing sequence in $(0,\infty)$ bounded below by t. Hence, $\{t_n\}$ is a convergent sequence. Therefore $\lim_{n\to\infty} P(a,b,t_n) = \lim_{n\to\infty} \alpha_n$ implies

$$P(a,b,\lim_{n\to\infty} t_n) = \alpha \quad (\text{ by (P5) }) \tag{3.8}$$

From the relations (3.7) and (3.8), we have

$$\begin{aligned} &P(a,b,\lim_{n\to\infty} t_n) = P(a,b,t)\\ \Longrightarrow &\lim_{n\to\infty} t_n = t\\ \Longrightarrow &\lim_{n\to\infty} d_{\alpha_n}(a,b) = d_\alpha(a,b) \end{aligned}$$

Similarly we can prove that for a decreasing sequence $\{\alpha_n\}$, converging to α, $\lim_{n\to\infty} d_{\alpha_n}(a,b) = d_\alpha(a,b)$. Hence the proof is complete. □

Theorem 3.29. (2nd Decomposition Theorem) Let $(X, P, \max)$ be a generalized parametric metric space and P be strictly decreasing which satisfies (P4) and (P5). If for each $\alpha > 0$, d_α be the α-metric on X induced by P and P' be the generalized parametric metric as defined in (3.2) then $P' = P$.

Proof . Let $x_0, y_0 \in X$ and $t_0 > 0$. Case I: $x_0 = y_0$, $t_0 > 0$ Then $P(x_0, y_0, t_0) = 0$. Now

$$\begin{aligned} P'(x_0,y_0,t_0) =&\inf\{\alpha > 0 : d_\alpha(x_0,y_0) \le t_0\}\\ =&0, \quad \text{since for each } \ \alpha > 0,\ d_\alpha(x_0,y_0) = 0 \end{aligned}$$

Case II: $x_0 \ne y_0$, $t_0 > 0$ such that $P(x_0, y_0, t_0) = 0$. For $\alpha \in (0,\infty)$, $d_\alpha(x_0, y_0) = \inf\{t > 0 : P(x_0, y_0, t) < \alpha\}$ Since $P(x_0, y_0, t_0) = 0 < \alpha$, $d_\alpha(x_0, y_0) \le t_0$, for all $\alpha \in (0,\infty)$. Therefore,

$$P'(x_0,y_0,t_0) = \inf\{\alpha > 0 : d_\alpha(x_0,y_0) \le t_0\} = 0.$$

Case III: $x_0 \ne y_0$, $t_0 > 0$ such that $P(x_0, y_0, t_0) > 0$. Let $P(x_0, y_0, t_0) = \alpha_0$, $\alpha_0 \in (0,\infty)$. Then from Lemma 3.25, we have $d_{\alpha_0}(x_0, y_0) = t_0$. Hence, $P'(x_0, y_0, t_0) = \inf\{\alpha > 0 : d_\alpha(x_0, y_0) \le t_0\}$ implies $P'(x_0, y_0, t_0) \le \alpha_0$. Thus,

$$P'(x_0,y_0,t_0) \le P. \tag{3.9}$$

Next, choose $0 < \alpha' < \alpha_0 < \infty$ and let $d_{\alpha'}(x_0, y_0) = t'$. Since $\{d_\beta : \beta \in (0,\infty)\}$ is a non-increasing family of metrics,

$$\begin{aligned} &\alpha' < \alpha_0\\ \Longrightarrow &d_{\alpha'}(x_0,y_0) \ge d_{\alpha_0}(x_0,y_0)\\ \Longrightarrow &t' \ge t. \end{aligned}$$

Again the Lemma 3.25 gives $P(x_0, y_0, t') = \alpha'$. Since $P(x_0, y_0, \cdot)$ is strictly decreasing, $P(x_0, y_0, t') = \alpha' < \alpha_0 = P(x_0, y_0, t_0)$, implies

$$t' > t_0. \tag{3.10}$$

So for all $0 < \alpha' < \alpha_0$, $d_{\alpha'}(x_0, y_0) = t' > t_0 \implies P'(x_0, y_0, t_0) \ge \alpha'$. If $P'(x_0, y_0, t_0) = \alpha'$, then

$$\begin{aligned} &\inf\{\beta > 0 : d_\beta(x_0,y_0) \le t_0\} = \alpha'\\ \Longrightarrow &\text{there exists a sequence } \ \{\alpha_n\} \ \text{ such that } \ \alpha_n \to \alpha' \ \text{ and } \ d_{\alpha_n}(x_0,y_0) \le t_0\\ \Longrightarrow &\lim_{n\to\infty} \ d_{\alpha_n}(x_0,y_0) \le t_0\\ \Longrightarrow &d_{\alpha'}(x_0,y_0) \le t_0 \ \ \text{(using Lemma 3.28).} \end{aligned}$$

A contradiction to our assumption. Since it is true for all $0 < \alpha' < \alpha_0$,

$$P'(x_0,y_0,t_0) \ge \alpha_0 = P(x_0,y_0,t_0) \tag{3.11}$$

(3.9) and (3.11) together implies, $P'(x_0, y_0, t_0) = P(x_0, y_0, t_0)$. Thus, for all $x, y \in X$ and for all $t > 0$, $P'(x, y, t) = P(x, y, t)$. □

Justification of the above Theorem 3.29.

***Remark* 3.30.** We consider the parametric metric P of Example 3.19 which is strictly decreasing and continuous function of t where we have shown that P satisfies (P4) and so we construct the family of metrics $\{d_\alpha : \alpha \in (0,\infty)\}$. Then in Example 3.24 we show that a parametric metric P' is induced which is identical with P.

Definition 3.31. Let (X, P, o) be a generalized parametric metric space. A sequence $\{x_n\} \subseteq X$ is said to be a

(i) convergent sequence if there exists $x \in X$ such that $\lim\limits_{n\to\infty} P(x_n, x, t) = 0$, for all $t > 0$. The point x is said to be limit of $\{x_n\}$ and denoted by $\lim\limits_{n\to\infty} x_n$.

(ii) Cauchy sequence if $\lim\limits_{m,n\to\infty} P(x_n, x_m, t) = 0$, for all $t > 0$.

Proposition 3.32. Let (X, P, o) be a generalized parametric metric space and $'o'$ be continuous. Then

(a) limit of a sequence is unique, if exist.
(b) every convergent sequence is Cauchy.
(c) every subsequence of a convergent sequence converges to same limit.

Proof .

(a) Suppose a sequence $\{x_n\}$ converges to two points x and y in X. Then $\lim\limits_{n\to\infty} P(x_n, x, t) = 0$, for all $t > 0$ and $\lim\limits_{n\to\infty} P(x_n, y, t) = 0$, for all $t > 0$. Now, for all $t > 0$,

$$\begin{aligned} & P(x, y, t) \leq P(x, x_n, \frac{t}{2}) \ o \ P(y, x_n, \frac{t}{2}), \ \forall n \\ \Longrightarrow & \lim_{n\to\infty} P(x, y, t) \leq \lim_{n\to\infty} P(x, x_n, \frac{t}{2}) \ o \ \mathit{lim}_{n\to\infty} P(y, x_n, \frac{t}{2}) \\ \Longrightarrow & P(x, y, t) \leq 0 \ o \ 0 = 0. \end{aligned}$$

Since P is non-negative real valued, $P(x, y, t) = 0$, for all $t > 0$ implies $x = y$.

(b) Let $\{x_n\} \subseteq X$ converges to $x \in X$. Then $\lim\limits_{k\to\infty} P(x_k, x, t) = 0$, for all $t > 0$. Now, for all $t > 0$

$$\begin{aligned} & P(x_m, x_n, t) \leq P(x_m, x, \frac{t}{2}) \ o \ P(x_n, x, \frac{t}{2}), \ \forall m, n \in \mathbb{N} \\ \Longrightarrow & \lim_{n,m\to\infty} P(x_m, x_n, t) \leq \mathit{lim}_{m\to\infty} P(x_m, x, \frac{t}{2}) \ o \ \lim_{n\to\infty} P(x_n, x, \frac{t}{2}) \leq 0 \ o \ 0 \\ \Longrightarrow & \lim_{n,m\to\infty} P(x_m, x_n, t) = 0. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in X.

(c) Let $\{x_{n_k}\}$ be a subsequence of a sequence $\{x_n\}$ which converges to $x \in X$ that is $\lim\limits_{k\to\infty} P(x_k, x, t) = 0$, for all $t > 0$. Then for all $t > 0$,

$$\begin{aligned} & P(x_{n_k}, x, t) \leq P(x_{n_k}, x_n, \frac{t}{2}) \ o \ P(x_n, x, \frac{t}{2}) \\ \Longrightarrow & \lim_{n\to\infty} P(x_{n_k}, x, t) \leq \lim_{n\to\infty} P(x_{n_k}, x_n, \frac{t}{2}) \ o \ \lim_{n\to\infty} P(x_n, x, \frac{t}{2}) \leq 0 \ o \ 0 \\ \Longrightarrow & \lim_{n\to\infty} P(x_{n_k}, x, t) = 0. \end{aligned}$$

Hence $\{x_{n_k}\}$ converges to $x \in X$.

□

***Remark* 3.33.** Every Cauchy sequence necessarily not a convergent sequence in generalized parametric metric spaces. To justify we consider the following example.

Example 3.34. We consider the set $X = \{x = \{x_i\} : x_i \in (0,\infty), \ \text{ for all } i\} \subset l_2(\mathbb{R})$. Then

$$d(x, y) = \sqrt{\sum_{i=1}^{\infty} |x_i - y_i|^2},$$

where $x = \{x_i\}, y = \{y_i\}$ is a metric on X and with the mapping $P(x,y,t) = \frac{d(x,y)}{t}$, for all $x, y \in X$ and $t > 0$, $(X, P, \max)$ is a generalized parametric metric space (see Example 3.19).

If we consider the sequence $\{x_n\} \subset X$, where $x_1 = \{1,0,0,\cdots\}, x_2 = \{0,\frac{1}{2},0,\cdots\}$ and so on. Then $\{x_n\}$ is Cauchy in (X,d) and converges to $x = \{0,0,0,\cdots\} \in l_2(\mathbb{R})$. Therefore, $\{x_n\}$ is a Cauchy sequence in $(X, P, \max)$, but does not converges in X.

Definition 3.35. A generalized parametric metric space (X, P, o) is said to be complete if every Cauchy sequence in X is convergent and converges to some point in it.

Proposition 3.36. Let (X, P, max) be a generalized parametric metric space satisfying (P4) and for each $\alpha \in (0,\infty)$, d_α be the α-metrics on X induced by P. Then for any sequence $\{x_n\} \subseteq X$, following results hold.

(i) $\{x_n\}$ is Cauchy in (X, P, max) if and only if $\{x_n\}$ is Cauchy in $(X, d_\alpha),\ \alpha \in (0,\infty)$.

(ii) $\{x_n\}$ converges to x in (X, P, max) if and only if $\{x_n\}$ converges to x in $(X, d_\alpha),\ \alpha \in (0,\infty)$.

Proof .

(i) Suppose $\{x_n\}$ is Cauchy in $(X, P, \max)$. Then $\lim_{m,n\to\infty} P(x_n, x_m, t) = 0$, for all $t > 0$. So for a given $\epsilon_0 > 0$, for each $t > 0$, there exists$N(t) \in \mathbb{N}$ such that

$$\begin{aligned} & P(x_n, x_m, t) < \epsilon_0,\ \forall m, n \geq N(t) \\ \Longrightarrow\ & \inf\{s > 0 : P(x_n, x_m, s) < \epsilon_0\} \leq t,\ \forall m, n \geq N(t) \\ \Longrightarrow\ & d_{\epsilon_0}(x_n, x_m) \leq t,\ \forall m, n \geq N. \end{aligned}$$

Since $t > 0$ is arbitrary, $\lim_{n,m\to\infty} d_{\epsilon_0}(x_n, x_m) = 0$. Again, since ϵ_0 arbitrarily chosen, $\lim_{n,m\to\infty} d_\alpha(x_n, x_m) = 0$, for any $\alpha \in (0,\infty)$. Therefore, $\{x_n\}$ is Cauchy in $(X, d_\alpha),\ \alpha \in (0,\infty)$.
Conversely suppose that, $\{x_n\}$ is Cauchy in $(X, d_\alpha),\ \alpha \in (0,\infty)$. Then $\lim_{n,m\to\infty} d_\alpha(x_n, x_m) = 0$, for any $\alpha \in (0,\infty)$. So for a given $\epsilon > 0$, for each $\alpha \in (0,\infty)$, $\exists N(\alpha) \in \mathbb{N}$ such that

$$\begin{aligned} & d_\alpha(x_n, x_m) < \epsilon, \forall m, n \geq N(\alpha) \\ \Longrightarrow\ & \inf\{s > 0 : P(x_n, x_m, s) < \alpha\} < \epsilon,\ \forall m, n \geq N(\alpha) \\ \Longrightarrow\ & P(x_n, x_m, \epsilon) < \alpha,\ \forall m, n \geq N(\alpha). \end{aligned}$$

Since $\alpha > 0$ is arbitrary, we have

$$\begin{aligned} & \lim_{n,m\to\infty} P(x_n, x_m, \epsilon) = 0 \\ \Longrightarrow\ & \lim_{n,m\to\infty} P(x_n, x_m, t) = 0,\ \forall t > 0 \quad \text{(Since, } \epsilon > 0 \text{ is arbitrary)} \\ \Longrightarrow\ & \{x_n\} \text{ is Cauchy in } (X, P, \max). \end{aligned}$$

(ii) First assume that $\{x_n\}$ converges to x in $(X, P, \max)$. Then $\lim_{n\to\infty} P(x_n, x, t) = 0$, for all $t > 0$. So for a given $\epsilon > 0$, for each $t > 0$, there exists $N(t) \in \mathbb{N}$ such that

$$\begin{aligned} & P(x_n, x, t) < \epsilon,\ \forall n \geq N(t) \\ \Longrightarrow\ & \inf\{s > 0 : P(x_n, x, s) < \epsilon\} \leq t,\ \forall n \geq N(t) \\ \Longrightarrow\ & d_\epsilon(x_n, x) \leq t,\ \forall n \geq N(t) \\ \Longrightarrow\ & \lim_{n\to\infty} d_\epsilon(x_n, x) = 0, \quad \text{(Since, } t > 0 \text{ is arbitrary)} \\ \Longrightarrow\ & \lim_{n\to\infty} d_\alpha(x_n, x) = 0,\ \alpha \in (0,\infty), \quad \text{(Since, } \epsilon > 0 \text{ is arbitrary).} \end{aligned}$$

Hence, $\{x_n\}$ converges to x in $(X, d_\alpha),\ \alpha \in (0,\infty)$.
Conversely suppose that, $\{x_n\}$ converges to x in $(X, d_\alpha),\ \alpha \in (0,\infty)$. Then $\lim_{n\to\infty} d_\alpha(x_n, x) = 0$, for each $\alpha \in (0,\infty)$. So for a given $\epsilon > 0$, for each $\alpha > 0$, there exists $N(\alpha) \in \mathbb{N}$ such that

$$\begin{aligned} & d_\alpha(x_n, x) < \epsilon,\ \forall n \geq N(\alpha) \\ \Longrightarrow\ & \inf\{t > 0 : P(x_n, x, t) < \alpha\} < \epsilon,\ \forall n \geq N(\alpha) \\ \Longrightarrow\ & P(x_n, x, \epsilon) < \alpha,\ \forall n \geq N(\alpha) \\ \Longrightarrow\ & \lim_{n\to\infty} P(x_n, x, \epsilon) = 0 \quad \text{(Since, } \alpha > 0 \text{ is arbitrary)} \\ \Longrightarrow\ & \lim_{n\to\infty} P(x_n, x, t) = 0,\ \forall t > 0, \quad \text{(Since, } \epsilon > 0 \text{ is arbitrary)} \end{aligned}$$

This completes the proof.

□

Definition 3.37. Let (X,P,o) be a generalized parametric metric space. $A \subseteq X$ is said to be bounded if for each $t>0$ there exist a non-negative real number K_t such that $P(x,y,t) \leq K_t$, for all $x,y \in A$.

Proposition 3.38. Every convergent sequence in a generalized parametric metric space (X,P,o) is bounded.

Proof . Let $\{x_n\}$ be a sequence in X converging to $x \in X$. Then $\underset{n\to\infty}{lim} P(x_n,x,t)=0,\ \forall t>0$. So for a given $t>0, \{P(x_n,x,t)\}$ is a bounded sequence. Thus for each $t>0,\ \exists K_t>0$ such that $P(x_n,x,t) \leq K_t,\quad$ for all n. Choose a fixed $y \in \{x_m\}$ and let $P(x,y,\frac{t}{2})=s_t$(depends on t). Now for all $t>0$ and for all $n \in \mathbb{N}$, we have

$$P(x_m,y,t) \leq P(x_m,x,\frac{t}{2})\ o\ P(x,y,\frac{t}{2}) \leq K_{\frac{t}{2}}\ o\ s_t = r_t.$$

This shows that $\{x_n\}$ is a bounded sequence in X. □

4 Banach type fixed point theorem and an application to integral equation

Theorem 4.1. Let (X,P,o) be a complete generalized parametric metric space and $'0'$ be continuous. If F be a self mapping on X which satisfies the contraction condition

$$P(Fx,Fy,t) \leq kP(x,y,t),$$

for all $x,y \in X$ and for all $t>0$ where $0<k<1$, then F has a unique fixed point in X.

Proof . To prove the existence of fixed point, let $x_0 \in X$ and consider the iterative sequence:

$$x_0,\ x_1=F(x_0),\ x_2=F(x_1)=F^2(x_0)\ \cdots,\ x_n=F(x_{n-1})=F^n(x_0),\cdots$$

First we prove that $\{x_n\}$ is a Cauchy sequence in X. Now for $n \in \mathbb{N}$ and $\forall t>0$,

$$P(x_{n+1},x_n,t) = P(F(x_n),F(x_{n-1}),t) \leq kP(x_n,x_{n-1},t).$$

Repeating this process we obtain, $P(x_{n+1},x_n,t) \leq k^n P(x_1,x_0,t),\ \forall t>0$ and for all $n \in \mathbb{N}$. Now $\forall t>0$ and for $m=n+p,\ p=1,2,\cdots,$

$$\begin{aligned}
&P(x_{n+p},x_n,t)\\
&\leq P(x_{n+p},x_{n+1},\frac{t}{2})\ o\ P(x_{n+1},x_n,\frac{t}{2})\\
&\leq P(x_{n+p},x_{n+2},\frac{t}{4})\ o\ P(x_{n+2},x_{n+1},\frac{t}{4})\ o\ P(x_{n+1},x_n,\frac{t}{2})\\
&\leq \cdots \leq P(x_{n+p},x_{n+p-1},\frac{t}{2^{p-1}})\ o\ P(x_{n+p-1},x_{n+p-2},\frac{t}{2^{p-1}})\ o\ \cdots o\ P(x_{n+2},x_{n+1},\frac{t}{4})\ o\ P(x_{n+1},x_n,\frac{t}{2})\\
&\leq k^{n+p-1}P(x_1,x_0,\frac{t}{2^{p-1}})\ o\ k^{n+p-2}P(x_1,x_0,\frac{t}{2^{p-1}})\ o\ \cdots\ o\ k^{n+1}P(x_1,x_0,\frac{t}{4})\ o\ k^nP(x_1,x_0,\frac{t}{2}).
\end{aligned}$$

Taking limit $n\to\infty$ on both side, we obtain, $\lim_{n\to\infty} P(x_{n+p},x_n,t)=0$, for all $t>0$ which proves that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $\{x_n\}$ converges to x. We prove that x is a fixed point for F. Now $\forall t>0$,

$$\begin{aligned}
&P(F(x),x,t) \leq P(F(x),x_n,\frac{t}{2})\ o\ P(x,x_n,\frac{t}{2})\\
\Longrightarrow\ &P(F(x),x,t) \leq kP(x,x_{n-1},\frac{t}{2})\ o\ P(x,x_n,\frac{t}{2})\\
\Longrightarrow\ &P(F(x),x,t) \leq \lim_{n\to\infty}[kP(x,x_{n-1},\frac{t}{2})\ o\ P(x,x_n,\frac{t}{2})]=0\\
\Longrightarrow\ &P(F(x),x,t)=0\\
\Longrightarrow\ &F(x)=.
\end{aligned}$$

So x is a fixed point of F. If possible suppose $\exists y \in X$ such that $Fy = y$. Then

$$\begin{aligned} &P(x,y,t) = P(F(x),F(y),t) \leq kP(x,y,t),\ \forall t>0 \\ \implies &P(x,y,t) = 0,\ \forall t>0 \quad \text{(Since } k<1\text{)} \\ \implies &x = y. \end{aligned}$$

This completes the proof of the theorem. □

Example 4.2. Let $X = C[-1,1]$. For all $f, g \in X$, $d(f,g) = \sup_{t\in[-1,1]} |f(t) - g(t)|$ is a metric on X. Next we consider the function $P(f,g,s) = \frac{d(f,g)}{s}$, for all $f, g \in X$ and $s > 0$. Then in Example 3.19, we have shown that $(X, P, \max)$ is a generalized parametric metric space and clearly it is complete. Now we define $F : X \to X$ by $F(f) = \frac{5f}{11}$, for all $f \in X$. Then for $k \in (\frac{5}{11}, 1)$, $P(F(f), F(g), t) \leq kP(f, g, t)$, for all $f, g \in X$ and $t > 0$. So by Theorem 4.1, F has a unique fixed point in X and here the fixed point is the null mapping.

Theorem 4.3. We consider the complete metric space $(C[0,a], d)$, $a > 0$ where the metric d is given by $d(f,g) = \sup_{t\in[0,a]} |f(t) - g(t)|$, for all $f, g \in C[0,a]$ and consider an integral operator η on $C[0,a]$ by

$$\eta(f(t)) = g(t) + \int_0^t \phi(t,s,f(s))\ ds,\ \forall t \in [0,a] \tag{4.1}$$

Now consider the complete generalized parametric metric space $(X, P, \max)$ where $X = C[0,a]$, $a > 0$ and $P(f,g,t) = \frac{d(f,g)}{t}$, for all $f, g \in X$ and $t > 0$. Let us choose a function $h : [0,a] \times [0,a] \to [0,\infty)$ which satisfies $\underset{t\in[0,a]}{sup} \int_0^t h(s,t)dt \leq k < 1$ and ϕ satisfies the following condition:

$$|\phi(s,t,f_1(t)) - \phi(s,t,f_2(t))| \leq h(s,t)|f_1(t) - f_2(t)|,\ \forall f_1, f_2 \in X,\ \forall s,t \in [0,a].$$

Then the integral equation (4.1) has a unique solution in $C[0,a]$.

Proof . We have,

$$\begin{aligned} |\eta f_1(t) - \eta f_2(t)| =& |\int_0^t [\phi(t,s,f_1(s)) - \phi(t,s,f_2(s))]\ ds|,\quad \forall t \in [0,a] \\ \leq& \int_0^t |\phi(t,s,f_1(s)) - \phi(t,s,f_2(s))|\ ds,\quad \forall t \in [0,a] \\ \leq& \int_0^t h(t,s)|f_1(s) - f_2(s)|\ ds,\quad \forall t \in [0,a] \\ \leq& d(f_1,f_2) \int_0^t h(t,s)ds,\quad \forall t \in [0,a] \\ \leq& kd(f_1,f_2),\quad \forall t \in [0,a]. \end{aligned}$$

Which implies

$$\begin{aligned} &d(\eta f_1, \eta f_2) \leq kd(f_1,f_2) \\ \implies &P(\eta f_1, \eta f_2, t) \leq kP(f_1,f_2),\ \forall t > 0 \end{aligned}$$

where $k < 1$. So η satisfies the contraction condition of Theorem (4.1) and hence it has a unique solution in X. □

5 Conclusion

We have introduced a new concept of generalized metric space by changing the condition of inequality in the definition of parametric metric space introduced by N. Hussian et al. It is possible to achieve a decomposition theorem from a generalized parametric metric into a family of crisp metrics. Banach type contraction principle is established and an application to find out the solution of Integral equation is given. We think this decomposition theorem will play a key role to develop further results in generalized parametric metric spaces. There are lot of scopes for researchers to study various results in compactness, completeness in such generalized parametric metric spaces and develop fixed point theorem for different types of contraction and expansion mappings in such spaces.

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