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Some new Ramsey families on natural numbers

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Abstract

In this paper, we present some new Ramsey families by using the van der Waerden theorem and affine topological correspondence principle.

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1 Introduction

A focal matter in Ramsey theory is to discover which patterns can be monochromatic of any finite coloring of $\mathbb{N}$. We start with following definition:

Definition 1.1. Let $k, s \in \mathbb{N}$, and let $f_1, \ldots, f_k : \mathbb{N}^s \to \mathbb{Z}$. We say that $f_1, \ldots, f_k$ is a Ramsey family if for any finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$, there exist $x \in \mathbb{N}^s$ and $i \in \{1, \ldots, r\}$ such that $\{f_1(x), \ldots, f_k(x)\} \subset C_i$.

I. Schur (1916) stated that for any finite partition of the natural numbers $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exist $x, y \in \mathbb{N}$ and $C \in \{C_1, \cdots, C_r\}$ such that $\{x, y, x+y\}$ [17]. A famous result in arithmetic Ramsey theory is van der Waerden's theorem(1927) on arithmetic progressions [18].

In this paper, we state generalized van der Waerden theorem and then we present Theorem 1.4 for some function that satisfy in Definition 2.1(such that does not necessarily apply to the assumption of Theorem 1.4). Also, we introduce some new Ramsey families such as we arrive that for $v, k \in \mathbb{N}$, the pattern $\{xy, x+v, x+y+v, x+2y+v, \ldots, x+ky+v\}$ is monochromatic.

Theorem 1.2 (van der Waerden theorem). For any finite partition of the natural numbers $\mathbb{N} = C_1 \cup \cdots \cup C_r$ and any $k \in \mathbb{N}$ there exist $x, y \in \mathbb{N}$ and $C \in \{C_1, \cdots, C_r\}$ such that $\{x, x+y, x+2y, ..., x+ky\} \subseteq C$.

After that, in [7] Brauer(1928) extended of Schur's theorem and van der Waerden theorem. Brauer's theorem stated for each $p \in \mathbb{N}$, the family $\{x, x+y, x+2y, \cdots, x+py\}$ is Ramsey in $\mathbb{N}$. Rado(1933) presented a theorem such that introduced necessary and sufficient conditions for a family of linear functions to be Ramsey [15]. Deuber (1973)

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presented the theorem such that contains Schur's theorem, van der Waerden's theorem, Brauer's theorem [8]. Deuber's theorem applies to many families of the form $\{f_1, \cdots, f_k\}$, such that f_i is a monomial, are Ramsey. Furstenberg and Sárközy (1977) proved that the family $\{x, x+y^2\}$ is Ramsey independently [10] and [16]. Afterward Bergelson (1987) proved that the family $\{x, y, x+y^2\}$ is Ramsey[4]. Bergelson and Leibman (1996) presented Polynomial van der Waerden theorem as follows [6].

Theorem 1.3 (Polynomial van der Waerden theorem). Let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be polynomials such that $f_i(0) = 0$ for all $i = 1, ..., k$. Then for any finite coloring of $\mathbb{N} = C_1 \cup \cdots \cup C_r$ there exist a color $C \in C_1, \cdots, C_r$ and $x, y \in \mathbb{N}$ such that

$$\{x, x+f_1(y), x+f_2(y), ..., x+f_k(y)\} \subset C.$$

The above theorem presented major advance towards Ramsey family problem. Many polynomial Ramsey present such as [1], [5], [3], [9], [13]. Afterward in [14], J. Moreira state an affine topological correspondence principle by Polynomial van der Waerden theorem and by using its proved the following theorem that introduce some Ramsey families.

Theorem 1.4. Let $s \in \mathbb{N}$ and, for each $i = 1, \ldots, s$, let F_i be a finite set of functions $\mathbb{N}^i \to \mathbb{Z}$ such that for all $f \in F_i$ and any $x_1, \ldots, x_{i-1} \in \mathbb{N}$, the function $x \to f(x_1, \ldots, x_{i-1}, x)$ is polynomial with 0 constant term. Then for any finite coloring of $\mathbb{N}$, there exist a color $C \subset \mathbb{N}$ and $(s+1)$-tuples $x_0, \ldots, x_s \in \mathbb{N}$ such that

$$\{x_0 \ldots x_s\} \cup \{x_0 \ldots x_j + f(x_{j+1}, \ldots, x_i) : 0 \leq j < i \leq s, f \in F_{i-j}\} \subset C.$$

Then he came to the following corollary.

Corollary 1.5. For any finite coloring of $\mathbb{N}$, there exist $x, y \in \mathbb{N}$ such that $\{x, xy, x+y\}$ is monochromatic.

First we define condition $(*)$(Definition 2.1) and by using this condition, we state generalized van der Waerden theorem.

Corollary 1.6 (generalized van der Waerden theorem). Let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{Z})$ and let F satisfy in condition $(*)$. Then for any finite coloring of $\mathbb{N}$, there exist x and $y \in \mathbb{N}$ such that $\{x+f(y) : f \in F\}$ is monochromatic.

We restate Theorem 1.4 for function that satisfies in condition $(*)$ by using 2.4 as follows. All family of functions that satisfy in the assumption of the following theorem are Ramsey family, including finite family of homomorphism functions.

Theorem 1.7. Let $s \in \mathbb{N}$ and, for each $i = 1, \ldots, s$, let F_i be a finite set of functions $\mathbb{N}^i \to \mathbb{Z}$ such that for all $f \in F_i$ and any $x_1, \ldots, x_{i-1} \in \mathbb{N}$, the function $x \to f(x_1, \ldots, x_{i-1}, x)$ satisfies in condition $(*)$. Then for any finite coloring of $\mathbb{N}$, there exist a color $C \subset \mathbb{N}$ and $(s+1)$-tuples $x_0, \ldots, x_s \in \mathbb{N}$ such that

$$\{x_0 \ldots x_s\} \cup \{x_0 \ldots x_j + f(x_{j+1}, \ldots, x_i) : 0 \leq j < i \leq s, f \in F_{i-j}\} \subset C.$$

Finally, we present a weak version of polynomial Ramsey family as follows.

Corollary 1.8. For $v \in \mathbb{N}$, $k \in \mathbb{N}$, we define $f_{v,k} : \mathbb{N} \to S$ by $f_{v,k}(x) = kx + v$. $F = \{f_{v,k} : k = n_1 < n_2 < \cdots < n_k\} \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ satisfies in condition $(*)$. Then for $v \in S$, $k \in \mathbb{N}$, the pattern $\{xy, x+v, x+y+v, x+2y+v, \ldots, x+ky+v\}$ is monochromatic.

2 Extend of the van der Waerden theorem

In this section, we defined condition $(*)$. Then we extend of the van der Waerden theorem.

Definition 2.1. Let $(S, +)$ be a commutative semigroup and $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$. We say that F satisfies in condition $(*)$, if
(1) for each $a, b \in \mathbb{N}$ there exist $c \in \mathbb{N}$ such that $f(a) + f(b) = f(c)$, for each $f \in F$ or
(2) for each $a, b \in \mathbb{N}$ there exist $c \in \mathbb{N}, d \in S$ such that $f(a) + f(b) = f(c) + d$, for each $f \in F$.

We present an example such that doesn't satisfy in polynomial van der Waerden theorem.

Example 2.2. (a) For $v \in S$, $k \in \mathbb{N}$, we define $f_{v,k} : \mathbb{N} \to S$ by $f_{v,k}(x) = kx + v$. Now let $F = \{f_{v,k} : k = n_1 < n_2 < \cdots < n_k\}$. Then F satisfies in condition $(*)$.
(b) Let $Hom(\mathbb{N}, S)$ denote the collection of all homomorphisms from $\mathbb{N}$ into S. Every $F \in \mathcal{P}_f(Hom(\mathbb{N}, S))$ satisfies in condition $(*)$.

Theorem 2.3. Let $(S, +)$ be a commutative semigroup and let A be a piecewise syndetic, if $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ satisfies in condition $(*)$, then there exist $a \in S$ and $b \in \mathbb{N}$ such that the set $\{a + f(b) : f \in F\} \subseteq A$.

Proof . Since A be a piecewise syndetic, by Theorem 14.8.3 in [12] A is J-set. So whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ there exist $a \in S, H \in \mathcal{P}_f(\mathbb{N})$ such that the set $a + \sum_{t \in H} f(t) \in A$ for each $f \in F$. According to assumption F satisfies in condition $(*)$ then there exist $c \in S, d \in \mathbb{N}$ such that $\{c + f(d) : f \in F\} \subseteq A$. □

Corollary 2.4 (generalized van der Waerden theorem). Let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{Z})$ such that if F satisfies in condition $(*)$, then for any finite coloring of $\mathbb{N}$, there exist $x, y \in \mathbb{N}$ such that $\{x + f(y) : f \in F\}$ is monochromatic.

3 Main results

The affine semigroup $(\mathcal{A}_{\mathbb{N}}^{-})$ is the semigroup consisting of all the linear maps from $\mathbb{Z}$ to $\mathbb{Z}$ such that $x \to ax + b$, where $a \in \mathbb{N}$ and $b \in \mathbb{Z}$, and the semigroup operation is composition of functions. Also, A_u is the map $x \to x + u$ for $u \in \mathbb{Z}$ and M_u is the map $x \to ux$ for $u > 0$. The following law satisfies:

$$\forall u \in \mathbb{N}, v \in \mathbb{Z}, \qquad M_u A_v = A_{uv} M_u. \tag{3.1}$$

Let $\mathcal{A}_{\mathbb{N}}^{-}$ acts on X via $(T_g)_{g \in \mathcal{A}_{\mathbb{N}}^{-}}$, for each $g \in \mathcal{A}_{\mathbb{N}}^{-}$, there is a map $T_g : X \to X$ and for any $g, h \in \mathcal{A}_{\mathbb{N}}^{-}$, $T_g \circ T_h = T_{gh}$, and let $u \in \mathbb{Z}$. The map T_{A_u} denote by A_u and, if $u > 0$, the map T_{M_u} by M_u.

Let G be a semigroup. A G-topological system denoted by a pair $(X, (T_g)_{g \in G})$, where X is a compact Hausdorff space, not necessarily metrizable, and let $(T_g)_{g \in G}$ be an action by continuous functions $T_g : X \to X$. We say system $(X, (T_g)_{g \in G})$ is minimal if X contains no proper nonempty closed invariant subsets. $x \in X$ is called a minimal point if its orbit closure $Y := \overline{\{T_g x : g \in G\}}$ is a minimal subsystem of X. We say that $A \subseteq \mathbb{N}$ is piecewise syndetic if $\overline{A} \cap K(\beta\mathbb{N}) \neq \emptyset$, where $K(\beta\mathbb{N})$ is minimal ideal of the Stone-Čech compactification of the natural numbers. It is well known that $K(\beta\mathbb{N})$ is union of all minimal sub dynamical system $(\beta\mathbb{N}, \{A_n\}_{n \in \mathbb{N}})$. In fact, minimal sub dynamical systems of $(\beta\mathbb{N}, \{A_n\}_{n \in \mathbb{N}})$ are precisely minimal left ideals of $\beta\mathbb{N}$. Since that $\beta\mathbb{N}$ is an additive compact semigroup, so there exists minimal idempotent. In this paper, minimal idempotent of $(\beta\mathbb{N}, +)$ is called additive minimal idempotent, and every element of minimal ideal of $\beta\mathbb{N}$ is called additive minimal point. For more detail see [4].

Proposition 3.1. Let $E \subset \mathbb{N}$ and $g \in \mathcal{A}_{\mathbb{N}}^{-}$. E is piecewise syndetic if and only if its image g(E) is piecewise syndetic.

Proof . See Proposition 2.1 in [14]. □

Proposition 3.2. Let $E \subset \mathbb{N}$ be a piecewise syndetic set. Then for any finite partition $\mathcal{C}$ of E, there exists $C \in \mathcal{C}$ that is piecewise syndetic.

Proof . See Theorem 1.24 in [11]. □

Theorem 3.3. There exists an $\mathcal{A}_{\mathbb{N}}^{-}$-topological system $(X, (T_g)_{g \in \mathcal{A}_{\mathbb{N}}^{-}})$ with a dense set of additively minimal points, such that all map $T_g : X \to X$ is open and injective, and for any finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$, there exists an open cover $X = U_1 \cup \ldots \cup U_r$ such that for each $g_1, \ldots, g_k \in \mathcal{A}_{\mathbb{N}}^{-}$and $t \in \{1, \ldots, r\}$,

$$\bigcap_{l=1}^{k} T_{g_l}(U_t) \neq \varnothing \qquad \Longrightarrow \qquad \mathbb{N} \cap \bigcap_{l=1}^{k} g_l(C_t) \neq \varnothing. \tag{3.2}$$

Proof . See the proof of Theorem 3.2 in [14]. □

Lemma 3.4. For each $g \in \mathcal{A}_{\mathbb{N}}^{-}$, the map $T_g : \beta\mathbb{N} \setminus \mathbb{N} \to \beta\mathbb{N} \setminus \mathbb{N}$ is continuous, open and injective. Morever, for $g, h \in \mathcal{A}_{\mathbb{N}}^{-}$, one has $T_g \circ T_h = T_{gh}$.

Proof . See the proof of Lemma 3.6 in [14]. □

Theorem 3.5. Let $(X, (T_g)_{g\in\mathcal{A}_{\mathbb{N}}^{-}})$ be an $\mathcal{A}_{\mathbb{N}}^{-}$-topological dynamical system, and let X contains a dense set of additively minimal points. Let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{Z})$ satisfy in condition $(*)$. Then for each nonempty open set $U \subset X$, there exists $n \in \mathbb{N}$ such that

$$\bigcap_{f\in F} A_{f(n)}U \neq \varnothing.$$

Proof . Let $y \in U$ be an additively minimal point and let $Y = \overline{\{A_n y : n \in \mathbb{Z}\}}$ be its additive orbit closure. $(Y, (A_n)_{n\in\mathbb{Z}})$ is a minimal topological system, then $\bigcup_n A_n U$ covers Y, and by ussing compactness there exists $r \in \mathbb{N}$, where $\bigcup_{n=1}^{r} A_n U$ covers Y. We define $\chi : \mathbb{N} \to \{1, \ldots, r\}$ of $\mathbb{N}$ by letting $\chi(n)$ be such that $A_n y \in A_{\chi(n)}U$.

Let $\mathbb{N} = \bigcup_{t=1}^{r} C_t$. By Corollary 2.4, if $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{Z})$ such that satisfies in condition $(*)$ then there exist $t \in \{1, \ldots, r\}, a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that $\chi(a + f(b)) = t$ for each $f \in F$. For any $f \in F$, let $\tilde{f} : n \to -f(n)$ and observe that $\tilde{f} \in^{\mathbb{N}} \mathbb{Z}$. Put $\tilde{F} = \{\tilde{f} : f \in F\}$ so find some $t \in \{1, \ldots, r\}$, $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that $\chi(a + \tilde{f}(b)) = t$ for every $f \in F$. In other words, $A_{a-f(b)}y \in A_t U$ for all $f \in F$ and so, $A_{a-t}y \in A_{f(b)}U$. Then we have

$$\bigcap_{f\in F} A_{f(b)}U \neq \varnothing.$$

□

Theorem 3.6. Let $(X, (T_g)_{g\in\mathcal{A}_{\mathbb{N}}^{-}})$ be an $\mathcal{A}_{\mathbb{N}}^{-}$-topological system with a dense set of additively minimal points, and let each map $T_g : X \to X$ be open and injective. Let $s \in \mathbb{N}$ and, for any $i = 1, \ldots, s$, let F_i be a finite set of functions $\mathbb{N}^i \to \mathbb{Z}$ such that for each $f \in F_i$ and each $x_1, \ldots, x_{i-1} \in \mathbb{N}$, function $x \to f(x_1, \ldots, x_{i-1}, x)$ satisfies in condition $(*)$. Then for each open cover $\mathcal{U}$ of X, there exists an open set $U \in \mathcal{U}$ in that cover and infinitely many s-tuples $x_1, \ldots, x_s \in \mathbb{N}$ such that

$$U \cap \bigcap_{0\leq j<i\leq s} \bigcap_{f\in F_{i-j}} M_{x_{j+1}\ldots x_s} A_{f(x_{j+1},\ldots,x_i)}U \neq \varnothing.$$

Proof . Let $\mathcal{U}$ be an open cover of X. We want to find $U \in \mathcal{U}$ and infinitely many s-tuples $x_1, \ldots, x_s \in \mathbb{N}$ such that

$$U \cap \bigcap_{0\leq j<i\leq s} \bigcap_{f\in F_{i-j}} M_{x_{j+1}\ldots x_s} A_{f(x_{j+1},\ldots,x_i)}U \neq \varnothing.$$

If X is compact, then there exists a finite subcover $U_1, \ldots, U_r$ of $\mathcal{U}$ such that $U_t \neq \varnothing$. We build four sequences $(t_n)_{n>0}, (y_n)_{n\geq 1}, (B_n)_{n\geq 0}$ and $(D_n)_{n\geq 0}$ as in the proof of Theorem 3.1 in [14]. For each $i \in 1, \ldots, s$ and each $f \in F_i$, we define the collection $G_n(f)$ of all functions $g : \mathbb{Z} \to \mathbb{Z}$ of the form

$$g : z \to y(m_1, n-1)f(y(m_1, m_2), y(m_2, m_3), \ldots, y(m_i, n-1)z)$$

for each $0 \leq m_1 < m_2 < \ldots < m_i < n$. If $i > n$, then we set $G_n(f) = \emptyset$. For each $g \in G_n(f)$ is function such that satisfies in condition $(*)$. By Theorem 3.5, we can detect $y_n \in \mathbb{N}$ satisfying

$$D_n := B_{n-1} \cap \bigcap_{i=1}^{s} \bigcap_{f\in F_i} \bigcap_{g\in G_n(f)} A_{g(y_n)}B_{n-1} \neq \varnothing.$$

Let $t_n \in \{1, \ldots, r\}$ such that $B_n := M_{y_n}D_n \cap U_{t_n} \neq \varnothing$. Since M_{y_n} is an open map, B_n is open. This finishes the construction of y_n, t_n, D_n, B_n . So $B_n \subset U_{t_n}$ for every $n \geq 0$. Also, $B_n \subset M_{y_n}D_n \subset M_{y_n}B_{n-1}$. By repeating this observation we arrive as follows.

$$\forall m \leqslant n, \qquad B_n \subset M_{y(m,n)} B_m.$$

The rest of the proof is the same as proof of Theorem 3.1 in [14].

□

Theorem 3.7. Let $s \in \mathbb{N}$ and, for any $i = 1, \ldots, s$, let F_i be a finite set of functions $\mathbb{N}^i \to \mathbb{Z}$ such that for all $f \in F_i$ and each $x_1, \ldots, x_{i-1} \in \mathbb{N}$, the function $x \to f(x_1, \ldots, x_{i-1}, x)$ satisfies in condition $(*)$. Then for each finite coloring of $\mathbb{N}$, there exists a color $C \subset \mathbb{N}$ and $(s+1)$-tuples $x_0, \ldots, x_s \in \mathbb{N}$ such that

$$\{x_0 \ldots x_s\} \cup \{x_0 \ldots x_j + f(x_{j+1}, \ldots, x_i) : 0 \leq j < i \leq s, f \in F_{i-j}\} \subset C.$$

Proof . Let $s \in \mathbb{N}$ and, for each $i = 1, \ldots, s$, let F_i be a finite set of functions $\mathbb{N}^i \to \mathbb{Z}$ such that for all $f \in F_i$ and any $x_1, \ldots, x_{i-1} \in \mathbb{N}$, the function $x \to f(x_1, \ldots, x_{i-1}, x)$ satisfies in condition $(*)$. Let $\mathbb{N} = C_1 \cup \ldots \cup C_r$ be a finite coloring of $\mathbb{N}$. We need to show that there exists a color C_t and (infinitely many) s + 1-tuples $x_0, \ldots, x_s \in \mathbb{N}$ such that $x_0 \ldots x_s \in C_t$ and, for every $0 \leq j < i \leq s$ and $f \in F_{i-j}$, we have $x_1 \ldots x_j + f(x_{j+1}, \ldots, x_i) \in C_t$.

We append to F_s the zero function $f : \mathbb{N}^s \to 0$ if necessary. Invoking Theorem 3.3 and then Theorem 3.6, the rest of the proof is similar to proof of Theorem 1.4 in [14]. □

We arrive as a result to corollary below.

Corollary 3.8. For $v \in \mathbb{N}$, $k \in \mathbb{N}$, we define $f_{v,k} : \mathbb{N} \to S$ by $f_{v,k}(x) = kx + v$. $F = \{f_{v,k} : k = n_1 < n_2 < \cdots < n_k\} \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ satisfies in condition $(*)$. Then for $v \in S$, $k \in \mathbb{N}$, the pattern $\{xy, x+v, x+y+v, x+2y+v, \ldots, x+ky+v\}$ is monochromatic.

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