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Bounds for the zeros of a quaternionic polynomial with restricted coefficients

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Abstract

The problem of determining the zeros of regular polynomials of a quaternionic variable with quaternionic coefficients is addressed in this study. We derive new bounds of the Eneström-Kakeya type for the zeros of these polynomials by virtue of a maximum modulus theorem and the structure of the zero sets in the newly developed theory of regular functions and polynomials of a quaternionic variable. Our findings generalise several newly proven conclusions concerning the distribution of zeros of a quaternionic polynomial.

Keywords: Quaternionic Polynomial, Zeros, Eneström-Kakeya Theorem
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1 Introduction

Polynomial zeros have a long and storied history in mathematics. This study has been the inspiration for many theoretical research (including being the original reason for contemporary algebra) and has a wide range of applications. Limiting polynomials is a good concept since reaching the zeros of a polynomial can be difficult using algebraic and analytic approaches. The fields first contributors were Gauss and Cauchy, and the subject dates back to approximately the time when the geometric representation of complex numbers was introduced into mathematics. Cauchy's [3] classic result on the distribution of zeros of a polynomial can be phrased as follows:

Theorem 1.1. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then all the zeros of p lie in

$$|z| < 1 + \max_{1 \le v \le n-1} \left| \frac{a_v}{a_n} \right|.$$

Although there are other results in the literature about the bounds for polynomial zeros (see [12], [13]), the striking property of the bound in Theorem 1.1 that distinguishes it from other such bounds is its ease of computation. This simplicity, however, comes at the expense of precision. The following elegant result on the location of zeros of a polynomial with restricted coefficients is known as Eneström-Kakeya Theorem (see [5], [12], [13]) which states that:

Theorem 1.2. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n such that $0 < a_0 \le a_1 \le ... \le a_n$, then all the zeros of p lie in $|z| \le 1$.

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G. Eneström appears to have been the first to obtain a finding of this sort while researching a problem in pension fund theory. S. Kakeya [11] presented a paper in the Tôhoku Mathematical Journal in 1912 that featured the following more comprehensive result:

Theorem 1.3. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n with real and positive coefficients, then all the zeros of p lie in the annulus $R_1 \leq |z| \leq R_2$, where $R_1 = \min_{0\leq v\leq n-1} a_v/a_{v+1}$ and $R_2 = \max_{0\leq v\leq n-1} a_v/a_{v+1}$.

The Eneström-Kakeya Theorem gives an upper bound on the modulus of the zeros of polynomials in a certain class (namely, those polynomials with real, non-negative, monotone increasing coefficients). We can easily obtain a zero free region for a related class of polynomials in the sense that we can get a lower bound on the modulus of the zeros. In the literature, for example see ([1], [9], [10], [12], [13]), there exist various extensions and generalizations of Eneström-Kakeya Theorem. In 1967, Joyal, Labelle, and Rahman [10] published a result which might be considered the foundation of the studies which we are currently studying. The Eneström-Kakeya Theorem, as stated in Theorem B, deals with polynomials with non-negative coefficients which form a monotone sequence. Joyal, Labelle, and Rahman generalized Theorem 1.2 by dropping the condition of non-negativity and maintaining the condition of monotonicity. Namely, they proved:

Theorem 1.4. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n such that $a_0 \leq a_1 \leq ... \leq a_n$, then all the zeros of p lie in $|z| \leq \frac{1}{|a_n|}(|a_0| + a_n - a_0)$.

Of course, when $a_0 \geq 0$, then Theorem 1.4 reduces to Theorem 1.2.
In this paper, we will prove some extensions and generalizations of above results for the class of polynomials with quaternionic variable and quaternionic coefficients.

2 Background

This section contains some preliminaries on regular functions of a quaternionic variable which will be useful in the sequel. In the recent study (for example, see [2], [6]-[8]), a new theory of regularity for functions, particularly for polynomials of a quaternionic variable was developed, and is extremely useful in replicating many important properties of holomorphic functions. One of the basic properties of holomorphic functions of a complex variable is the discreteness of their zero sets (except when the function vanishes identically). Given a regular function of a quaternionic variable, all its restrictions to complex lines are holomorphic and hence either have a discrete zero set or vanishes identically. In the preliminary steps, the structure of the zero sets of a quaternionic regular function and the factorization property of zeros was described. In this regard, Gentili and Stoppato [6] gave a necessary and sufficient condition for a quaternionic regular function to have a zero at a point in terms of the coefficients of the power series expansion of the function. Before we state our results, we need to introduce some preliminaries on quaternions and quaternionic polynomials.
Quaternions are the extension of complex numbers to four dimensions, introduced by William Rowan Hamilton in 1843. The set of all quaternions are denoted by $\mathbb{H}$ in honour of Sir Hamilton and are generally represented in the form $q = \alpha + i\beta + j\gamma + k\delta \in \mathbb{H}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and i, j, k are the fundamental quaternion units, such that $i^2 = j^2 = k^2 = ijk = -1$. Each quaternion q has a conjugate. The conjuate of a quaternion $q = \alpha + i\beta + j\gamma + k\delta$ is denoted by q^* and is defined as $q^* = \alpha - i\beta - j\gamma - k\delta$. Morever, the norm (or length) of a quaternion q is given by

$$||q|| = \sqrt{qq^*} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}.$$

The quaternions are the standard example of a noncommutative division ring and also forms a four dimensional vector space over $\mathbb{R}$ with $\{1, i, j, k\}$ as a basis.
In 2020, Carney et al. [2] proved the following extension of Theorem 1.2 for the quaternionic polynomial $p(q)$. More prescisely they proved the following result:

Theorem 2.1. If $p(q) = \sum_{v=0}^{n} q^v a_v$ is a quaternionic polynomial of degree n with real coefficients satisfying $0 < a_0 \leq a_1 \leq ... \leq a_n$, then all the zeros of p lie in $|q| \leq 1$.

They also proved by using Lemma 4.2 the following result:

Theorem 2.2. If $p(q) = \sum\limits_{v=0}^{n} q^v a_v$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non-zero quaternion and suppose $\angle(a_v, b) \leq \theta \leq \pi/2$ for some θ and $v = 0, 1, 2, ..., n$. Assume

$$|a_n| \geq |a_{n-1}| \geq ... \geq |a_0|,$$

then all the zeros of p lie in

$$|q| \leq \cos\theta + \sin\theta + \frac{2\sin\theta}{|a_n|}\sum_{v=0}^{n-1}|a_v|.$$

We state our main results about quaternionic polynomials with restricted coefficients and the location of their zeros. We start with the following generalization of Theorem 2.2.

3 Main Results

Theorem 3.1. If $p(q) = \sum_{v=0}^{n} q^v a_v$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non-zero quaternion and suppose $\angle(a_v, b) \leq \theta \leq \pi/2$ for some θ, $v = 0, 1, 2, ..., n$. Assume

$$|a_n| \leq |a_{n-1}| \leq ... \leq |a_{r+1}| \leq \lambda|a_r| \geq |a_{r-1}| \geq ... \geq |a_1| \geq |a_0|,$$

where $0 \leq r \leq n$ and $\lambda \geq 1$, then all the zeros of p lie in

$$|q| \leq \frac{1}{|a_n|}\left\{|a_n|(\sin\theta - \cos\theta) + 2\lambda|a_r|\cos\theta + 2(\lambda-1)|a_r|\sin\theta + 2\sin\theta\sum_{v=0}^{n-1}|a_v| + 2|1-\lambda||a_r|\right\}.$$

If we take $r = n$ in Theorem 3.1, we get the following result:

Corollary 3.2. If $p(q) = \sum_{v=0}^{n} q^v a_v$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non-zero quaternion and suppose $\angle(a_v, b) \leq \theta \leq \pi/2$ for some θ, $v = 0, 1, 2, ..., n$. Assume

$$\lambda|a_n| \geq |a_{n-1}| \geq ... \geq |a_1| \geq |a_0|,$$

where $\lambda \geq 1$, then all the zeros of p lie in

$$|q| \leq (\sin\theta - \cos\theta) + 2\lambda\cos\theta + 2(\lambda-1)\sin\theta + \frac{2\sin\theta}{|a_n|}\sum_{v=0}^{n-1}|a_v| + 2|1-\lambda|.$$

By taking $\lambda = 1$ in above corollary, we get Theorem 2.2. Similarly by taking $r = 0$ and $\lambda = 1$ in Theorem 3.1, we get the following result:

Corollary 3.3. If $p(q) = \sum_{v=0}^{n} q^v a_v$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non-zero quaternion and suppose $\angle(a_v, b) \leq \theta \leq \pi/2$ for some θ, $v = 0, 1, 2, ..., n$. Assume

$$|a_n| \leq |a_{n-1}| \leq ... \leq |a_1| \leq |a_0|,$$

then all the zeros of p lie in

$$|q| \leq (\sin\theta - \cos\theta) + 2\left|\frac{a_0}{a_n}\right|\cos\theta + \frac{2\sin\theta}{|a_n|}\sum_{v=0}^{n-1}|a_v|.$$

Theorem 3.4. If $p(q) = \sum_{v=0}^{n} q^v a_v$ is a quaternionic polynomial of degree n with quaternionic coefficients $a_v = \alpha_v + i\beta_v + j\gamma_v + k\delta_v$, $v = 0, 1, 2, ..., n$ and for some $k_1, k_2, k_3, k_4 \geq 1$, $0 < \rho \leq 1$, $0 \leq \lambda \leq n$, $0 \leq \mu \leq n$, $0 \leq \zeta \leq n$ and $0 \leq l \leq n$ satisfying

$$k_1\alpha_n \geq \alpha_{n-1} \geq ... \geq \rho\alpha_\lambda \leq \alpha_{\lambda-1} \leq ... \leq \alpha_1 \leq \alpha_0,$$
$$k_2\beta_n \geq \beta_{n-1} \geq ... \geq \rho\beta_\mu \leq \beta_{\mu-1} \leq ... \leq \beta_1 \leq \beta_0,$$
$$k_3\gamma_n \geq \gamma_{n-1} \geq ... \geq \rho\gamma_\zeta \leq \gamma_{\zeta-1} \leq ... \leq \gamma_1 \leq \gamma_0,$$

$$k_4\delta_n \geq \delta_{n-1} \geq ... \geq \rho\delta_l \leq \delta_{l-1} \leq ... \leq \delta_1 \leq \delta_0,$$

then all the zeros of p lie in

$$|q| \leq \frac{1}{|a_n|}\Big\{(|\alpha_0| + \alpha_0 - |\alpha_n|) + (|\beta_0| + \beta_0 - |\beta_n|) + (|\gamma_0| + \gamma_0 - |\gamma_n|) + (|\delta_0| + \delta_0 - |\delta_n|)$$
$$+ k_1(\alpha_n + |\alpha_n|) + k_2(\beta_n + |\beta_n|) + k_3(\gamma_n + |\gamma_n|) + k_4(\delta_n + |\delta_n|) - \rho(\alpha_\lambda + \beta_\mu + \gamma_\zeta + \delta_l)$$
$$+ (1-\rho)(|\alpha_\lambda| + |\beta_\mu| + |\gamma_\zeta| + |\delta_l|)\Big\}.$$

Applying Theorem 3.4 to the polynomial $p(q)$ having real coefficients, i.e., $\beta = \gamma = \delta = 0$, we have the following result:

Corollary 3.5. If $p(q) = \sum_{v=0}^{n} q^v a_v$ is a quaternionic polynomial of degree n with real coefficients $a_v,\ v = 0, 1, 2, ..., n$ and for some $k_1 \geq 1$, $0 < \rho \leq 1$ and $0 \leq \lambda \leq n$ satisfying

$$k_1 a_n \geq a_{n-1} \geq ... \geq \rho a_\lambda \leq a_{\lambda-1} \leq ... \leq a_1 \leq a_0,$$

then all the zeros of p lie in

$$|q| \leq \frac{1}{|a_n|}\Big\{(|a_0| + a_0 - |a_n|) + k_1(a_n + |a_n|) - \rho a_\lambda + (1-\rho)|a_\lambda|\Big\}.$$

4 Lemmas

We need the following lemmas for the proofs of the main results. The first lemma is due to Gentili and Stoppato [6].

Lemma 4.1. If $f(q) = \sum_{v=0}^{\infty} q^v a_v$ and $g(q) = \sum_{v=0}^{\infty} q^v b_v$ be two given quaternionic power series with radii of convergence greater than R. The regular product of $f(q)$ and $g(q)$ is defined as $(f \star g)(q) = \sum_{v=0}^{\infty} q^v c_v$, where $c_v = \sum_{k=0}^{v} a_k b_{v-k}$. Let $|q_0| < R$, then $(f \star g)(q_0) = 0$ if and only if either $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g(f(q_0)^{-1} q_0 f(q_0)) = 0$.

The following lemma is due to Carney et al. [2].

Lemma 4.2. Let $q_1,\ q_2 \in \mathbb{H}$, where $q_1 = \alpha_1 + i\beta_1 + j\gamma_1 + k\delta_1$ and $q_2 = \alpha_2 + i\beta_2 + j\gamma_2 + k\delta_2$, $\angle(q_1, q_2) = 2\theta' \leq 2\theta$, and $|q_1| \leq |q_2|$, then

$$|q_2 - q_1| \leq (|q_2| - |q_1|)\cos\theta + (|q_2| + |q_1|)\sin\theta.$$

5 Proofs of the Theorems

Proof of Theorem 3.1. Consider the polynomial

$$f(q) = \sum_{v=1}^{n} q^v (a_v - a_{v-1}) + a_0.$$

Let $p(q) \star (1-q) = f(q) - q^{n+1}a_n$, therefore by Lemma 4.1, $p(q) \star (1-q) = 0$ if and only if either $p(q) = 0$ or $p(q) \neq 0$ implies $p(q)^{-1}qp(q) - 1 = 0$, that is, $p(q)^{-1}qp(q) = 1$. If $p(q) \neq 0$, then $q = 1$. Therefore, the only zeros of $p(q) \star (1-q)$ are $q = 1$ and the zeros of $p(q)$. For $|q| = 1$, we have

$$\begin{aligned}
|f(q)| &= |a_0 + \sum_{v=1}^{n}(a_v - a_{v-1})| \\
&\leq |a_0| + \sum_{v=1}^{n}|a_v - a_{v-1}| \\
&= |a_0| + |a_n - a_{n-1}| + ... + |a_{r+1} - a_r| + |a_r - a_{r-1}| + ... + |a_1 - a_0| \\
&= |a_0| + |a_n - a_{n-1}| + ... + |a_{r+1} - \lambda a_r + \lambda a_r - a_r| + |a_r - \lambda a_r + \lambda a_r - a_{r-1}| + ... + |a_1 - a_0| \\
&\leq |a_0| + |a_n - a_{n-1}| + ... + |a_{r+1} - \lambda a_r| + |\lambda a_r - a_{r-1}| + 2|1-\lambda||a_r| + ... + |a_1 - a_0|.
\end{aligned}$$

Now using Lemma 4.2, it follows that

$$\begin{aligned}
|f(q)| &\leq 2|1-\lambda||a_r| + (|a_{n-1}| - |a_n|)\cos\theta + (|a_{n-1}| + |a_n|)\sin\theta \\
&+ (|a_{n-2}| - |a_{n-1}|)\cos\theta + ... + (|a_{n-2}| + |a_{n-1}|)\sin\theta + (\lambda|a_r| - |a_{r+1}|)\cos\theta \\
&+ (\lambda|a_r| + |a_{r+1}|)\sin\theta + (\lambda|a_r| - |a_{r-1}|)\cos\theta + (\lambda|a_r| + |a_{r-1}|)\sin\theta \\
&+ ... + (|a_2| - |a_1|)\cos\theta + (|a_2| + |a_1|)\sin\theta + (|a_1| - |a_0|)\cos\theta + (|a_1| + |a_0|)\sin\theta + |a_0| \\
&= |a_n|(\sin\theta - \cos\theta) + 2\lambda|a_r|\cos\theta + 2(\lambda-1)|a_r|\sin\theta \\
&+ 2\sin\theta\sum_{v=0}^{n-1}|a_v| - |a_0|(\cos\theta + \sin\theta - 1) + 2|1-\lambda||a_r| \\
&\leq |a_n|(\sin\theta - \cos\theta) + 2\lambda|a_r|\cos\theta + 2(\lambda-1)|a_r|\sin\theta + 2\sin\theta\sum_{v=0}^{n-1}|a_v| + 2|1-\lambda||a_r|.
\end{aligned}$$

Since

$$\max_{|q|=1}\left|q^n \star f\Big(\frac{1}{q}\Big)\right| = \max_{|q|=1}\left|f\Big(\frac{1}{q}\Big)\right| = \max_{|q|=1}|f(q)|,$$

therefore, $q^n \star f\left(\frac{1}{q}\right)$ has the same bound on $|q|=1$ as $f(q)$, that is

$$\begin{aligned}
\left|q^n \star f\Big(\frac{1}{q}\Big)\right| &\leq |a_n|(\sin\theta - \cos\theta) + 2\lambda|a_r|\cos\theta + 2(\lambda-1)|a_r|\sin\theta \\
&+ 2\sin\theta\sum_{v=0}^{n-1}|a_v| + 2|1-\lambda||a_r| \quad \text{for} \quad |q|=1.
\end{aligned}$$

Applying maximum modulus theorem ([7], Theorem 3.4), it follows that

$$\begin{aligned}
\left|q^n \star f\Big(\frac{1}{q}\Big)\right| &\leq |a_n|(\sin\theta - \cos\theta) + 2\lambda|a_r|\cos\theta + 2(\lambda-1)|a_r|\sin\theta \\
&+ 2\sin\theta\sum_{v=0}^{n-1}|a_v| + 2|1-\lambda||a_r| \quad \text{for} \quad |q|\leq 1.
\end{aligned}$$

Replacing q by $\frac{1}{q}$, we get for $|q|\geq 1$

$$\begin{aligned}
&|f(q)| \\
&\leq \left\{|a_n|(\sin\theta - \cos\theta) + 2\lambda|a_r|\cos\theta + 2(\lambda-1)|a_r|\sin\theta + 2\sin\theta\sum_{v=0}^{n-1}|a_v| + 2|1-\lambda||a_r|\right\}|q|^n. \qquad (5.1)
\end{aligned}$$

But $|p(q) \star (1-q)| = |f(q) - q^{n+1}a_n| \geq |a_n||q|^{n+1} - |f(q)|$.
Using (5.1), we have for $|q|\geq 1$

$$\begin{aligned}
&|p(q) \star (1-q)| \geq |a_n||q|^{n+1} \\
&- \left\{|a_n|(\sin\theta - \cos\theta) + 2\lambda|a_r|\cos\theta + 2(\lambda-1)|a_r|\sin\theta + 2\sin\theta\sum_{v=0}^{n-1}|a_v| + 2|1-\lambda||a_r|\right\}|q|^n.
\end{aligned}$$

This implies that $|p(q) \star (1-q)| > 0$, i.e., $p(q) \star (1-q) \neq 0$ if

$$|q| > \frac{1}{|a_n|}\left\{|a_n|(\sin\theta - \cos\theta) + 2\lambda|a_r|\cos\theta + 2(\lambda-1)|a_r|\sin\theta + 2\sin\theta\sum_{v=0}^{n-1}|a_v| + 2|1-\lambda||a_r|\right\}.$$

Since the only zeros of $p(q) \star (1-q)$ are $q=1$ and the zeros of $p(q)$. Therefore, $p(q) \neq 0$ for

$$|q| > \frac{1}{|a_n|}\left\{|a_n|(\sin\theta - \cos\theta) + 2\lambda|a_r|\cos\theta + 2(\lambda-1)|a_r|\sin\theta + 2\sin\theta\sum_{v=0}^{n-1}|a_v| + 2|1-\lambda||a_r|\right\}.$$

Hence all the zeros of $p(q)$ lie in

$$|q| \leq \frac{1}{|a_n|}\left\{|a_n|(\sin\theta - \cos\theta) + 2\lambda|a_r|\cos\theta + 2(\lambda-1)|a_r|\sin\theta + 2\sin\theta\sum_{v=0}^{n-1}|a_v| + 2|1-\lambda||a_r|\right\}.$$

This completes the proof of Theorem 3.1.

Proof of Theorem 3.4. Consider the polynomial

$$f(q) = \sum_{v=1}^{n} q^v(a_v - a_{v-1}) + a_0.$$

Let $p(q)\star(1-q) = f(q) - q^{n+1}a_n$, therefore by Lemma 4.1, $p(q)\star(1-q) = 0$ if and only if either $p(q) = 0$ or $p(q) \neq 0$ implies $p(q)^{-1}qp(q) - 1 = 0$, that is, $p(q)^{-1}qp(q) = 1$. If $p(q) \neq 0$, then $q = 1$. Therefore the only zeros of $p(q)\star(1-q)$ are $q = 1$ and the zeros of $p(q)$. For $|q| = 1$, we have

$$\begin{aligned}
|f(q)| &\leq |a_0| + \sum_{v=1}^{n}|a_v - a_{v-1}| \\
&\leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{v=1}^{n}\left\{|\alpha_v - \alpha_{v-1}| + |\beta_v - \beta_{v-1}| + |\gamma_v - \gamma_{v-1}| + |\delta_v - \delta_{v-1}|\right\} \\
&= |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + |\alpha_n - \alpha_{n-1}| + |\alpha_{\lambda+1} - \alpha_\lambda| \\
&+ \sum_{v=1}^{\lambda}|\alpha_v - \alpha_{v-1}| + \sum_{v=\lambda+2}^{n-1}|\alpha_v - \alpha_{v-1}| + |\beta_n - \beta_{n-1}| + |\beta_{\mu+1} - \beta_\mu| \\
&+ \sum_{v=1}^{\mu}|\beta_v - \beta_{v-1}| + \sum_{v=\mu+2}^{n-1}|\beta_v - \beta_{v-1}| + |\gamma_n - \gamma_{n-1}| + |\gamma_{\zeta+1} - \gamma_\zeta| \\
&+ \sum_{v=1}^{\zeta}|\gamma_v - \gamma_{v-1}| + \sum_{v=\zeta+2}^{n-1}|\gamma_v - \gamma_{v-1}| + |\delta_n - \delta_{n-1}| + |\delta_{l+1} - \delta_l| \\
&+ \sum_{v=1}^{l}|\delta_v - \delta_{v-1}| + \sum_{v=l+2}^{n-1}|\delta_v - \delta_{v-1}| \\
&= |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + |k_1\alpha_n + \alpha_n - k_1\alpha_n - \alpha_{n-1}| + |\rho\alpha_\lambda + \alpha_{\lambda+1} - \rho\alpha_\lambda - \alpha_\lambda| \\
&+ \sum_{v=1}^{\lambda}|\alpha_v - \alpha_{v-1}| + \sum_{v=\lambda+2}^{n-1}|\alpha_v - \alpha_{v-1}| + |k_2\beta_n + \beta_n - k_2\beta_n - \beta_{n-1}| + |\rho\beta_\mu + \beta_{\mu+1} - \rho\beta_\mu - \beta_\mu| \\
&+ \sum_{v=1}^{\mu}|\beta_v - \beta_{v-1}| + \sum_{v=\mu+2}^{n-1}|\beta_v - \beta_{v-1}| + |k_3\gamma_n + \gamma_n - k_3\gamma_n - \gamma_{n-1}| + |\rho\gamma_\zeta + \gamma_{\zeta+1} - \rho\gamma_\zeta - \gamma_\zeta| \\
&+ \sum_{v=1}^{\zeta}|\gamma_v - \gamma_{v-1}| + \sum_{v=\zeta+2}^{n-1}|\gamma_v - \gamma_{v-1}| + |k_4\delta_n + \delta_n - k_4\delta_n - \delta_{n-1}| + |\rho\delta_l + \delta_{l+1} - \rho\delta_l - \delta_l| \\
&+ \sum_{v=1}^{l}|\delta_v - \delta_{v-1}| + \sum_{v=l+2}^{n-1}|\delta_v - \delta_{v-1}|
\end{aligned}$$

$$\leq |\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+|k_1\alpha_n-\alpha_{n-1}|+(k_1-1)|\alpha_n|+|\alpha_{\lambda+1}-\rho\alpha_\lambda|+(1-\rho)|\alpha_\lambda|$$
$$+\sum_{v=1}^{\lambda}|\alpha_{v-1}-\alpha_v|+\sum_{v=\lambda+2}^{n-1}|\alpha_v-\alpha_{v-1}|+|k_2\beta_n-\beta_{n-1}|+(k_2-1)|\beta_n|+|\beta_{\mu+1}-\rho\beta_\mu|+(1-\rho)|\beta_\mu|$$
$$+\sum_{v=1}^{\mu}|\beta_{v-1}-\beta_v|+\sum_{v=\mu+2}^{n-1}|\beta_v-\beta_{v-1}|+|k_3\gamma_n-\gamma_{n-1}|+(k_3-1)|\gamma_n|+|\gamma_{\zeta+1}-\rho\gamma_\zeta|+(1-\rho)|\gamma_\zeta|$$
$$+\sum_{v=1}^{\zeta}|\gamma_{v-1}-\gamma_v|+\sum_{v=\zeta+2}^{n-1}|\gamma_v-\gamma_{v-1}|+|k_4\delta_n-\delta_{n-1}|+(k_4-1)|\delta_n|+|\delta_{l+1}-\rho\delta_l|+(1-\rho)|\delta_l|$$
$$+\sum_{v=1}^{l}|\delta_{v-1}-\delta_v|+\sum_{v=l+2}^{n-1}|\delta_v-\delta_{v-1}|$$

$$=|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+|k_1\alpha_n-\alpha_{n-1}|+(k_1-1)|\alpha_n|+|\alpha_{\lambda+1}-\rho\alpha_\lambda|+(1-\rho)|\alpha_\lambda|$$
$$+|k_2\beta_n-\beta_{n-1}|+(k_2-1)|\beta_n|+|\beta_{\mu+1}-\rho\beta_\mu|+(1-\rho)|\beta_\mu|+|k_3\gamma_n-\gamma_{n-1}|+(k_3-1)|\gamma_n|$$
$$+|\gamma_{\zeta+1}-\rho\gamma_\zeta|+(1-\rho)|\gamma_\zeta|+|k_4\delta_n-\delta_{n-1}|+(k_4-1)|\delta_n|+|\delta_{l+1}-\rho\delta_l|+(1-\rho)|\delta_l|$$
$$+\alpha_0-\alpha_1+\alpha_1-\alpha_2+...+\alpha_{\lambda-2}-\alpha_{\lambda-1}+\alpha_{\lambda-1}-\alpha_\lambda+\alpha_{\lambda+2}-\alpha_{\lambda+1}+\alpha_{\lambda+3}-\alpha_{\lambda+2}$$
$$+...+\alpha_{n-1}-\alpha_{n-2}+\beta_0-\beta_1+\beta_1-\beta_2+...+\beta_{\mu-2}-\beta_{\mu-1}+\beta_{\mu-1}-\beta_\mu+\beta_{\mu+2}$$
$$-\beta_{\mu+1}+\beta_{\mu+3}-\beta_{\mu+2}+...+\beta_{n-1}-\beta_{n-2}+\gamma_0-\gamma_1+\gamma_1-\gamma_2+...+\gamma_{\zeta-2}-\gamma_{\zeta-1}$$
$$+\gamma_{\zeta-1}-\gamma_\zeta+\gamma_{\zeta+2}-\gamma_{\zeta+1}-\gamma_{\zeta+3}-\gamma_{\zeta+2}+...+\gamma_{n-1}-\gamma_{n-2}+\delta_0-\delta_1+\delta_1-\delta_2$$
$$+...+\delta_{l-2}-\delta_{l-1}+\delta_{l-1}-\delta_l+\delta_{l+2}-\delta_{l+1}-\delta_{l+3}-\delta_{l+2}+...+\delta_{n-1}-\delta_{n-2}$$
$$=(|\alpha_0|+\alpha_0-|\alpha_n|)+(|\beta_0|+\beta_0-|\beta_n|)+(|\gamma_0|+\gamma_0-|\gamma_n|)+(|\delta_0|+\delta_0-|\delta_n|)$$
$$+k_1(\alpha_n+|\alpha_n|)+k_2(\beta_n+|\beta_n|)+k_3(\gamma_n+|\gamma_n|)+k_4(\delta_n+|\delta_n|)-\rho(\alpha_\lambda+\beta_\mu+\gamma_\zeta+\delta_l)$$
$$+(1-\rho)(|\alpha_\lambda|+|\beta_\mu|+|\gamma_\zeta|+|\delta_l|).$$

Since

$$\max_{|q|=1}\left|q^n\star f\left(\frac{1}{q}\right)\right|=\max_{|q|=1}\left|f\left(\frac{1}{q}\right)\right|=\max_{|q|=1}|f(q)|,$$

therefore, $q^n\star f\left(\frac{1}{q}\right)$ has the same bound on $|q|=1$ as $f(q)$, that is

$$\left|q^n\star f\left(\frac{1}{q}\right)\right|\leq(|\alpha_0|+\alpha_0-|\alpha_n|)+(|\beta_0|+\beta_0-|\beta_n|)+(|\gamma_0|+\gamma_0-|\gamma_n|)+(|\delta_0|+\delta_0-|\delta_n|)$$
$$+k_1(\alpha_n+|\alpha_n|)+k_2(\beta_n+|\beta_n|)+k_3(\gamma_n+|\gamma_n|)+k_4(\delta_n+|\delta_n|)-\rho(\alpha_\lambda+\beta_\mu+\gamma_\zeta+\delta_l)$$
$$+(1-\rho)(|\alpha_\lambda|+|\beta_\mu|+|\gamma_\zeta|+|\delta_l|)\quad\text{for}\quad|q|=1.$$

After few steps as in the proof of Theorem 3.1, we conclude that all the zeros of $p(q)$ lie in

$$|q| \leq \frac{1}{|a_n|}\Big\{(|\alpha_0| + \alpha_0 - |\alpha_n|) + (|\beta_0| + \beta_0 - |\beta_n|) + (|\gamma_0| + \gamma_0 - |\gamma_n|) + (|\delta_0| + \delta_0 - |\delta_n|)$$
$$+ k_1(\alpha_n + |\alpha_n|) + k_2(\beta_n + |\beta_n|) + k_3(\gamma_n + |\gamma_n|) + k_4(\delta_n + |\delta_n|) - \rho(\alpha_\lambda + \beta_\mu + \gamma_\zeta + \delta_l)$$
$$+ (1-\rho)(|\alpha_\lambda| + |\beta_\mu| + |\gamma_\zeta| + |\delta_l|)\Big\}.$$

This completes the proof of Theorem 3.4.

6 Conclusions

Some fresh findings on Eneström-Kakeya Theorem for quaternionic polynomials has been discovered that are useful in determining the regions containing all the zeros of a polynomial.

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