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Stability analysis of harvested fractional-order prey-predator model with Holling type IV response

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Abstract

We consider a fractional-order prey-predator model with Holling type IV functional response and the effect of harvesting on both populations. The sufficient condition for the existence, uniqueness, non-negativity and boundedness of the solutions are discussed. Moreover, we found all possible equilibrium points and the local and global stability behavior are investigated. Finally, numerical results are presented with some examples.

Keywords: Caputo fractional dynamical system; Harvesting; Stability analysis; Holling type IV response
2020 MSC: 26A33, 37C75, 65L07, 65P10, 65P40

1 Introduction

The study of dynamics of the prey-predator model gets more attention among researchers due to its global importance and existence. Lotka and Volterra were the first to present this interaction mathematically [20, 34]. The qualitative properties of the prey-predator model in population dynamics have been studied by many ecologists. [12, 2, 24, 22]. Several factors impact the dynamics of prey-predator interactions: prey refuge, harvesting, disease, delay, and many other factors. Among them, researchers are particularly interested in examining the impact of harvesting. Chakraborty et al. described the constant rate harvesting [6], Leard et al. and Lenzini et al. presented the proportional harvesting [17, 18], Das et al. and Jana et al. discussed the nonlinear harvesting [8, 13], which are the three different types of harvesting have been investigated by several researchers. Some authors considered the case that any one of the species is harvested, while others considered both the prey and predator species are harvested [13, 21, 28].

In ecology, the functional response of a prey-predator system is significant. A model with functional response help to describe two important parameters: handling time and attack rate. The population dynamics models with different functional responses have been widely studied in the ecological literature, especially prey-predator models with Holling types, which were extensively examined by several authors [6, 8, 13, 21, 28, 11]. In 1968, Andrews [5] first derived a function

$$H(u) = \frac{mu}{a + bu + u^2}.$$

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The above function is said to be the Monod-Haldane function or Holling type IV function. This functional response indicates the predator's per capita predation rate decreasing due to the cooperative defense of prey species and high prey densities. M. C. Kohnke et al. derived the prey-predator model with Holling type IV response with different shapes [16]. In [32], J. Song et al. proposed the Leslie-Gower model with type IV functional response.

Researchers have become more interested in developing mathematical models involving fractional-order differential equations in recent years [9, 7, 3, 31, 26, 27, 23]. Many methods exist to solve the fractional-order differential equations, for example, laplace transform method [27], homotopy perturbation method [35], fourier transformation method [30], fourier sine transformation method [1] and many others. Furthermore, due to memory effects, fractional-order systems are more realistic in biological modeling than integer-order systems. Fractional operators are nonlocal and so they are more suitable for modeling the system with memory effect. In the literature, various studies have been done on the stability analysis of fractional-order system. M. Das et al. [9] discussed the fractional-order modified Holling type IV functional response. R. Chinnathambi et al. [7], studied the fractional-order time-delay with Holling type IV functional response and many others [14, 29, 3, 4]. In [19], X. Liu et al. discussed the integer-order predator-prey model with Holling type IV response and also they found at an equilibrium point, the system stabilizes by varying the harvesting efforts and initial value. By the above motivation, we consider the fractional-order prey-predator model with group defense and harvesting on both populations.

The rest of this article is organized as follows: In Section 2, we formulate the fractional-order system. Some preliminary results have been given in Section 3. The property of uniformly bounded of the system is shown in Section 4. In Section 5, existence and uniqueness conditions are proved. Section 6, provided the stability behavior of all feasible equilibrium points. In Section 7, we analyzed the global behavior of predator-free equilibrium point and coexisting equilibrium point. We present numerical examples in Section 8 to validate our theoretical findings.

2 Fractional-order model

The following is the prey-predator model with Holling type IV functional response:

$$\begin{aligned}\frac{du}{dt} &= rx\left(1-\frac{u}{K}\right)-\frac{uv}{a+bu+u^2}-\gamma_1 u,\\ \frac{dv}{dt} &= \frac{\tau uv}{a+bu+u^2}-dv-\gamma_2 v.\end{aligned} \tag{2.1}$$

Here, $u(t)$ is the prey density and $v(t)$ is the predator density, r is the intrinsic growth rate of prey and K denotes the environmental carrying capacity. Let d and τ be the death rate and maximum predation rate respectively, $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$ decribes the harvesting efforts for both the species. We take these terms as $H_1\gamma_1 u$ and $H_2\gamma_2 v$, where H_1 and H_2 denotes the catchability coefficients of the prey and predator respectively. For our convenience, we assume that H_1 and H_2 as unity. The term $\frac{uv}{a+bu+u^2}$ gives the Holling functional response of type IV , where a denotes the half-saturation constant and $b > \sqrt{2}a$ is the denominator of the functional response.

In this paper, we consider the fractional-order prey-predator system with harvesting as the following form:

$$\begin{aligned}{}^cD^\alpha u(t) &= ru(1-\frac{u}{K})-\frac{uv}{a+bu+u^2}-\gamma_1 u,\\ {}^cD^\alpha v(t) &= \frac{\tau uv}{a+bu+u^2}-dv-\gamma_2 v.\end{aligned} \tag{2.2}$$

with the non-negative initial values $u(0) = u_0$ and $v(0) = v_0$, where $0 < \alpha < 1$ and $r, K, a, b, d, \tau, \gamma_1, \gamma_2$ are non-negative.

3 Preliminary results

Definition 3.1. [26] The Caputo differential operator for $\alpha > 0$ is given by

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t f^m(s)(t-s)^{m-\alpha-1}ds,$$

where $\Gamma(.)$ is the Euler's gamma function. For $\alpha \in (0, 1)$

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t f^{'}(s)(t-s)^{-\alpha}ds.$$

Definition 3.2. A point Z^* is an equilibrium point of the following system

$$^cD^\alpha Z(t) = f(t, Z(t)), \quad \text{with } Z(0) = Z_0, \tag{3.1}$$

if and only if, $f(t, Z^*) = 0$.

Lemma 3.3. [25] Assume $0 < \alpha \leq 1$, $f(t)$ and $^cD^\alpha f(t) \in C[a, b]$. For $t \in (a, b)$,

(a) If $^cD^\alpha\ f(t) \geq 0$ is a non-decreasing function $\forall\ a \leq t \leq b$.

(b) If $^cD^\alpha f(t) \leq 0$ is a non-increasing function $\forall\ a \leq t \leq b$.

Lemma 3.4. [15] Let $u(t)$ be a continuous function on $(0, T]$ and also satisfy

$$^cD^\alpha u(t) \leq -au(t) + b, \quad u(0) = u_0 > 0, \quad 0 < \alpha < 1,$$

where a, b $\in \mathbb{R}^2$, a $\neq 0$. Then

$$u(t) \leq \left(u_0 - \frac{b}{a}\right) E_\alpha[-at^\alpha] + \frac{b}{a}.$$

Lemma 3.5. [27] Consider the system (3.1), where $f : \aleph \times (0, T] \to \mathbb{R}^n$, $\aleph \subset \mathbb{R}^n$. If $f(t, X)$ satisfies the locally Lipschitz condition then there exist a unique solution of (3.1) on $\aleph \times (0, T]$.

Theorem 3.6. [26] The equilibrtum points of the system (3.1) are locally asymptotically stable if all eigenvalues λ_i $(i = 1, ..., n)$ of the jacobian matrix $j = \frac{\partial f}{\partial x}$ which satisfy $|arg(\lambda_i)| > \alpha\frac{\pi}{2}$ and it is unstable if all eigenvalues λ_i $(i = 1, ..., n)$ satisfy $|arg(\lambda_i)| < \alpha\frac{\pi}{2}$.

4 Non-negativity and boundedness of the solution

Here we show that the system (2.2) solutions are non-negative. Assume that $\Omega^+ = \{(u, v) \in \Omega : u, v \in \mathbb{R}_+\}$.

Theorem 4.1. All solutions of system (2.2) which initiates in $\mathbb{R}^2_+$ are non-negative and uniformly bounded.

Proof . For any solution $u(t) \in \mathbb{R}_+$ we need to show that it is nonnegative. Assume $u_0 > 0$ and $v_0 > 0$ for $t = 0$ and for all $t \in \mathbb{R}^+$. If $u(t) > 0$ is not true, then there exists t_1 be a real number satisfying $0 \leq t < t_1$, we have $u(t) > 0$ for $0 \leq t < t_1$, $u(t_1) = 0$ and $u(t) < 0$ for $t > t_1$. According to the first equation of (2.2),

$$^cD^\alpha u(t)\Big|_{u(t_1)} = 0.$$

By Lemma 3.3, $u(t_1^+) = 0$, which contradicts our assumption $u(t_1^+) < 0$. Therefore, for all $t \in (0, T]$, we obtain $u(t) \geq 0$. Similarly, $v(t) \geq 0\ \forall\ t \in (0, T]$. Now, we formulate a function $P(t) = u(t) + \frac{v(t)}{\tau}$ to show the boundedness of the solution.

$$\begin{aligned}
^cD^\alpha P(t) &= (r - \gamma_1)u - r\frac{u^2}{K} - \frac{1}{\tau}(d + \gamma_2)v \\
^cD^\alpha P(t) + \eta P(t) &= (r - \gamma_1)u - r\frac{u^2}{K} - \frac{1}{\tau}(d + \gamma_2)v + \eta u + \frac{\eta}{\tau}v \\
&= (r + \eta - \gamma_1)u - r\frac{u^2}{K} + \left(\frac{\eta}{\tau} - \frac{d + \gamma_2}{\tau}\right)v \\
^cD^\alpha P(t) + \eta P(t) &\leq \frac{k(r + \eta - \gamma_1)^2}{4r}.
\end{aligned}$$

By using the Lemma 3.4, we have

$$P(t) \leq \left(P(0) - \frac{k(r + \eta - \gamma_1)^2}{4r}\right) E_\alpha\big[-\delta t^\alpha\big] + \frac{k(r + \eta - \gamma_1)^2}{4r} \to \frac{k(r + \eta - \gamma_1)^2}{4r}\ as\ t\ \to\ \infty.$$

That is to say, every solutions of the system (2.2) initiating in Ω^+ are remains in $\sigma = \{(u, v) \in \Omega^+ : u + \frac{v}{\tau} \leq \frac{k(r+\eta-\gamma_1)^2}{4r} + \epsilon,\ \epsilon > 0\}$. □

5 Existence and uniqueness of the solution

In this section, we investigate the existence of model (2.2) and also which is unique in $\aleph \times (0,T]$, where $\aleph = \{(u,v) \in \mathbb{R}^2 : \max\ \{\ |u|,\ |v|\ \} \leq \kappa\}$.

Let $\mathbb{V} = (u,v)$ and $\overline{\mathbb{V}} = (u_1,v_1)$ be two points in $\aleph$. Consider the mapping $\chi : \aleph \to \mathbb{R}^2$ defined by $\chi(\mathbb{V}) = (\chi_1(\mathbb{V}), \chi_2(\mathbb{V}))$, where

$$\chi_1(\mathbb{V}) = ru\left(1 - \frac{u}{K}\right) - \frac{uv}{a+bu+u^2} - \gamma_1 u,$$
$$\chi_2(\mathbb{V}) = \frac{\tau uv}{a+bu+u^2} - dv - \gamma_2 v.$$

For any $\mathbb{V}, \overline{\mathbb{V}} \in \aleph$ be arbitrary, it follows that

$$\begin{aligned}
\|\chi(\mathbb{V}) - \chi(\overline{\mathbb{V}})\| &= |\chi_1(\mathbb{V}) - \chi_1(\overline{\mathbb{V}})| + |\chi_2(\mathbb{V}) - \chi_2(\overline{\mathbb{V}})| \\
&= \left|ru\Big(1-\frac{u}{K}\Big) - \frac{uv}{a+bu+u^2} - \gamma_1 u - ru_1\Big(1-\frac{u_1}{K}\Big) + \frac{u_1v_1}{a+bu_1+u_1^2} + \gamma_1 u_1\right| \\
&\quad + \left|\frac{\tau uv}{a+bu+u^2} - dv - \gamma_2 v - \frac{\tau u_1 v_1}{a+bu_1+u_1^2} + dv_1 + \gamma_2 v_1\right| \\
&\leq \left|r(u-u_1) - \frac{r}{K}(u^2-u_1^2) - \left[\frac{uv(a+bu_1+u_1^2) - u_1v_1(a+bu+u^2)}{(a+bu+u^2)(a+bu_1+u_1^2)}\right] - \gamma_1(u-u_1)\right| \\
&\quad + \left[\frac{\tau uv(a+bu_1+u_1^2) - \tau u_1v_1(a+bu+u^2)}{(a+bu+u^2)(a+bu_1+u_1^2)}] - d(v-v_1) - \gamma_2(v-v_1)\right] \\
&\leq \left[r - \frac{2\kappa r}{K} + \gamma_1 + \frac{\kappa(1+\tau)(a+\kappa^2)}{(a+b\kappa+\kappa^2)^2} + d + \gamma_2\right]|u-u_1| \\
&\quad + \left[\frac{\kappa(1+\tau)(a+b\kappa+\kappa^2)}{(a+b\kappa+\kappa^2)^2}\right]|v-v_1| \\
&= \left[r + \frac{2\kappa r}{K} + \kappa(1+\tau)(a+\kappa^2) + \gamma_1\right]|u-u_1| \\
&\quad + \left[\kappa(1+\tau)(a+b\kappa+\kappa^2) + d + \gamma_2\right]|v-v_1| \\
&\leq \mathbb{K}\|\mathbb{V} - \overline{\mathbb{V}}\|.
\end{aligned}$$

where $\mathbb{K} = max\ \{r + \frac{2\kappa r}{K} + \kappa(1+\tau)(a+\kappa^2) + \gamma_1,\ \kappa(1+\tau)(a+b\kappa+\kappa^2) + d + \gamma_2\}$. Thus, $\chi(\mathbb{V})$ satisfies Lipschitz condition, it follows from Lemma 3.5 with initial condition $\mathbb{V}_0 = (u_0, v_0)$ has a unique solution $\mathbb{V}(t)$. As a result, the following theorem is established:

Theorem 5.1. The fractional-order model (2.2) with any nonnegative initial value (u_0, v_0) has a unique solution $\mathbb{V}(t) \in \aleph\ \ \forall\ t > 0$.

6 Local stability of equilibria

By solving the following equation we can find the equilibrium points of the system (2.2)

$$\begin{cases} {}^cD^\alpha u(t) = 0, \\ {}^cD^\alpha v(t) = 0, \end{cases}$$

that is,

$$ru(1 - \frac{u}{K}) - \frac{uv}{a+bu+u^2} - \gamma_1 u = 0,$$
$$\frac{\tau uv}{a+bu+u^2} - dv - \gamma_2 v = 0.$$

The equilibrium points are as follows:

(i) $E_0(0,0)$,

(ii) $E_1\left(K\left(1-\frac{\gamma_1}{r}\right),0\right)$ exist if $\gamma_1 < r$,

(iii) $E_i^*(u_i^*, v_i^*)$ (i= 1, 2), where (u_i^*, v_i^*) is given by the equation

$$(d+\gamma_2)u^2 + [(d+\gamma_2)b - \tau]u + (d+\gamma_2)a = 0. \tag{6.1}$$

$$\begin{aligned} u_i^* &= \frac{\tau-(d+\gamma_2)b \pm \sqrt{[(d+\gamma_2)b-\tau]^2 - 4(d+\gamma_2)^2 a}}{2(d+\gamma_2)} \quad (i=1,2), \\ v_i^* &= (a+bu_i^* + {u_i^*}^2)\left[r\left(1-\frac{u_i^*}{K}\right)-\gamma_1\right] \quad (i=1,2). \end{aligned} \tag{6.2}$$

E_i^* (i= 1, 2) exists under the following conditions,

$$\xi = \frac{\tau}{2\sqrt{a}+b} - d, \quad \rho_1 = r\left(1-\frac{u_1^*}{K}\right), \quad \rho_2 = r\left(1-\frac{u_2^*}{K}\right). \tag{6.3}$$

where $\rho_2 < \rho_1$.

Now, we have

(i) If $0 < \gamma_2 < \xi$ then, we have
   (a) two coexisting equilibrium points for $0 < \gamma_1 < \rho_2$,
   (b) unique coexisting equilibrium point for $\rho_2 \le \gamma_1 < \rho_1$,
   (c) no coexisting equilibrium point for $\gamma_1 \ge \rho_1$.

(ii) For $\gamma_2 > \xi$, we have no coexisting equilibrium point.

(iii) For $\gamma_2 = \xi$, we have unique coexisting equilibrium point if $0 < \gamma_1 < \xi = r\left(1-\frac{\sqrt{a}}{K}\right)$, where the special coexisting equilibrium point is $X^*\left(\sqrt{a}, (2a+b\sqrt{a})\left[r\left(1-\frac{\sqrt{a}}{K}\right)-\gamma_1\right]\right)$.

Next, we check the local stability behavior of all feasible equilibrium points by using the standard linearization method. The jacobian matrix of the model (2.2) is provided by

$$J(u,v) = \begin{pmatrix} r\left(1-\frac{2u}{K}\right) + \frac{(u^2-a)v}{(a+bu+u^2)^2} - \gamma_1 & -\frac{u}{(a+bu+u^2)} \\ \frac{(a-u^2)v\tau}{(a+bu+u^2)^2} & \frac{u\tau}{(a+bu+u^2)} - d - \gamma_2 \end{pmatrix}. \tag{6.4}$$

Theorem 6.1. The trivial equilibrium point E_0 of system (2.2) is locally asymptotically stable if $r < \gamma_1$.

Proof . For $E_0 = (0,0)$, the jacobian matrix is given by

$$J(E_0) = \begin{pmatrix} r-\gamma_1 & 0 \\ 0 & -d-\gamma_2 \end{pmatrix}.$$

Hence, we obtain $\lambda_1 = r - \gamma_1$ and $\lambda_2 = -m - \gamma_2$. It is clear that $|arg(\lambda_1)| = \pi > \alpha\frac{\pi}{2}$ for $r < \gamma_1$, otherwise we get $|arg(\lambda_1)| = 0 < \alpha\frac{\pi}{2}$ and also $|arg(\lambda_2)| = \pi$, which leads to $|arg(\lambda_2)| > \alpha\frac{\pi}{2}$ for $0 < \alpha < 1$. Therefore, we conclude that E_0 is locally asymptotically stable by Theorem 3.6. □

Theorem 6.2. The predator-free equilibrium point E_1 of system (2.2) is locally asymptotically stable if $r > \gamma_1$ and $\zeta < \gamma_2$, where $\zeta = \frac{\tau K r(r-\gamma_1)}{ar^2+bkr(r-\gamma_1)+k^2(r-\gamma_1)^2} - m$.

Proof . For E_1, the jacobian matrix is as follows:

$$J(E_1) = \begin{pmatrix} -(r-\gamma_1) & \frac{Kr(\gamma_1-r)}{ar^2+bKr(r-\gamma_1)+k^2(r-\gamma_1)^2} \\ 0 & \frac{\tau K r(r-\gamma_1)}{ar^2+bKr(r-\gamma_1)+K^2(r-\gamma_1)^2} - d - \gamma_2 \end{pmatrix}.$$

Hence, the eigenvalues are $\lambda_1 = -(r-\gamma_1)$ and $\lambda_2 = \frac{\tau K r(r-\gamma_1)}{ar^2+bKr(r-\gamma_1)+K^2(r-\gamma_1)^2} - m - \gamma_2$. If $r > \gamma_1$, we have $|arg(\lambda_1)| = \pi > \alpha\frac{\pi}{2}$, otherwise $|arg(\lambda_1)| = 0 < \alpha\frac{\pi}{2}$ and if $\zeta < \gamma_2$, then $|arg(\lambda_2)| = \pi > \alpha\frac{\pi}{2}$, otherwise $|arg(\lambda_2)| = 0 < \alpha\frac{\pi}{2}$ for $0 < \alpha < 1$. Therefore, by Theorem 3.6, we conclude that E_1 is locally asymptotically stable. ☐

Theorem 6.3. The coexisting equilibrium point $E_1^*(u_1^*, v_1^*)$ of system (2.2) is locally asymptotically stable if any one of the following conditions holds:

(i) if $A_{11} < 0$ and $A_{11}^2 \geq 4A_{12}A_{21}$.

(ii) if $A_{11} > 0$ and $A_{11}^2 < 4A_{12}A_{21}$.

(iii) if $A_{11} < 0$ and $A_{11}^2 < 4A_{12}A_{21}$.

(iv) if $A_{11} = 0$ and $A_{11}^2 < 4A_{12}A_{21}$.

Proof . For $E_i^*(u_i^*, v_i^*)(i = 1, 2)$, the jacobian matrix is as follows:

$$J(E_i^*) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \text{ (i= 1, 2).}$$

$$\begin{aligned}
A_{11} &= r\Big(1 - \frac{2u_i^*}{K}\Big) + \frac{({u_i^*}^2 - a)(d+\gamma_2)\Big[u\Big(1 - \frac{u_i^*}{K}\Big) - \gamma_1\Big]}{\tau u_i^*} - \gamma_1, \\
A_{12} &= \frac{\gamma_2 + m}{\tau}, \\
A_{21} &= \frac{(a - {u_i^*}^2)(d+\gamma_2)\Big[r\big(1 - \frac{u_i^*}{K}\big) - \gamma_1\Big]}{u_i^*}, \\
A_{22} &= 0.
\end{aligned}$$

The determinant (Det) and trace (Tr) of $J(E_i^*)$ $(i = 1, 2)$ are

$$\begin{aligned}
Det &= A_{12}A_{21} = \frac{(a - {u_i^*}^2)(d+\gamma_2)^3 v_i^*}{\tau^2 {u_i^*}^2}. \\
Tr &= A_{11} = r(1 - \frac{2u_i^*}{K}) + \frac{({u_i^*}^2 - a)v_i^*}{(a + bu_i^* + {u_i^*}^2)^2} - \gamma_1.
\end{aligned} \tag{6.5}$$

Therefore, the eigen values are

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}\Big(A_{11} + \sqrt{A_{11}^2 - 4A_{12}A_{21}}\Big), \\
\lambda_2 &= \frac{1}{2}\Big(A_{11} - \sqrt{A_{11}^2 - 4A_{12}A_{21}}\Big).
\end{aligned} \tag{6.6}$$

Case (i) If $A_{11} < 0$ and $A_{11}^2 \geq 4A_{12}A_{21}$, then both the eigen values λ_1, λ_2 are negative. Therefore, we obtain the condition $|arg(\lambda_1)| = \pi > \alpha\frac{\pi}{2}$ and $|arg(\lambda_2)| = \pi > \alpha\frac{\pi}{2}$. Hence, by Theorem 3.6, $E_1^*(u_1^*, v_1^*)$ is asymptotically stable.

Case (ii) If $A_{11} > 0$ and $A_{11}^2 < 4A_{12}A_{21}$, then we have both λ_1 and λ_2 are pair of complex conjugate roots:

$$\lambda_{1,2} = \frac{1}{2}\left(A_{11} \pm i\sqrt{4A_{12}A_{21} - A_{11}^2}\right) \text{ where } (i = \sqrt{-1}).$$

We have $|arg(\lambda_{1,2})| = tan^{-1}\left|\frac{\sqrt{4A_{12}A_{21}-A_{11}^2}}{A_{11}}\right|$. Therefore, $E_1^*(u_1^*, v_1^*)$ is asymptotically stable, if $tan^{-1}\left|\frac{\sqrt{4A_{12}A_{21}-A_{11}^2}}{A_{11}}\right| > \frac{\alpha\pi}{2}$ in the interval $0 < \alpha < \frac{2}{\pi}tan^{-1}\left|\frac{\sqrt{4A_{12}A_{21}-A_{11}^2}}{A_{11}}\right|$.

Case (iii) If $A_{11} < 0$ and $A_{11}^2 < 4A_{12}A_{21}$, then $|arg(\lambda_{1,2})| = \pi - tan^{-1}\left|\frac{\sqrt{4A_{12}A_{21}-A_{11}^2}}{A_{11}}\right|$. We obtain the condition $|arg(\lambda_{1,2})| > \alpha\frac{\pi}{2}$. Then $E_1^*(u_1^*, v_1^*)$ is asymptotically stable if $\pi - tan^{-1}\left|\frac{\sqrt{4A_{12}A_{21}-A_{11}^2}}{A_{11}}\right| > \frac{\alpha\pi}{2}$ in the interval $0 < \alpha < 2 - \frac{2}{\pi}tan^{-1}\left|\frac{\sqrt{4A_{12}A_{21}-A_{11}^2}}{A_{11}}\right|$.

Case (iv) If $A_{11} = 0$ and $A_{11}^2 < 4A_{12}A_{21}$, then $|arg(\lambda_{1,2})| = \frac{\pi}{2}$. We obtain the condition $|arg(\lambda_{1,2})| > \alpha\frac{\pi}{2}$. Hence, by Theorem 3.6, the equilibrium point $E_1^*(u_1^*, v_1^*)$ is asymptotically stable. □

Theorem 6.4. The coexisting equilibrium point $E_1^*(u_1^*, v_1^*)$ is unstable if $A_{11} \geq 0$ and $A_{11}^2 \geq 4A_{12}A_{21}$.

Proof . If $A_{11} \geq 0$ and $A_{11}^2 \geq 4A_{12}A_{21}$, then one of the eigenvalues $\lambda_1 > 0$ or $\lambda_2 > 0$. Therefore, we obtain the condition $|arg(\lambda_{1,2})| < \alpha\frac{\pi}{2}$. Hence, by Theorem 3.6, the $E_1^*(u_1^*, v_1^*)$ is unstable. □

7 Global stability of equilibria

We establish the global behavior of the system (2.2) at the equilibrium points E_1 and $E_1^*(u_1^*, v_1^*)$ in this section.

Lemma 7.1. [33] Let $u(t)$ be a continuously differentiable function. Then, for any time instant $t > 0$,

$$ {}^cD^\alpha\left(u(t) - \omega - \omega\log\frac{u(t)}{\omega}\right) \leq \left(1 - \frac{\omega}{u(t)}\right) {}^cD^\alpha u(t),\ u(t),\ \omega \in \mathbb{R}^+, \tag{7.1}$$

where $o < \alpha < 1$.

Theorem 7.2. The predator-free equilibrium point E_1 is globally asymptotically stable if $2r - \gamma_1 < 0$ and $K < \frac{1}{\tau}(d + \gamma_2)$.

Proof . The positive definite Lyapunov function is defined as follows:

$$\mathbb{W}(u, v) = u - K\left(1 + \log\frac{u}{K}\right) + \frac{v}{\tau}. \tag{7.2}$$

By using the Lemma 7.1, we get

$$\begin{aligned}
{}^cD^\alpha\mathbb{W}(u, v) &\leq \left(1 - \frac{u}{K}\right) {}^cD^\alpha u(t) + \frac{1}{\tau}\ {}^cD^\alpha v(t)\\
&= (u - K)\left[-\frac{(u-K)r}{K} - \frac{v}{a + bu + u^2} - \gamma_1\right] + \frac{uv}{a + bu + u^2} - \frac{d}{\tau}y - \frac{\gamma_2}{\tau v}\\
&\leq \frac{-ru^2}{K} - rK + 2ru + \frac{vK}{a + bu + u^2} - \gamma_1 u + \gamma_1 K - \frac{d}{\tau}v - \frac{\gamma_2}{\tau v}\\
&= \frac{-ru^2}{K} + (2r - \gamma_1)u + \left(K - \frac{1}{\tau}(d + \gamma_2)\tau\right)v + (\gamma_1 - r)K\\
{}^cD^\alpha\mathbb{W}(u, v) &\leq 0.
\end{aligned}$$

Since $\gamma_1 - r < 0$ for the existence of E_1. ${}^cD^\alpha\mathbb{W}(u, v) \leq 0$ if $2r - \gamma_1 < 0$ and $K < \frac{1}{\tau}(d + \gamma_2)$. Hence E_1 is globally asymptotically stable. □

Theorem 7.3. The coexisting equilibrium point $E_1^*(u_1^*, v_1^*)$ is globally asymptotically stable if $\frac{v^*(b+(u+u^*))}{(a+bu+u^2)(a+bu^*+u^{*2})} < \frac{r}{K}$.

Proof . First, we define the Lyapunov function as follows:

$$\mathbb{W}(u,v) = u - u^*\Big(1+\log\frac{u}{u^*}\Big) + \frac{1}{\tau}\Big(v - v^*\Big(1+\log\frac{v}{v^*}\Big)\Big). \tag{7.3}$$

Using Lemma 7.1, we can show that

$$\begin{aligned}
{}^cD^\alpha\mathbb{W}(u,v) &\le \left(\frac{u-u^*}{u}\right){}^cD^\alpha u(t) + \frac{1}{\tau}\left(\frac{v-v^*}{v}\right){}^cD^\alpha v(t)\\
&= \left(\frac{u-u^*}{u}\right)\left[ru\Big(1-\frac{u}{K}\Big) - \frac{uv}{a+bu+u^2} - \gamma_1 u\right] + \frac{1}{\tau}\left(\frac{v-v^*}{v}\right)\left[\tau\frac{uv}{a+bu+u^2} - dv - \gamma_2 v\right]\\
&\le (u-u^*)\left[-\frac{r}{K}(u-u^*) - \left[\frac{(va+bvu^*+vu^{*2})-(av^*+buv^*+u^2v^*)}{(a+bu+u^2)(a+bu^*+u^{*2})}\right]\right]\\
&\quad + (v-v^*)\left[\frac{(au+buu^*+uu^{*2})-(au^*+buu^*+u^2u^*)}{(a+bu+u^2)(a+bu^*+u^{*2})}\right]\\
&\le \frac{r}{K}(u-u^*)^2 - \frac{bu^*(u-u^*)(v-v^*)}{(a+bu+u^2)(a+bu^*+u^{*2})} + \frac{bv^*(u-u^*)^2}{(a+bu+u^2)(a+bu^*+u^{*2})}\\
&\quad + \frac{v^*(u-u^*)^2(v-v^*)}{(a+bu+u^2)(a+bu^*+u^{*2})} - \frac{u^*(u+u^*)(u-u^*)(v-v^*)}{(a+bu+u^2)(a+bu^*+u^{*2})}\\
&\le \left[\frac{v^*(b+(u+u^*))}{(a+bu+u^2)(a+bu^*+u^{*2})} - \frac{r}{K}\right](u-u^*)^2\\
&\quad - \left[\frac{u^*(b+(u+u^*))}{(a+bu+u^2)(a+bu^*+u^{*2})}\right](u-u^*)(v-v^*)\\
{}^cD^\alpha\mathbb{W}(u,v) &\le 0.
\end{aligned}$$

Therefore, ${}^cD^\alpha\mathbb{W}(u,v) \le 0$ if $\frac{v^*(b+(u+u^*))}{(a+bu+u^2)(a+bu^*+u^{*2})} < \frac{r}{K}$, Hence $E_1^*(u_1^*, v_1^*)$ is globally asymptotically stable. □

8 Numerical Simulations

The numerical simulations for the system (2.2) are performed by generalized Adams-Bashforth-Moulton Predictor Corrector method [10]. We provide some examples to verify the analytical results of our formulated model through MATLAB.

Example 1. We take $r = 2,\ K = 45,\ d = 0.1,\ \tau = 4,\ a = 0.95,\ b = 0.9,\ \gamma_1 = 1.93,\ \gamma_2 = 1$ and the initial time $u_0 = 0.5,\ v_0 = 0.9$ with step size 2^{-6}. The dynamical behavior of the positive interior equilibrium point E_1^* is presented in Figure 1, 2, 3. When the value of α is decreasing, limit cycle disappears and coexisting equilibrium point becomes stable.

Example 2. In Figure 4, We verified the global stability of the positive equilibrium point E_1^* with the parameters set $r = 2,\ K = 20,\ d = 0.1,\ \tau = 4,\ a = 0.9,\ b = 0.9,\ \gamma_1 = 1.9,\ \gamma_2 = 1$. Here, we have taken the value $\alpha = 0.90$. Therefore, Theorem 7.3 is satisfied with this set of parameters.

9 Conclusion

A Caputo fractional-order predator-prey system with Holling response function of type IV and harvesting on species was investigated. Also, we derived the sufficient condition for existence and uniqueness, non-negativity and boundedness of the solution. This model (2.2) has three positive equilibrium points and we have investigated the local behavior of all possible positive equilibrium points. In further, the global stability of the predator-free equilibrium

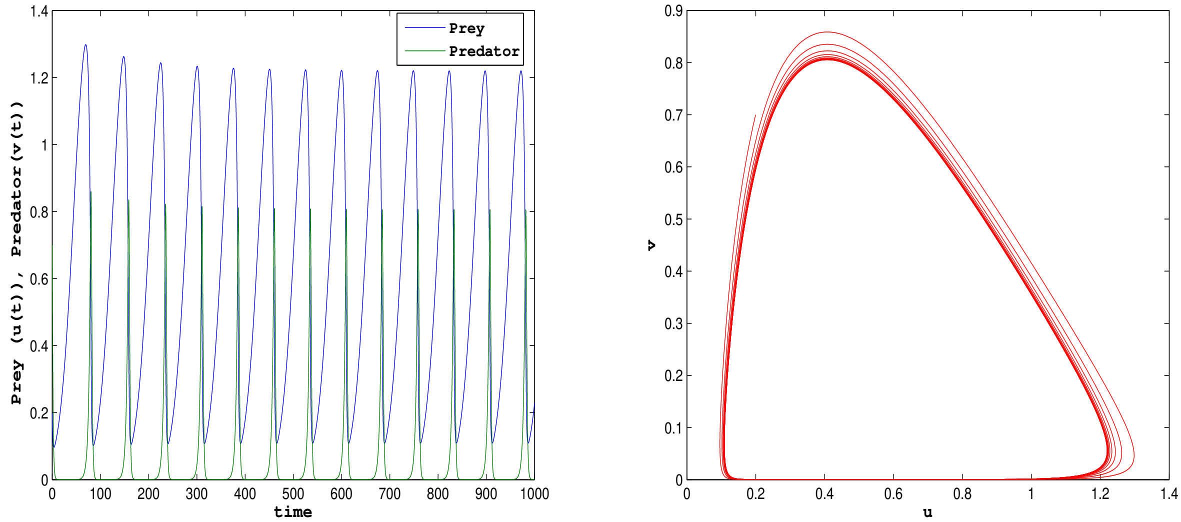

Figure 1: Numerical simulations showing the trajectory and phase portrait of the system (2.2) with value $\alpha = 1$ around the equilibrium point E_1^*.

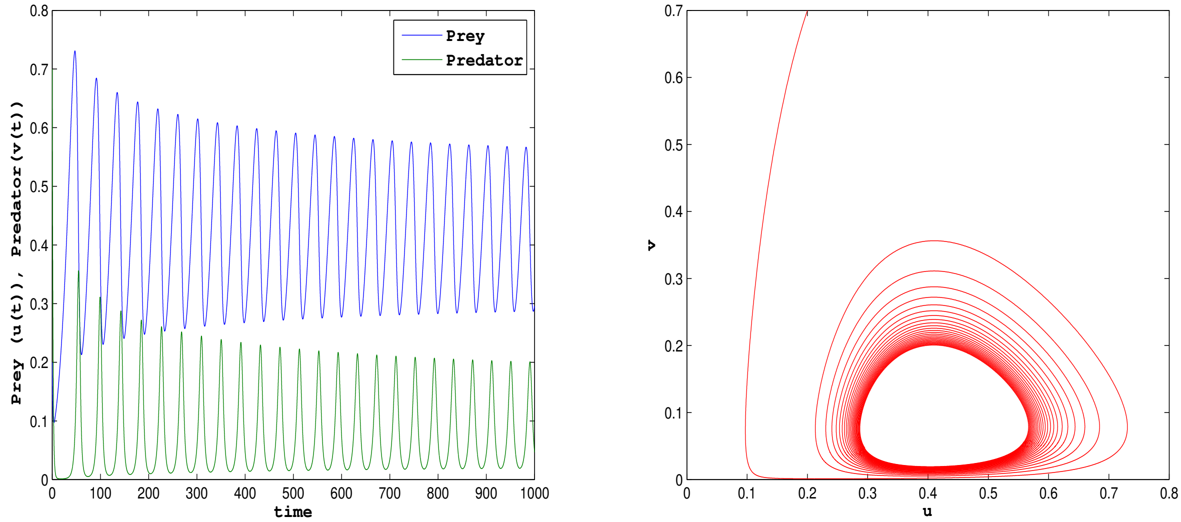

Figure 2: Numerical simulations showing the trajectory and phase portrait of the system (2.2) with value $\alpha = 0.99$ around the equilibrium point E_1^*.

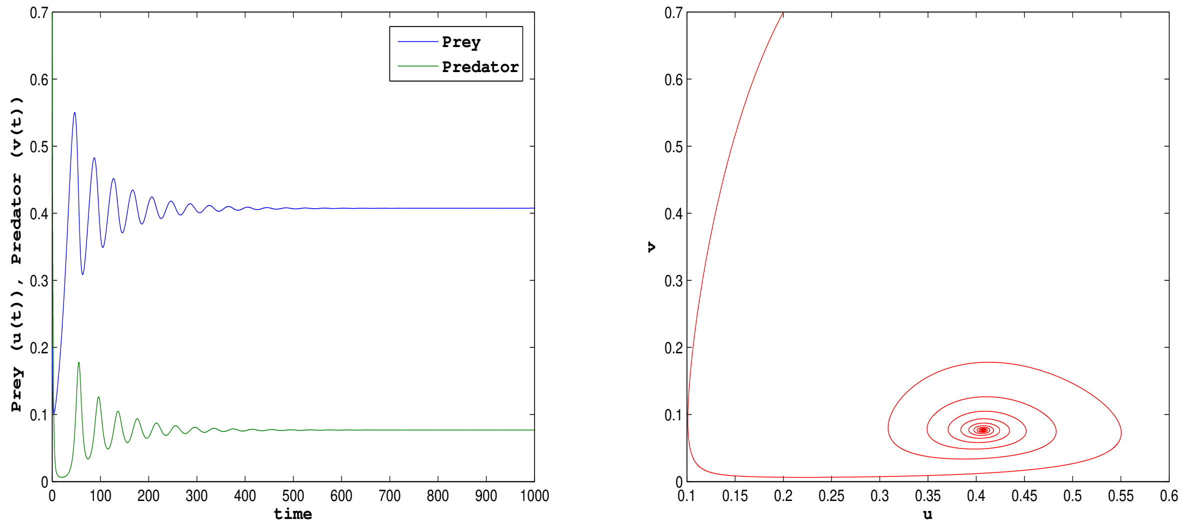

Figure 3: Numerical simulations showing the trajectory and phase portrait of the system (2.2) with value $\alpha = 0.95$ around the equilibrium point E_1^*.

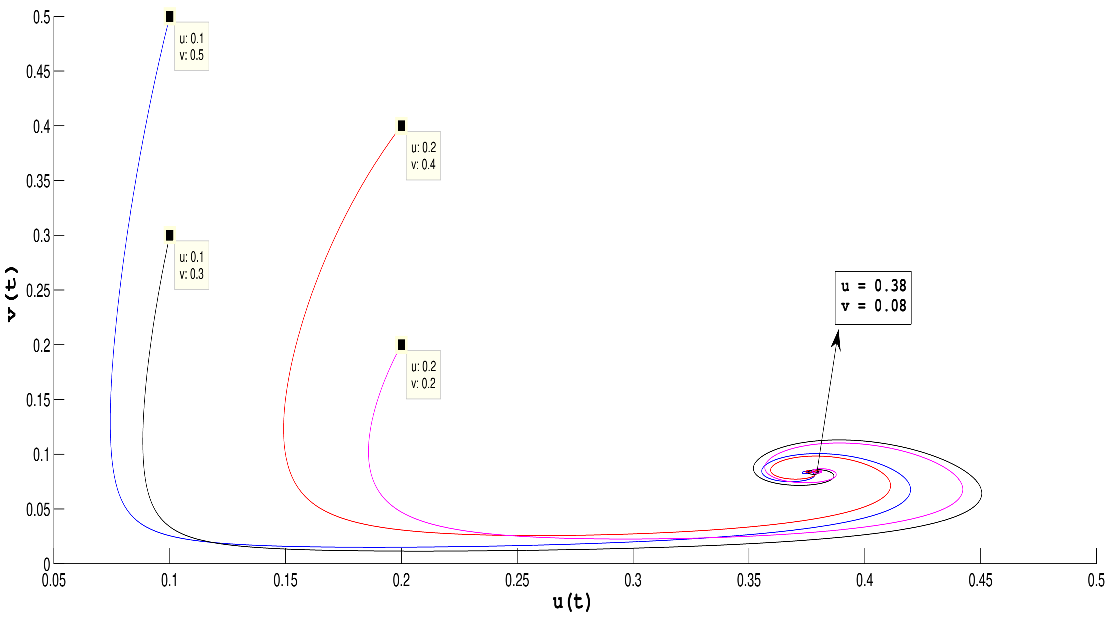

Figure 4: Global behavior of the system (2.2) at E_1^*. From the figure we observed that the trajectories with different initial conditions converge to the $E_1^*(0.38, 0.08)$ (shown in different colors) and is globally asymptotically stable.

point E_1 and the coexisting equilibrium point E_1^* were derived by formulating certain Lyapunov functions. Finally, numerical simulations have been used to verify these theoretical findings. From the numerical simulations we observed that, if we decrease the value α, the coexisting equilibrium point E_1^* gradually changes from unstable to locally asymptotically stable and for the integer-order case ($\alpha = 1$) the system is unstable which is shown in the Figure 1. Several prey species have defensive capabilities in real life due to the effect of the type IV functional response. As a result, the number of prey species rises for a long time, limiting the growth of predators. At last, the predator approaches extermination when the prey population stabilize at the equilibrium point.

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