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Operational matrix and their applications for solving time-varying delay systems

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Abstract

The purpose of this paper is to provide a numerical method based on the Hat basis functions as well as the operational matrices for finding the approximate solution of time-invariant delay systems. From this point of view, the operational matrices of integration, delay, product, and dual of Hat basic functions are derived, which are utilized to practically reduce the time-varying delay systems solution to the simplest system of algebraic equations. The numerical results have been compared and tabulated with previous results. The simplicity, clarity and effectiveness of the proposed method are shown through three examples.

Keywords: Hat basis functions, Delay operational matrix, Delay systems
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1 Introduction

The theory and applications of delay systems are encountered in numerous branches of technology and science; additionally, most of the practical problems related to the industrial units have inherent time-varying delays, and knowing about those delays is important for a number of reasons. The analysis and monitoring of the delay value can easily lead to the development of efficient procedures [2, 3, 8].

Engineers and mathematicians studied time delay dynamics primarily in relation to linear dynamical systems and the principles of infinite-dimensional dynamical systems. In the past few decades, interdisciplinary studies using nonlinear and chaotic dynamical systems have become popular in the study of complex dynamical scenarios. There has been a great deal of theoretical detail explained, but there is still substantial room for improvement, particularly with regard to the delays with time extensions over spatial space, and delay systems based on state variables [16, 21].

Over the last decade, time delay dynamics have shown increasing interest across a number of disciplines, including robotics, electronics, hydraulic and pneumatic networks, chemical processes, long transmission lines, etc [1, 16, 22].

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Numerous operational matrices have defined through orthogonal polynomials and its characteristics for integration. For example, Haar wavelet technique [14], CAS Wavelets [25], Triangular functions [7, 24, 28], Block pulse [20, 6], Boubaker functions [13, 15]. Recently, orthogonal functions have been used to solve delay systems more effectively. The most frequent and reliable applications of orthogonal functions can be classified in a vast number of ways. For instance, Imran Aziz extended the via hybrid functions for the numerical solution of optimal control for delays time-varying systems in [30], Safaie in [27] utilized Bernstein polynomials to solve a multi-dimensional delay fractional optimal control, problem, and Babolian in [23] used a Bernoulli.wavelet. Bhrawy in [4], proposed a Legendre operational technique for addressing fractional optimal control problems with delays. Hat functions (HFs) were formerly a great instrument for solving a broad range of problems in applied science and mathematics., M. H. Heydari et al. used HFs to solve nonlinear stochastic Ito-Volterra integral equations [9, 10]. F. Mirzaee solved space-time integral equations using 2DHFs [19]. M. P. Tripathi et al. solved fractional differential equations using HFs [29].

This paper propose a numerical method based on the HFs to approximate the solution of the linear time-varying delay system

$$X^{'}(t) = B(t)X(t) + G(t)X(t-\tau) + F(t)U(t), \tag{1.1}$$

$$X(0) = X_0, \quad X(t) = \Psi(t), \quad -\tau \leq t < 0, \tag{1.2}$$

where X is a constant specified vector, $B(t)$, $G(t)$, $F(t)$, $X(t) \in \mathbb{R}^l$ and $U(t) \in \mathbb{R}^q$ defined as a matrix and $\Psi(t)$ is an arbitrary known function.

The rest of the paper is organized: In Section 2, we give a brief review of HFs and their properties that are used throughout the paper, followed by constructing the delay operation matrix based on HFs. Of functions using HFs has been discussed and presented in Section 3. In general, we use the Hat functions in order to approximate the system dynamics in this section. Afterward, Section 4 examines the presented approach by solving three systems numerically and to better interpretation the efficiency and accuracy of the proposed scheme, the results have been tabled and compared with previous works. Finally, in Section 5 we state a brief outline of the results.

2 Brief review of HFs and their properties

In this section, we represent the HFs as great mathematical tools in order to approximate the solution of Eqs. (1.1)-(1.2). We usually define a set of HFs on $[0,1]$ such as:

$$\phi_0(t) = \begin{cases} \dfrac{h-t}{h}, & 0 \leq t < h, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_i(t) = \begin{cases} \frac{t-(i-1)h}{h}, & (i-1)h \leq t < ih, \\ \frac{(i+1)h-t}{h}, & ih \leq t < (i+1)h, \quad i = 1, 2, \cdots, n-1. \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_n(t) = \begin{cases} \frac{t-(T-h)}{h}, & T-h \leq t \leq T, \\ 0, & \text{otherwise}, \end{cases}$$

where $h = \frac{1}{n}$ and n should be an arbitrary positive integer. Actually, n is the number of subdivided equidistant of the interval $[0,1]$ Based on the properties of HFs, we have

$$\phi_i(jh) = \delta_{ij}, \tag{2.1}$$

where δ is the Kronecker delta function. Also

$$\phi_i(t)\phi_j(t) = 0, \quad \mid i-j \mid \geq 2, \tag{2.2}$$

and

$$\sum_{i=0}^{n} \phi_i(t) = 1. \tag{2.3}$$

We can expand an vector form of an arbitrary function $f \in C^2([0,1])$ as:

$$f(t) \simeq F^T \Phi(t) = \Phi^T(t) F, \tag{2.4}$$

where $\Phi(t) = [\phi_0(t), \phi_1(t), \cdots, \phi_n(t)]^T$, $F = [f_0, f_1, \cdots, f_n]^T$, and

$$f_i = f(ih), \quad \forall i = 0, 1, \cdots, n. \tag{2.5}$$

Clearly, from relation (2.2), we have:

$$\phi(t)\phi^T(t) = \begin{pmatrix} \phi_0^2(t) & \phi_0(t)\phi_1(t) & & & \\ \phi_0(t)\phi_1(t) & \phi_1^2(t) & \phi_0(t)\phi_1(t) & & \\ & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \phi_{n-1}(t)\phi_n(t) \\ & & & \phi_{n-1}(t)\phi_n(t) & \phi_n^2(t) \end{pmatrix},$$

and

$$\int_0^t \Phi(t)d\tau \simeq E\Phi(t), \tag{2.6}$$

where E is defined as $((n+1) \times (n+1))$ matrix with the following form

$$E = \frac{h}{6}\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 1 & 2 & \cdot & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

By considering (2.1) and developing entries of $\Phi(t)\Phi^T(t)$ by HFs, we can write:

$$\Phi(t)\Phi^T(t) \simeq diag(\Phi). \tag{2.7}$$

Now assume that Λ is an (n+1)-vector. By applying (2.7), we have:

$$\Phi(t)\Phi^T(t)\Lambda \simeq \tilde{\Lambda}\Phi(t), \tag{2.8}$$

where $\tilde{\Lambda} = diag(\Lambda)$ represents an diagonal matrix of $(n+1) \times (n+1)$ order. Also, if A would be a $(n+1) \times (n+1)$ matrix we have

$$\Phi^T(t)A\Phi(t) \simeq \Phi^T(t)\hat{A}, \tag{2.9}$$

where $\hat{A}$ represent an $(n+1)$-vector such that its elements precisely the same as the diagonal of matrix A.

2.1 Delay operational matrix of HFs

In this section, we deal with the construction of the delay operational matrix based on HFs. Clearly, the delay functions $\Phi(t-\tau)$ derive by shifting the function $\Phi(t)$. We have the general expression by

$$\Phi(t-\tau) = \Upsilon\Phi(t), \quad t > \tau, \quad 0 \le t \le 1, \tag{2.10}$$

where Υ represents the delay operational matrix of HFs. If τ is as $\tau = kh = \frac{k}{n}$, where $1 \le k < m-1$, one could write that

$$\begin{aligned} \phi_0(t-\tau) = \phi_0(t-kh) &= \begin{cases} \dfrac{h-t-kh}{h}, & 0 \le t-kh < h, \\ 0, & \text{otherwise}, \end{cases} \\ &= \begin{cases} \dfrac{h(1+kh)-t}{h}, & kh \le t < h(1+k), \\ 0, & \text{otherwise}, \end{cases} \\ &= \phi_k(t), \end{aligned}$$

$$\begin{aligned}
\phi_i(t-\tau)=\phi_i(t-kh) &= \begin{cases} \frac{t-kh-(i-1)h}{h}, & (i-1)h \le t-kh < ih, \\ \frac{(i+1)h-t-kh}{h}, & ih \le t-kh < (i+1)h, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} \frac{t-(k+i-1)h}{h}, & (i+k-1)h \le t < (i+k)h, \\ \frac{(i-k+1)h-t}{h}, & (i+k+1)h \le t < (i+k)h, \\ 0, & \text{otherwise}, \end{cases} \\
&= \phi_{i+k}(t),
\end{aligned}$$

where $i=1,2,\cdots,n-1$.

$$\begin{aligned}
\phi_n(t-\tau)=\phi_n(t-kh) &= \begin{cases} \frac{t-kh-(1-h)}{h}, & T-h \le t-kh < T, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} \frac{t+h(1-k)-1}{h}, & T-h(1-k) \le t < T+kh, \\ 0, & \text{otherwise}, \end{cases} \\
&= \phi_{n+k}(t),
\end{aligned}$$

where the first element 1 in the first row is in the $(k+1)th$ column. While an integer N could be choose for other states of τ by the following rule [17],

$$N=\begin{cases} \frac{1}{\tau}, & \frac{1}{\tau} \in Z, \\ [\frac{1}{\tau}]+1, & otherwise, \end{cases}$$

one has the various intervals $[0,\tau],[0,2\tau],...,[(N-1)\tau,N\tau]$, where $N\tau \ge 1$ and n could be selected as $n=kN, k=1,2,...$ Thus

$$\tau = kh = k\frac{N\tau}{n}. \tag{2.11}$$

So, Υ could be represented by the following form

$$\Upsilon = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{M\times M}.$$

3 Construction of approximation using HFs

In this section, we use the HFs to find the approximate solution of (1.1)-(1.2) as follows.

Let

$$\begin{aligned} X(t) &= [x_1(t),x_2(t),\cdots,x_l(t)]^T, \\ U(t) &= [u_1(t),u_2(t),\cdots,u_q(t)]^T, \end{aligned} \tag{3.1}$$

$$\hat{\Phi}_l(t)=I_l\otimes\Phi(t), \quad \hat{\Phi}_q(t)=I_q\otimes\Phi(t),$$

where I_l and I_q are identity matrices of order l and q, respectively. Additionally, the $\otimes$ notation is used to indicate the well -known Kronecker product. $\hat{\Phi}(t)$ and $\hat{\Phi}_q(t)$ are $l2^kM\times l$ and $q^{2^k}M\times q$ matrices, respectively. By using Eq. (2.4), we can write

$$x_i(t)=\Phi^T(t)X_i, \quad u_j(t)=\Phi^T(t)U_j, \quad i=1,2,...,l, \quad j=1,...,q. \tag{3.2}$$

So, by using Eq. (3.1), we get

$$\begin{aligned} X(t) &= \hat{\Phi}_l^T(t)X, && (3.3)\\ U(t) &= \hat{\Phi}_l^T(t)U, && (3.4)\end{aligned}$$

where $X = [X_1^T, X_2^T, \cdots, X_l^T]^T$, and $U = [U_1^T, U_2^T, \cdots, U_q^T]^T$.

In a similar way, we have

$$X(0) = \hat{\Phi}_1^T h, \tag{3.5}$$

$$\psi(t-\tau) = \hat{\Phi}_l^T(t)P, \tag{3.6}$$

where h and P are vectors of order $l2^kM \times 1$ given by

$$h = [h_1^T, h_2^T, \cdots, h_l^T]^T, \quad P = [P_1^T, P_2^T, \cdots, P_l^T]^T,$$

respectively. Now, we can use the HFs to expand $B(t)$, $G(t)$, and $F(t)$ as follows:

$$B(t) = B^T\hat{\Phi}_l(t), \quad G(t) = G^T\hat{\Phi}_l(t) \quad F(t) = F^T\hat{\Phi}_q(t), \tag{3.7}$$

where B^T, G^T and F^T are $l \times l2^kM$, $l \times 2^kM$, $l \times q2^kM$. The $X(t-\tau)$ can be write in terms of HFs such as

$$X(t-\tau) = \begin{cases} \hat{\Phi}_l^T P, & 0 \le t \le \tau, \\ \hat{\Phi}_l^T(t)\hat{\Upsilon}^T X, & \tau \le t \le 1, \end{cases}$$

where

$$\hat{\Upsilon} = I_l \otimes \Upsilon, \tag{3.8}$$

and Υ represent the delay operational matrix in Eq. (2.10). Now, by using (2.8), we can write

$$B(t)X(t) = B^T\hat{\Phi}_l^T(t)\hat{\Phi}_l^T(t)X = \hat{\Phi}_l^T(t)\tilde{B}^T X, \tag{3.9}$$

$$F(t)U(t) = F^T\hat{\Phi}_q^T(t)\hat{\Phi}_q^T(t)X = \hat{\Phi}_q^T(t)\tilde{F}^T U. \tag{3.10}$$

Moreover,

$$\int_0^t \Phi_l^T(s)ds = \left(I_l \otimes \Phi^T(t)\right)\left(I_l \otimes E^T\right) = \hat{\Phi}_l^T(t)\hat{E}^T, \tag{3.11}$$

$$\int_0^t G(s)X(t-\tau)ds = \begin{cases} \hat{\Phi}_l^T(t)P\hat{E}^T\tilde{G}^T, & 0 \le t \le \tau, \\ \hat{\Phi}_l^T(t)PZ\tilde{G}^T + \hat{\Phi}_l^T(t)\tilde{G}^T\hat{\Upsilon}^T\hat{E}^T X, & \tau \le t \le 1, \end{cases} \tag{3.12}$$

where E represent a constant matrix of order $l2^kM \times l2^kM$, which can be obtained as follows:

$$\int_0^\tau \hat{\Phi}_l^T ds = \hat{\Phi}_l^T Z. \tag{3.13}$$

Integrating Eq. (1.1) from zero to t, and applying Eqs. (1.2), (3.1)-(3.12), we conclude that conclude that:

$$\hat{\Phi}_l^T(t)(X-h) = \hat{\Phi}_l^T(t)\left(\hat{E}^T X\tilde{B}^T + \tilde{G}^T\hat{E}^T P + \tilde{G}^T PZ + \tilde{G}^T\hat{\Upsilon}^T X\hat{E}^T + \tilde{F}^T U\hat{E}^T\right), \tag{3.14}$$

using Eq. (3.14) we get

$$X\left(I - \tilde{B}^T\hat{E}^T - \tilde{G}^T\hat{\Upsilon}^T\hat{E}^T\right) = \tilde{G}^T\hat{E}^T P + \tilde{G}^T PZ + \tilde{F}^T U\hat{E}^T + h. \tag{3.15}$$

Clearly, the above equation represents a system of algebraic equations. By solving Eq. (3.15) we can obtain the approximate solution of Eq. (1.1)-(1.2).

4 Numerical examples

In this section, three test examples were used for illustrating the effectiveness of the proposed method. Also, comparing the results with the results of previous works is done in this section.

Example 4.1. In this first example, we will discuss the delay system presented in [17]

$$\begin{aligned} x^{'}(t) &= 4x(t-\frac{1}{4}), \\ x(0) &= 1, \\ x(t) &= 0, \quad \frac{-1}{4} \le t < 0. \end{aligned} \tag{4.1}$$

The exact solution is

$$x(t) = \begin{cases} 1, & 0 \le t < \frac{1}{4}, \\ 1+4(t-\frac{1}{4}), & \frac{1}{4} \le t < \frac{1}{2}, \\ 1+4(t-\frac{1}{4})+8(t-\frac{1}{2})^2, & \frac{1}{2} \le t < \frac{3}{4}, \\ 1+4(t-\frac{1}{4})+8(t-\frac{1}{2})^2+\frac{32}{3}(t-\frac{3}{4})^2, & \frac{3}{4} \le t < 1. \end{cases}$$

Here, we use HFs by selecting $k = 2$ and $M = 4$. Clearly

$$x(t) = X^T \Phi(t). \tag{4.2}$$

Now, we can use the HFs to expand the $x(0)$. So we get:

$$x(0) = h_1^T H_1^T(t). \tag{4.3}$$

From Eqs. (2.10) and (4.2), we have:

$$x(t-\frac{1}{4}) = X_1^T \Upsilon \Phi(t), \quad t > \frac{1}{4}, \tag{4.4}$$

where Υ represents the delay operational matrix and defined as follows:

$$\boldsymbol{\Upsilon} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By use of the integral operator from 0 to t on Eq. (4.1) as well as applying (4.4), we can write

$$X_1^T - h_1^T = 4(X_1^T)\Upsilon E, \tag{4.5}$$

where E is the operational matrix which define the integration. Table 1 displays comparison of the results of proposed method and the method of [11] as well as block-pulse method presented in [17].

Example 4.2. Consider the following delay system with delay in both control and state [11]

$$x^{'}(t) = -x(t) - 2x(t-\frac{1}{4}) + 2u(t-\frac{1}{4}), \quad 0 \le t \le 1, \tag{4.6}$$
$$x(t) = u(t) = 0, \quad for \quad -1/4 \le t \le 0 \tag{4.7}$$
$$u(t) = 1, \quad for \quad t > 0. \tag{4.8}$$

The exact solution is

$$x(t) = \begin{cases} 0, & 1 \le t < \frac{1}{4}, \\ 2-2e^{-(t-\frac{1}{4})}, & \frac{1}{4} \le t < \frac{1}{2}, \\ -2-2e^{-(t-\frac{1}{4})}+(2+4t)e^{-(t-\frac{1}{2})}, & \frac{1}{2} \le t < \frac{3}{4}, \\ 6-2e^{-(t-\frac{1}{4})}+(2+4t)e^{-(t-\frac{1}{2})}-(\frac{17}{4}+2t+4t^2)e^{-(t-\frac{3}{4})}, & \frac{3}{4} \le t < 1. \end{cases}$$

In the following, we find the approximate solution of this example by employing the HFs and selecting $l = 4$. Let

$$x(t) = X^T \Phi(t). \tag{4.9}$$

By expanding $u(t)$ in terms of HFs, we get

$$u(t) = U^T \Phi(t). \tag{4.10}$$

Using Eqs. (2.10), (4.9) and (4.11), we can write

$$x(t - \frac{1}{4}) = X^T \Upsilon \Phi(t), \quad t > \frac{1}{4}, \tag{4.11}$$

$$u(t - \frac{1}{4}) = U^T \Upsilon \Phi(t), \quad t > \frac{1}{4}, \tag{4.12}$$

where Υ represents delay operational matrix. By integrating on the interval $[0, 1]$ from Eq. (4.1) and using the Eqs. (4.7)-(4.12) we have

$$X^T = -X^T E - 2X^T \Upsilon E + 2U^T \Upsilon E, \tag{4.13}$$

where E represents the operational matrix of integration. The results for both approximate and exact solutions of $X(t)$ for case $l = 4$, have displayed in Table 2. Regarding the results, the approximate solution is the precisely similar to the exact solution in the interval $(0, 1/4)$.

Example 4.3. In the third example, we consider the system that represents time-varying delays as follows [18, 11, 12]:

$$\begin{pmatrix} x'(t) \\ x'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -25 & -5t \end{pmatrix} \begin{pmatrix} x_1(t - \frac{1}{4}) \\ x_2(t - \frac{1}{4}) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with the initial condition

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \in [-\frac{1}{4}, 0].$$

The exact solutions represents as follows:

$$x_1(t) = \begin{cases} 0, & 0 \le t < \frac{1}{4}, \\ \frac{1}{32} - \frac{1}{4}t + \frac{1}{2}t^2, & \frac{1}{4} \le t < \frac{1}{2}, \\ \frac{1}{32} - \frac{19}{96}t + \frac{3}{16}t^2 + \frac{5}{8}t^3 - \frac{5}{12}t^4, & \frac{1}{2} \le t < \frac{3}{4}, \\ -\frac{9641}{32768} + \frac{37391}{24576}t - \frac{3183}{1024}t^2 + \frac{7065}{2304}t^3 - \frac{135}{384}t^4 - \frac{85}{96}t^5 + \frac{5}{18}t^6, & \frac{3}{4} \le t < 1, \end{cases}$$

$$x_2(t) = \begin{cases} t, & 0 \le t < \frac{1}{4}, \\ -\frac{5}{3}t^3 + \frac{5}{8}t^2 + t - \frac{5}{384}, & \frac{1}{4} \le t < \frac{1}{2}, \\ \frac{5}{3}t^5 - \frac{75}{32}t^4 - \frac{115}{24}t^3 + \frac{1295}{192}t^2 - \frac{17}{8}t + \frac{775}{1536} & \frac{1}{2} \le t < \frac{3}{4}, \\ -\frac{25}{21}t^7 + \frac{2125}{576}t^6 + \frac{335}{96}t^5 - \frac{55325}{3075}t^4 + \frac{21515}{1536}t^3 - \frac{95755}{49152}t^2 - \frac{1051}{1024}t + \frac{87997}{132120}, & \frac{3}{4} \le t < 1. \end{cases}$$

Tables 3 displays and compare the presented Hat Function method with Shifted Legendre series method in [12].

5 Conclusion

In this paper, the Hfs and their associated operational matrices are presented and used to find the numerical solution of time-varying systems with delays. The operational matrices of integration, delay product and dual are obtained. By the help of the proposed method, the original problem reduced to a linear algebraic equations. Comparison the results of the proposed method and other existing numerical methods was conducted through the three examples. The results indicated that the proposed method can be simple, clear and effective.

Table 1: The absolute errors estimation and their comparison with other methods for Example1.

	$l=4$	$l=2$	$l=2$	
t	$BPFs$ [18]	TFs [11]	HFs	Exact
0.0	1	1	1	1
0.125	1	1	1	1
0.25	1	1	1	1
0.375	1.5	1.5	1.5	1.5
0.5	1.5	2	2	2
0.625	2.75	2.625	2.625	2.625
0.75	2.75	3.5	3.5	3.5
0.875	4.875	4.656	4.676	4.646
1	4.875	6.188	6.148	6.168

Table 2: The absolute errors estimation and their comparison with other methods for Example 2.

		$l=4$	$l=4$	
t	LaP [5]	TFs [11]	HFs	Exact
0.0	0	0	0	0
0.125	0	0	0	0
0.25	0	0	0	0
0.375	0.23486	0.23508	0.23510	0.23501
0.5	0.40827	0.44253	0.44248	0.44240
0.625	0.54855	0.59710	0.59628	0.59666
0.75	0.65916	0.68158	0.68014	0.68094
0.875	0.74333	0.71284	071179	0.71194
1.0	0.80414	0.71615	0.71419	0.71174

Table 3: The absolute errors estimation and their comparison with other methods for Example 3.

	Exact	$l=12$	$l=12$	Exact	$l=12$	$l=12$
t	$x_1(t)$	SLs [12]	HFs	$x_2(t)$	SLs [12]	HFs
0.0	0.000000	0.000000	0.000073	0.000000	0.001169	0.000128
0.10	0.000000	0.000000	0.000014	0.100000	0.100294	0.100075
0.20	0.000000	0.000000	0.000327	0.200000	0.199902	0.200026
0.30	0.001250	0.001400	0.000128	0.298229	0.298229	0.298229
0.40	0.011250	0.011157	0.011210	0.380313	0.380186	0.380108
0.50	0.031250	0.031304	0.031205	0.434896	0.435025	0.434820
0.60	0.061000	0.060991	0.061517	0.448532	0.448483	0.448379
0.70	0.098917	0.098901	0.0989587	0.395846	0.395868	0.395219
0.80	0.142244	0.142266	0.142908	0.254052	0.254038	0.254098
0.90	0.186819	0.186803	0.186821	0.011316	0.011295	0.011360
0.1	0.226030	0.226030	0.226032	-0.322405	-0.322386	-0.322458

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