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On the Fekete-Szegö problem associated with generalized fractional operator

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Abstract

In this paper, the classical Fekete-Szegö problem is studied regarding a class of univalent functions generated using a generalized fractional differential operator. The results presented in the main theorem are new generalizations for well-known results.

Keywords: Analytic functions, fractional operator, univalent functions, normalized functions, Fekete-Szegö problems

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1 Introduction

We denote by A the functions in the unit disc $U = \{z : 0 < |z| < 1\}$ that has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \tag{1.1}$$

Let S denote the class of univalent functions in the unit disk. The Hadamard convolution of the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ is given by:

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For a class S of analytic functions, Fekete and Szegö [5] obtained sharp upper bound for $\left|a_3 - \mu a_2^2\right|$, when μ is real. Later on, this inequality has received a huge interest from many researchers. For very recent results regarding Fekete-Szegö problem studied for different new classes of functions see the papers [2, 7, 8, 13, 16] and [17].
In this paper, we obtain sharp upper bound for the Fekete-Szegö inequality regarding a certain subclass of S, generated using the generalized fractional operator that has been introduced by Issa and Darus [9], given as follows:

$$D_z^{\nu,m} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n z^n,$$

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where, applications of this operator can be found on [8] and [10]. Now we define the class by $\mathcal{M}^{\nu,m}(\Phi,\Psi;\lambda,\alpha,\beta)$, as follows:

Definition 1.1. A function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n, a_n\geq 0$ belongs to the class $\mathcal{M}^{\nu,m}(\Phi,\Psi;\lambda,\alpha,\beta); 0\leq\alpha<1, 0\leq\lambda\leq 1, \beta>0, 0\leq\nu<1$, and $m=1,2,\ldots$, if there exist a function $g\in A$ such that:

$$\operatorname{Re}\left(\frac{\lambda z^2\left(D_z^{\nu,m}f(z)\right)''+z\left(D_z^{\nu,m}f(z)\right)'}{\lambda zg'(z)+(1-\lambda)g(z)}\right)>\alpha, z\in U \tag{1.2}$$

where $g(z)=z+b_2z+b_3z^2+\cdots$, is analytic and satisfies

$$\left|\arg\left(\frac{g(z)*\Phi(z)}{g(z)*\Psi(z)}\right)\right|<\frac{\pi\beta}{2}, \tag{1.3}$$

for some $\Phi(z)=z+\sum_{n=2}^{\infty}\varpi_nz^n$, and $\Psi(z)=z+\sum_{n=2}^{\infty}\gamma_nz^n$, both are analytic in U such that $g(z)*\Psi(z)\neq 0, \varpi_n\geq 0, \gamma_n\geq 0$, and $\varpi_n\geq\gamma_n\ (n\geq 2)$.

Note that $\mathcal{M}^{\nu,0}(\Phi,\Psi;\lambda,\alpha,\beta)$ give us the class $\mathcal{M}(\Phi,\Psi;\lambda,\alpha,\beta)$ that was defined by Darus in [4],

$$\mathcal{M}^{\nu,0}\left(\frac{z}{(1-z)^2},\frac{z}{1-z};0,0,\beta\right)=K(\beta),$$

the class of closed-to-convex that was defined by Chonweerayoot et al. in [1], and $\mathcal{M}^{\nu,0}\left(\frac{z}{(1-z)^2},\frac{z}{1-z};0,0,1\right)$ is the class of normalized close-to-convex functions was defined by Kaplan in [12].

Furthermore, $\mathcal{M}^{\nu,0}\left(\frac{z}{(1-z)^2},\frac{z}{1-z};0,\alpha,\beta\right)=K(\alpha,\beta)$ is the class of normalized close-to-convex defined by Darus and Thomas in [3]. We have to mention that the result we introduce in the following section generalizes various results obtained by many researchers.

2 Main result

To get our main result we need the following definition and lemma:

Definition 2.1. The analytic function $p(z)$ in U belongs to the class P, if $p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots$ and $\operatorname{Re}p(z)>0$ for all z in the open unit disk.

Lemma 2.1. [15] Let $h(z)=1+c_1z+c_2z^2+\cdots$, be analytic in the open unit disk with $\operatorname{Re}h(z)>0$ for all $z\in U$. Then

$$\left|c_2-\frac{c_1^2}{2}\right|\leq 2-\frac{|c_1|^2}{2}. \tag{2.1}$$

Now we present and prove our main result in the following theorem.

Theorem 2.2. Let $f(z)=z+a_2z^2+a_3z^3+\cdots$, be a function in the class $\mathcal{M}^{\nu,m}(\Phi,\Psi;\lambda,\alpha,\beta); 0\leq\alpha<1, 0\leq\lambda\leq 1, \beta>0, 0\leq\nu<1$, and $m=1,2,\ldots$, If $3\eta\mu\geq 2\delta^2+4\delta\gamma_2$, where $\delta=\varpi_2-\gamma_2, \eta=\varpi_3-\gamma_3$ and $\mu\geq 1$. Then we have the sharp inequality

$$\begin{aligned}\left|a_3-\mu a_2^2\right|\leq{}&\frac{\beta^2}{3\delta^2\eta C_4(\nu,m)C_3^2(\nu,m)}\left\{3\mu\eta C_4(\nu,m)-2C_3^2(\nu,m)\delta(\delta+2\gamma_2)\right\}\\&-\frac{3\delta\mu\alpha(1-\alpha)C_4(\nu,m)(1+2\lambda)}{3\delta C_3^2(\nu,m)C_4(\nu,m)(1+2\lambda)(1+\lambda)^2}\\&+\frac{2\beta(1-\alpha)(1+\lambda)}{3\delta C_3^2(\nu,m)C_4(\nu,m)(1+2\lambda)(1+\lambda)^2}\left\{3\mu C_4(\nu,m)-2C_3^2(\nu,m)(1+\lambda)^2\right\}\\&+\frac{(1-\alpha)}{3\delta C_3^2(\nu,m)C_4(\nu,m)(1+2\lambda)(1+\lambda)^2}\left\{3\mu\delta C_4(\nu,m)(1+2\lambda)-2C_3^2(\nu,m)(1+\lambda)^2\right\},\end{aligned}$$

where, $C_3(\nu,m)=\frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)}$ and $C_4(\nu,m)=\frac{\Gamma(4)\Gamma(2-m\nu)}{\Gamma(4-m\nu)}$.

Proof . Since $f(z) \in \mathcal{M}^{\nu,m}(\Phi,\Psi;\lambda,\alpha,\beta)$, it follows from (1.2) that there exist $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$, with $q(z) \in P$, and $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \in A$, such that

$$\lambda z^2 \left(D_z^{\nu,m} f(z)\right)'' + z \left(D_z^{\nu,m} f(z)\right)' = \{\alpha + (1-\alpha) q(z)\} \left\{\lambda z g'(z) + (1-\lambda) g(z)\right\},$$

which implies

$$\begin{aligned} & z + 2C_3(\nu,m)(\lambda+1) a_2 z^2 + 3C_4(\nu,m)(1+2\lambda) a_3 z^3 + \cdots \\ & = z + \{(1+\lambda) b_2 + (1-\alpha) q_1\} z^2 + \{(1+2\lambda) b_3 + (1-\alpha)(1+\lambda) q_1 b_2 + (1-\alpha) q_2\} z^3 + \cdots . \end{aligned}$$

Equating the coefficients in the last equation implies

$$2C_3(\nu,m)(1+\lambda) a_2 = (1+\lambda) b_2 + (1-\alpha) q_1, \tag{2.2}$$

and

$$3C_4(\nu,m)(1+2\lambda) a_3 = (1+2\lambda) b_3 + (1-\alpha)(1+\lambda) q_1 b_2 + (1-\alpha) q_2. \tag{2.3}$$

Moreover, from (1.3), there exist a function in P say, $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, such that

$$g(z) * \Phi(z) = (g(z) * \Psi(z))\, p^{\beta}(z),$$

which implies

$$\begin{aligned} & \left(z + b_2 z^2 + b_3 z^3 + \cdots\right) * \left(z + \varpi_2 z^2 + \varpi_3 z^3 + \cdots\right) \\ & = \left\{\left(z + b_2 z^2 + b_3 z^3 + \cdots\right) * \left(z + \gamma_2 z^2 + \gamma_3 z^3 + \cdots\right)\right\} \left(1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots\right)^{\beta}. \end{aligned}$$

So,

$$\begin{aligned} z + b_2\varpi_2 z^2 + b_3\varpi_3 z^3 + \cdots &= \left\{z + b_2\gamma_2 z^2 + b_3\gamma_3 z^3 + \cdots\right\} \\ & \quad \left\{1 + \beta p_1 z + \left(\beta p_2 + \left(\frac{\beta^2}{2} - \frac{\beta}{2}\right) p_1^2\right) z^2 + \cdots\right\} \\ &= z + (p_1\beta + \gamma_2 b_2) z^2 \\ & \quad + \left(p_1\beta b_2\gamma_2 + \beta p_2 + \left(\frac{\beta^2}{2} - \frac{\beta}{2}\right) p_1^2 + \gamma_3 b_3\right) z^3 + \cdots . \end{aligned}$$

Equating the coefficients in the last equality implies

$$b_2\varpi_2 = p_1\beta + \gamma_2 b_2,$$

which give us

$$b_2 = \frac{p_1\beta}{\delta}. \tag{2.4}$$

Also,

$$\begin{aligned} b_3\varpi_3 &= p_1\beta b_2\gamma_2 + \beta p_2 + \left(\frac{\beta^2}{2} - \frac{\beta}{2}\right) p_1^2 + \gamma_3 b_3 \\ b_3\varpi_3 - \gamma_3 b_3 &= p_1\beta b_2\gamma_2 + \beta p_2 + \frac{\beta^2}{2} p_1^2 - \frac{\beta}{2} p_1^2 \\ (\varpi_3 - \gamma_3) b_3 &= \beta\left(p_1 b_2\gamma_2 + p_2 + \frac{\beta}{2} p_1^2 - \frac{1}{2} p_1^2\right) \end{aligned}$$

$$\eta b_3 = \beta\left(p_1\frac{p_1\beta}{\delta}\gamma_2 + p_2 + \frac{\beta}{2}p_1^2 - \frac{1}{2}p_1^2\right) = \beta\left(p_2 + p_1^2\frac{\beta}{\delta}\gamma_2 + \frac{\beta}{2}p_1^2 - \frac{1}{2}p_1^2\right)$$
$$= \beta\left(p_2 + p_1^2\left(\frac{\beta}{\delta}\gamma_2 + \frac{\beta}{2} - \frac{1}{2}\right)\right) = \beta\left(p_2 + p_1^2\left(\frac{2\beta\gamma_2}{2\delta} + \frac{\delta\beta}{2\delta} - \frac{\delta}{2\delta}\right)\right),$$

which implies

$$b_3 = \frac{\beta}{\eta}\left\{p_2 + \left(\frac{\beta\left(\delta + 2\gamma_2\right) - \delta}{2\delta}\right)p_1^2\right\}. \tag{2.5}$$

So from (2.1),(2.2),(2.3),(2.4) and (2.5) we get

$$a_2 = \frac{p_1\beta}{2\delta C_3\left(\nu, m\right)} + \frac{\left(1-\alpha\right)q_1}{2\left(1+\lambda\right)C_3\left(\nu, m\right)}, \tag{2.6}$$

and

$$a_3 = \frac{1}{3C_4\left(\nu, m\right)\left(1+2\lambda\right)}\left\{\left(1+2\lambda\right)b_3 + \left(1-\alpha\right)\left(1+\lambda\right)q_1b_2 + \left(1-\alpha\right)q_2\right\}$$
$$= \frac{\left(1+2\lambda\right)b_3}{3C_4\left(\nu, m\right)\left(1+2\lambda\right)} + \frac{\left(1-\alpha\right)\left(1+\lambda\right)q_1b_2}{3C_4\left(\nu, m\right)\left(1+2\lambda\right)} + \frac{\left(1-\alpha\right)q_2}{3C_4\left(\nu, m\right)\left(1+2\lambda\right)}.$$

Indeed,

$$a_3 = \frac{\beta}{3\eta C_4\left(\nu, m\right)}\left\{p_2 + \left(\frac{\beta\left(\delta+2\gamma_2\right)-\delta}{2\delta}\right)p_1^2\right\} + \frac{\left(1-\alpha\right)\left(1+\lambda\right)q_1p_1\beta}{3\delta C_4\left(\nu, m\right)\left(1+2\lambda\right)} + \frac{\left(1-\alpha\right)q_2}{3C_4\left(\nu, m\right)\left(1+2\lambda\right)}. \tag{2.7}$$

Now, we will use (2.1),(2.2),(2.3),(2.4),(2.5),(2.6) and (2.7) to get the required result as follows:

$$a_3 - \mu a_2^2 = \frac{\beta p_2}{3\eta C_4\left(\nu, m\right)} + \frac{\beta^2\left(\delta+2\gamma_2\right)p_1^2}{6\delta\eta C_4\left(\nu, m\right)} - \frac{\beta\delta p_1^2}{6\delta\eta C_4\left(\nu, m\right)} + \frac{\left(1-\alpha\right)\left(1+\lambda\right)q_1p_1\beta}{3\delta C_4\left(\nu, m\right)\left(1+2\lambda\right)} + \frac{\left(1-\alpha\right)q_2}{3C_4\left(\nu, m\right)\left(1+2\lambda\right)}$$
$$- \frac{\mu\beta^2p_1^2}{4\delta^2C_3^2\left(\nu, m\right)} - \frac{2\beta\left(1-\alpha\right)\mu q_1p_1}{4\delta C_3^2\left(\nu, m\right)\left(1+\lambda\right)} - \frac{\left(1-\alpha\right)^2\mu q_1^2}{4\left(1+\lambda\right)^2C_3^2\left(\nu, m\right)}$$
$$+ \frac{2\left(1-\alpha\right)q_1^2}{12C_4\left(\nu, m\right)\left(1+2\lambda\right)} - \frac{2\left(1-\alpha\right)q_1^2}{12C_4\left(\nu, m\right)\left(1+2\lambda\right)}$$
$$= \left\{\frac{\left(1-\alpha\right)q_2}{3C_4\left(\nu, m\right)\left(1+2\lambda\right)} - \frac{2\left(1-\alpha\right)q_1^2}{12C_4\left(\nu, m\right)\left(1+2\lambda\right)}\right\} + \left\{\frac{\beta p_2}{3\eta C_4\left(\nu, m\right)} - \frac{\beta\delta p_1^2}{6\delta\eta C_4\left(\nu, m\right)}\right\}$$
$$+ \left\{\frac{2\left(1-\alpha\right)q_1^2}{12C_4\left(\nu, m\right)\left(1+2\lambda\right)} - \frac{\left(1-\alpha\right)^2\mu q_1^2}{4\left(1+\lambda\right)^2C_3^2\left(\nu, m\right)}\right\} + \left\{\frac{\left(1-\alpha\right)\left(1+\lambda\right)q_1p_1\beta}{3\delta C_4\left(\nu, m\right)\left(1+2\lambda\right)} - \frac{2\beta\left(1-\alpha\right)\mu q_1p_1}{4\delta C_3^2\left(\nu, m\right)\left(1+\lambda\right)}\right\}$$
$$+ \left\{\frac{\beta^2\left(\delta+2\gamma_2\right)p_1^2}{6\delta\eta C_4\left(\nu, m\right)} - \frac{\mu\beta^2p_1^2}{4\delta^2C_3^2\left(\nu, m\right)}\right\},$$

implies

$$a_3 - \mu a_2^2 = \frac{\left(1-\alpha\right)}{3C_4\left(\nu, m\right)\left(1+2\lambda\right)}\left\{q_2 - \frac{q_1^2}{2}\right\} + \frac{\beta}{3\eta C_4\left(\nu, m\right)}\left\{p_2 - \frac{p_1^2}{2}\right\}$$
$$+ \left(1-\alpha\right)q_1^2\left\{\frac{2\left(1+\lambda\right)^2C_3^2\left(\nu, m\right) - 3\mu\left(1-\alpha\right)C_4\left(\nu, m\right)\left(1+2\lambda\right)}{12C_3^2\left(\nu, m\right)C_4\left(\nu, m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^2}\right\}$$
$$+ \left(1-\alpha\right)q_1p_1\beta\left\{\frac{2\left(1+\lambda\right)^2C_3^2\left(\nu, m\right) - 3C_4\left(\nu, m\right)\mu\left(1+2\lambda\right)}{6\delta C_3^2\left(\nu, m\right)C_4\left(\nu, m\right)\left(1+2\lambda\right)\left(1+\lambda\right)}\right\}$$
$$+ \beta^2p_1^2\left\{\frac{2C_3^2\left(\nu, m\right)\delta\left(\delta+2\gamma_2\right) - 3\mu\eta C_4\left(\nu, m\right)}{12\delta^2\eta C_4\left(\nu, m\right)C_3^2\left(\nu, m\right)}\right\}. \tag{2.8}$$

Secondly, we will complete the proof by applying the technique introduced by London [14]. Now, assume that $a_3-\mu a_2^2$ is positive, then from (2.8) and by substituting $p_1=2re^{i\theta}$ and $q_1=2Re^{i\varphi}, 0\le r\le 1, 0\le R\le 1, 0\le\theta\le 2\pi$, and $0\le\varphi\le 2\pi$, we obtain:

$$\begin{aligned}
3\operatorname{Re}\left(a_3-\mu a_2^2\right)=&\frac{(1-\alpha)}{C_4(\nu,m)(1+2\lambda)}\operatorname{Re}\left\{q_2-\frac{q_1^2}{2}\right\}+\frac{\beta}{\eta C_4(\nu,m)}\operatorname{Re}\left\{p_2-\frac{p_1^2}{2}\right\}\\
&+(1-\alpha)\left\{\frac{2(1+\lambda)^2C_3^2(\nu,m)-3\mu(1-\alpha)C_4(\nu,m)(1+2\lambda)}{4C_3^2(\nu,m)C_4(\nu,m)(1+2\lambda)(1+\lambda)^2}\right\}\operatorname{Re}q_1^2\\
&+\beta(1-\alpha)\left\{\frac{2(1+\lambda)^2C_3^2(\nu,m)-3C_4(\nu,m)\mu(1+2\lambda)}{2\delta C_3^2(\nu,m)C_4(\nu,m)(1+2\lambda)(1+\lambda)}\right\}\operatorname{Re}p_1q_1\\
&+\beta^2\left\{\frac{2C_3^2(\nu,m)\delta(\delta+2\gamma_2)-3\mu\eta C_4(\nu,m)}{4\delta^2\eta C_4(\nu,m)C_3^2(\nu,m)}\right\}\operatorname{Re}p_1^2.
\end{aligned}$$

Consequently, by Lemma 2.1 we get

$$\begin{aligned}
3\operatorname{Re}\left(a_3-\mu a_2^2\right)\le&\frac{2(1-\alpha)}{C_4(\nu,m)(1+2\lambda)}\left(1-R^2\right)+\frac{2\beta}{\eta C_4(\nu,m)}\left(1-r^2\right)\\
&+(1-\alpha)\left\{\frac{2(1+\lambda)^2C_3^2(\nu,m)-3\mu(1-\alpha)C_4(\nu,m)(1+2\lambda)}{4C_3^2(\nu,m)C_4(\nu,m)(1+2\lambda)(1+\lambda)^2}\right\}4R^2\cos 2\theta\\
&+\beta(1-\alpha)\left\{\frac{2(1+\lambda)^2C_3^2(\nu,m)-3C_4(\nu,m)\mu(1+2\lambda)}{2\delta C_3^2(\nu,m)C_4(\nu,m)(1+2\lambda)(1+\lambda)}\right\}4rR\cos(\varphi+\theta)\\
&+\beta^2\left\{\frac{2C_3^2(\nu,m)\delta(\delta+2\gamma_2)-3\mu\eta C_4(\nu,m)}{4\delta^2\eta C_4(\nu,m)C_3^2(\nu,m)}\right\}4r^2\cos 2\varphi\\
\le&\frac{2(1-\alpha)}{C_4(\nu,m)(1+2\lambda)}\left(1-R^2\right)+\frac{2\beta}{\eta C_4(\nu,m)}\left(1-r^2\right)\\
&+(1-\alpha)\left\{\frac{3\mu(1-\alpha)C_4(\nu,m)(1+2\lambda)-2(1+\lambda)^2C_3^2(\nu,m)}{C_3^2(\nu,m)C_4(\nu,m)(1+2\lambda)(1+\lambda)^2}\right\}R^2\\
&+2\beta(1-\alpha)\left\{\frac{3C_4(\nu,m)\mu(1+2\lambda)-2(1+\lambda)^2C_3^2(\nu,m)}{\delta C_3^2(\nu,m)C_4(\nu,m)(1+2\lambda)(1+\lambda)}\right\}rR\\
&+\beta^2\left\{\frac{3\mu\eta C_4(\nu,m)-2C_3^2(\nu,m)\delta(\delta+2\gamma_2)}{\delta^2\eta C_4(\nu,m)C_3^2(\nu,m)}\right\}r^2\\
=&H(r,R).
\end{aligned}$$

Next, we will identify the sign of $H_{rr}H_{RR}-\left(H_{rR}\right)^2$ to illustrate the maximum value of $H(r,R)$, as follows:

$$\begin{aligned}
H_{rr}H_{RR}-\left(H_{rR}\right)^2=&\left\{\frac{-4\beta}{\eta C_4(\nu,m)}+2\beta^2\left\{\frac{3\mu\eta C_4(\nu,m)-2C_3^2(\nu,m)\delta(\delta+2\gamma_2)}{\delta^2\eta C_4(\nu,m)C_3^2(\nu,m)}\right\}\right\}\\
&\left\{\begin{array}{c}\dfrac{-4(1-\alpha)}{C_4(\nu,m)(1+2\lambda)}\\ +2(1-\alpha)\left\{\dfrac{3\mu(1-\alpha)C_4(\nu,m)(1+2\lambda)-2(1+\lambda)^2C_3^2(\nu,m)}{C_3^2(\nu,m)C_4(\nu,m)(1+2\lambda)(1+\lambda)^2}\right\}\end{array}\right\}\\
&-\left\{2\beta(1-\alpha)\frac{3C_4(\nu,m)\mu(1+2\lambda)-2(1+\lambda)^2C_3^2(\nu,m)}{\delta C_3^2(\nu,m)C_4(\nu,m)(1+2\lambda)(1+\lambda)}\right\}^2\\
=&\frac{16\beta(1-\alpha)}{\eta(1+2\lambda)C_4^2(\nu,m)}-8\beta(1-\alpha)\left\{\frac{3\mu(1-\alpha)C_4(\nu,m)(1+2\lambda)-2(1+\lambda)^2C_3^2(\nu,m)}{C_3^2(\nu,m)C_4^2(\nu,m)\eta(1+2\lambda)(1+\lambda)^2}\right\}
\end{aligned}$$

$$
\begin{aligned}
&- 8\left(1-\alpha\right)\beta^{2}\left\{\frac{3\mu\eta C_{4}\left(\nu,m\right)-2C_{3}^{2}\left(\nu,m\right)\delta\left(\delta+2\gamma_{2}\right)}{\delta^{2}\eta C_{4}^{2}\left(\nu,m\right)C_{3}^{2}\left(\nu,m\right)\left(1+2\lambda\right)}\right\}\\
&+ 4\beta^{2}\left(1-\alpha\right)\left[\begin{array}{c}\left\{\dfrac{3\mu\eta C_{4}\left(\nu,m\right)-2C_{3}^{2}\left(\nu,m\right)\delta\left(\delta+2\gamma_{2}\right)}{\delta^{2}\eta C_{4}\left(\nu,m\right)C_{3}^{2}\left(\nu,m\right)}\right\}\\ \left\{\dfrac{3\mu\left(1-\alpha\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)-2\left(1+\lambda\right)^{2}C_{3}^{2}\left(\nu,m\right)}{C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}\right\}\end{array}\right]\\
&- 4\beta^{2}\left(1-\alpha\right)^{2}\left\{\frac{3C_{4}\left(\nu,m\right)\mu\left(1+2\lambda\right)-2\left(1+\lambda\right)^{2}C_{3}^{2}\left(\nu,m\right)}{\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)}\right\}^{2}\\
&<0.
\end{aligned}
$$

Hence the function $H\left(r,R\right)$ attains its maximum value on the boundary of the unit disk. Therefore our inequality follows by observing that

$$
\begin{aligned}
\left|a_{3}-\mu a_{2}^{2}\right| \leq&\frac{1}{3}H\left(r,R\right)\leq\frac{1}{3}H\left(1,1\right)\leq\beta^{2}\left\{\frac{3\mu\eta C_{4}\left(\nu,m\right)-2C_{3}^{2}\left(\nu,m\right)\delta\left(\delta+2\gamma_{2}\right)}{3\delta^{2}\eta C_{4}\left(\nu,m\right)C_{3}^{2}\left(\nu,m\right)}\right\}\\
&+\left(1-\alpha\right)\left\{\frac{3\mu\left(1-\alpha\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)-2\left(1+\lambda\right)^{2}C_{3}^{2}\left(\nu,m\right)}{3C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}\right\}\\
&+2\beta\left(1-\alpha\right)\left\{\frac{3C_{4}\left(\nu,m\right)\mu\left(1+2\lambda\right)-2\left(1+\lambda\right)^{2}C_{3}^{2}\left(\nu,m\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)}\right\}\\
=&\frac{\beta^{2}}{3\delta^{2}\eta C_{4}\left(\nu,m\right)C_{3}^{2}\left(\nu,m\right)}\left\{3\mu\eta C_{4}\left(\nu,m\right)-2C_{3}^{2}\left(\nu,m\right)\delta\left(\delta+2\gamma_{2}\right)\right\}\\
&+\frac{6\beta\mu\left(1-\alpha\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}-\frac{4\beta\left(1-\alpha\right)\left(1+\lambda\right)^{3}C_{3}^{2}\left(\nu,m\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}\\
&+\frac{3\delta\mu\left(1-\alpha\right)^{2}C_{4}\left(\nu,m\right)\left(1+2\lambda\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}-\frac{2\delta\left(1-\alpha\right)\left(1+\lambda\right)^{2}C_{3}^{2}\left(\nu,m\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}\\
=&\frac{\beta^{2}}{3\delta^{2}\eta C_{4}\left(\nu,m\right)C_{3}^{2}\left(\nu,m\right)}\left\{3\mu\eta C_{4}\left(\nu,m\right)-2C_{3}^{2}\left(\nu,m\right)\delta\left(\delta+2\gamma_{2}\right)\right\}\\
&+\frac{6\beta\mu\left(1-\alpha\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}-\frac{4\beta\left(1-\alpha\right)\left(1+\lambda\right)^{3}C_{3}^{2}\left(\nu,m\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}\\
&+\frac{3\delta\mu\left(1-2\alpha+\alpha^{2}\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}-\frac{2\delta\left(1-\alpha\right)\left(1+\lambda\right)^{2}C_{3}^{2}\left(\nu,m\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}\\
=&\frac{\beta^{2}}{3\delta^{2}\eta C_{4}\left(\nu,m\right)C_{3}^{2}\left(\nu,m\right)}\left\{3\mu\eta C_{4}\left(\nu,m\right)-2C_{3}^{2}\left(\nu,m\right)\delta\left(\delta+2\gamma_{2}\right)\right\}\\
&+\frac{6\beta\mu\left(1-\alpha\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}-\frac{4\beta\left(1-\alpha\right)\left(1+\lambda\right)^{3}C_{3}^{2}\left(\nu,m\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}\\
&+\frac{3\delta\mu\left(1-\alpha\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}+\frac{3\delta\mu\alpha\left(\alpha-1\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}\\
&-\frac{2\delta\left(1-\alpha\right)\left(1+\lambda\right)^{2}C_{3}^{2}\left(\nu,m\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}\\
=&\frac{\beta^{2}}{3\delta^{2}\eta C_{4}\left(\nu,m\right)C_{3}^{2}\left(\nu,m\right)}\left\{3\mu\eta C_{4}\left(\nu,m\right)-2C_{3}^{2}\left(\nu,m\right)\delta\left(\delta+2\gamma_{2}\right)\right\}-\frac{3\delta\mu\alpha\left(1-\alpha\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}\\
&+\frac{6\beta\mu\left(1-\alpha\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}-\frac{4\beta\left(1-\alpha\right)\left(1+\lambda\right)^{3}C_{3}^{2}\left(\nu,m\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}\\
&+\frac{3\delta\mu\left(1-\alpha\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}-\frac{2\delta\left(1-\alpha\right)\left(1+\lambda\right)^{2}C_{3}^{2}\left(\nu,m\right)}{3\delta C_{3}^{2}\left(\nu,m\right)C_{4}\left(\nu,m\right)\left(1+2\lambda\right)\left(1+\lambda\right)^{2}}.
\end{aligned}
$$

Finally, we get:

$$\begin{aligned}\left|a_3-\mu a_2^2\right| \leq & \frac{\beta^2}{3\delta^2\eta C_4(\nu,m)\,C_3^2(\nu,m)}\left\{3\mu\eta C_4(\nu,m)-2C_3^2(\nu,m)\,\delta(\delta+2\gamma_2)\right\}\\ &-\frac{3\delta\mu\alpha(1-\alpha)\,C_4(\nu,m)(1+2\lambda)}{3\delta C_3^2(\nu,m)\,C_4(\nu,m)(1+2\lambda)(1+\lambda)^2}\\ &+\frac{2\beta(1-\alpha)(1+\lambda)}{3\delta C_3^2(\nu,m)\,C_4(\nu,m)(1+2\lambda)(1+\lambda)^2}\left\{3\mu C_4(\nu,m)-2C_3^2(\nu,m)(1+\lambda)^2\right\}\\ &+\frac{(1-\alpha)}{3\delta C_3^2(\nu,m)\,C_4(\nu,m)(1+2\lambda)(1+\lambda)^2}\left\{3\mu\delta C_4(\nu,m)(1+2\lambda)-2C_3^2(\nu,m)(1+\lambda)^2\right\},\end{aligned}$$

as required. The equality is attained by choosing $\nu=0$, $p_1=q_1=2i$ and $p_2=q_2=-2$ in (2.8). □

Remark 2.1. Letting $\nu=0$ in Theorem 2.2, we have the result by Darus [4].

Remark 2.2. Letting $\nu=0$ and $\lambda=0$ in Theorem 2.2, we have the result by Frasin and Darus [6].

Remark 2.3. Letting $\Phi(z)=\frac{z}{(1-z)^2}$, $\Psi(z)=\frac{z}{1-z}$, $\lambda=\alpha=1$, and $\nu=0$ in Theorem 2.2 , we have the result by Jahangiri [11].

Finally, we ought to mention that we can gain a lot of results from Theorem 2.2 by various choices of $\Phi(z)$ and $\Psi(z)$.

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