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Fractional ordered thermoelastic stress analysis of a thin circular plate under axi-symmetric heat supply

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Abstract

The main objective of the current study is to investigate the fractional ordered thermoelastic stress analysis of a thin circular plate under axi-symmetric heat supply. Initially, the plate is characterized by the initial temperature $T_0(r,z)$. The boundary value problem is formulated with a circular plate model where the perimetric edge is clamped and convection, and the upper and lower surfaces are subjected to heat convection with convection coefficient h_c and fluid temperature T_∞. The variable separable technique and Green's function approach scheme have been employed to solve the heat conduction equation. The impacts of the fractional ordered derivative of some other parameters on temperature, deflection, and stress profiles will be analyzed in detail. For instance, the results indicate that the temperature and thermal deflection are directly proportional to the fractional order parameter α. Also, the parameter α represents the weak, normal, and strong conductivity, within the range of $0 < \alpha < 1$, $\alpha = 1$ and $1 < \alpha < 2$ respectively.

Keywords: Caputo fractional derivative, Green's function, Axi-symmetric heat source, Thin circular plate, Mittag-Leffler functions
2020 MSC: 35B07, 35G30, 35K05, 44A10

1 Introduction

Povstenko [26, 27] solved some thermoelastic problems based on the equation of heat conduction in 1D as well as 2D with a time-fractional derivative and associated thermal stresses. In four distinct thermoelasticity theories, Roushan Kumar and Mukhopadhyay [23] explored general thermoelastic interactions in unbounded elastic media and spherical cavities. Avijit and Kanoria [16] presented thermoelasticity theories for a hollow sphere with a thermal shock problem. In the fractional calculus technique, Sherief et. al. [34] introduced the novel coupling between thermoelasticity and widespread thermoelasticity with one relaxation cycle. Sur and Kanoria [36] developed the new theory of thermoelastic distribution of two temperature with new heat conduction equation with fractional order. Youssef [39] solved the generalized thermoelasticity theory of a half-space filled with an elastic material, which has constant elastic parameters in the context of the fractional order derivative. Gaikwad and Ghadle [8] presented the thermoelastic problem for the thick circular plate subjected to an interior heat flux under an unsteady-state, the

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determination of unknown temperature, displacement and thermal stresses on the upper surface of a plate. Gaikwad and Ghadle [7] studied the nonhomogeneous heat conduction problem and its thermal deflection due to internal heat generation in a thin hollow circular disk. Gaikwad [4] analysed the thermoelastic deformation of a thin hollow circular disk due to partially distributed heat supply. Sur and Kanoria [37] introduced the fractional order generalized thermoelastic functionally graded solid with variable material properties.

Hussain [15] developed the fractional order thermoelastic problem for an infinitely long solid circular cylinder. Raslan [30] addressed the 1D issue utilising the Laplace transform technique of the thermoelasticity fractional order of an infinitely long cylindrical cavity. Gaikwad [5] developed the thermoelastic mathematical model for circular sector disk subject to internal heat generation. Bayat et. al. [1] analysed the unsteady state thermo-mechanical problem of the FGM thick sphere. Raslan [31] resolved a 2D problem of an axi-symmetric temperature distribution fractional thermoelasticity order theory of a thick plate. Tripathi [38] showed the impact of an axisymmetric supply of heat on the diffusion phenomena of an infinite and finite thick thermoelastic platform and the theory of widespread thermoelastic diffusion with a one-time interval of relaxation. Magdy [3] developed a 3D thermoelasticity model with time-dependent thermal shock issue, utilising a fractional thermoelasticity order theory for a half-space. Also, some contributions to this theory are the work in [6, 9, 17, 10, 12, 13, 18, 20, 11, 19, 21, 14, 2, 32, 33].

The solution of partial differential equations of heat conduction by the classical method of separation of variables is not always convenient when the equation and the boundary conditions involve non-homogeneities. It is for this reason that we considered the Green's function approach for the solution of linear, nonhomogeneous boundary value problems of heat conduction with the help of the Caputo time-fractional derivative.

Based on the above literature review, the main novelty and research gap are elucidated in the following. Although many numerical analyses have been performed on thin circular plates, there is a gap in the simultaneous use of the Green's function approach and the method of separation of variables for solving the fractional ordered thermoelastic stress analysis of a thin circular plate under axi-symmetric heat supply. The analytical solution of the circular plate under axi-symmetric heat supply will be utilized to solve the issue. This is the main motivation behind this computational study, which will be very helpful in closing some research gaps in this field. Another novelty of our present work is to evaluate the accuracy of these methods to obtain analytic solutions and compare them with each other.

It is believed that, this particular problem has not been considered by any one. This is a new and novel contribution to the field of thermoelasticity. The results presented here will be useful in studying the thermal characteristics of various bodies in real-life engineering problems, and those working to further develop the theory of fractional order thermoelasticity. To verify the accuracy and validity of the approach, numerical results are presented for an easy comparison with those found in the literatures [9, 17, 10, 12, 13, 18, 20, 11].

***List of Symbols*:**

E	Young's modulus(N/m^2)
k	Thermal diffusivity(m^2/s)
c	Specific heat constant(J/kg.K)
k_t	Thermal conductivity (W/m.K)
k_c	Heat transfer coefficient
g_1	Instantaneous line heat source
T_∞	Ambient temperature(K)
α_t	Thermal expansion coefficient of the plate material(1/K)

***Greek Symbols*:**

ν	Poisson's ratio
ρ	Density(kg/m^3)
μ	Lame's constants(GPa)
$\overline{\delta}$	Dirac delta function
σ_{ij}	Components of stress tensor

***Abbreviations*:**

$1D$	One-dimensional(m)
$2D$	Two-dimentional(m)
$3D$	Three-dimentional(m).

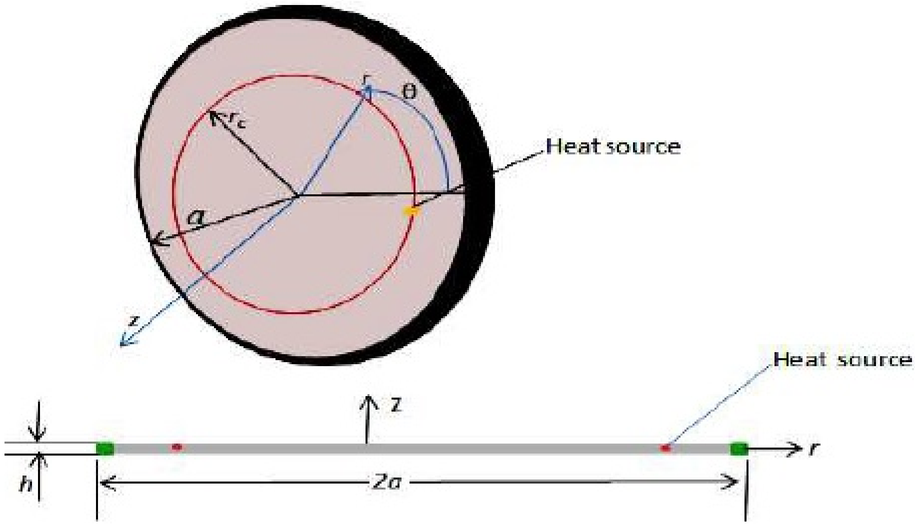

Figure 1: Schematic of the problem.

2 Problem Formulation

In the present research, the behaviour of fractional ordered thermoelastic stress analysis of a thin circular plate under axi-symmetric heat supply is investigated. As schematically depicted in figure 1, the attendance of heat source and heat convection are propounded to affect on temperature, deflection, and stresses of a thin circular plate.

We consider a thin circular plate of radius a and thickness h with occupying the region $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$, $-h/2 \leq z \leq h/2$ in the cylindrical coordinate. It is assumed that the upper surfaces ($z = h/2$) and lower surfaces ($z = -h/2$) of the plate are subjected to heat convection. Mathematical model is prepared considering nonlocal Caputo type time fractional heat conduction equation of order α for a thin circular disk.

The Gamma function, is denoted by $\Gamma(z)$, is a generalization of the factorial function $n!$, i.e., $\Gamma(n) = (n-1)!$ for $n \in N$. For complex arguments with positive real parts it is defined in [25] as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \text{Re z} > 0$$

The definition of Caputo type fractional derivative given by [25]

$$D^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int_0^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n; \\ \dfrac{df(t)}{dt}, & n = 1. \end{cases}$$

For finding the Laplace transform, the Caputo derivative requires information of the initial values of the function $f(t)$ and its integer derivative of the order $k = 1, 2, ..., n-1$

$$L\{D^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \qquad n-1 < \alpha < n".$$

The governing heat conduction equation in the form of fractional order parameter for a thin circular plate satisfies the differential equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g}{k_t} = \frac{1}{k}\frac{\partial^\alpha T}{\partial t^\alpha} \tag{1}$$

here $g(r,z,t) = g_1(t)\bar{\delta}(r - r_c)\bar{\delta}(z - z_c)$ is an axisymmetric heat source and $k = k_t/\rho c$. where g_1 is an instantaneous line heat source, and $\bar{\delta}$ is a Dirac delta function that characterizes the location of the line heat source at r_c and z_c. with the boundary conditions,

$$T < \infty, \qquad \text{at } r = 0, \text{ for } t > 0, \tag{2}$$

$$-k_t\frac{\partial T}{\partial r} = k_c(T - T_\infty), \qquad \text{at } r = a, \text{ for } t > 0, \tag{3}$$

$$-k_t\frac{\partial T}{\partial z} = k_c(T - T_\infty), \qquad \text{at } z = h/2, \text{ for } t > 0, \tag{4}$$

$$k_t\frac{\partial T}{\partial z} = k_c(T - T_\infty), \qquad \text{at } z = -h/2, \text{ for } t > 0. \tag{5}$$

The initial condition are assumed,

$$T = T_0(r, z), \qquad \text{at } t = 0,\ 0 < \alpha < 2\ , \tag{6}$$

Equations (1) to (6) constitute the mathematical formulation of the problem under consideration.

3 Determination of Temperature field

We solved the formulated BVP using Green's function approach, the temperature $T(r, z, t)$ is divided into two components, $T = T_\infty + \chi$, where T_∞ is the ambient component satisfies the equation (1) with $g(t, r, z) = 0$ and χ is the dynamic component satisfies the following equation:

$$\frac{\partial^2\chi}{\partial r^2} + \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{\partial^2\chi}{\partial z^2} + \frac{g}{k_t} = \frac{1}{k}\frac{\partial^\alpha T}{\partial t^\alpha} \tag{7}$$

$$\chi < \infty, \qquad \text{at } r = 0, \text{ for } t > 0, \tag{8}$$

$$k_t\frac{\partial\chi}{\partial r} + k_c\chi = 0, \qquad \text{at } r = a, \text{ for } t > 0, \tag{9}$$

$$k_t\frac{\partial\chi}{\partial z} + k_c\chi = 0, \qquad \text{at } z = h/2, \text{ for } t > 0, \tag{10}$$

$$k_t\frac{\partial\chi}{\partial z} - k_c\chi = 0, \qquad \text{at } z = -h/2, \text{ for } t > 0. \tag{11}$$

$$\chi = T_0 - T_\infty, \qquad \text{at } t = 0,\ 0 < \alpha < 2\ , \tag{12}$$

Using the Variable separation method $\chi(t, r, z) = \chi_1(t)\chi_2(r)\chi_3(z)$, the homogeneous form of equation (7) with $g(t, r, z) = 0$ is converted as:

$$\frac{d^\alpha\chi_1}{dt^\alpha} + k\lambda^2\chi_1 = 0 \tag{13}$$

$$\frac{d^2\chi_2}{dr^2} + \frac{1}{r}\frac{d\chi_2}{dr} + \delta^2\chi_2 = 0 \tag{14}$$

$$\frac{d^2\chi_3}{dz^2} + \beta^2\chi_3 = 0 \tag{15}$$

where λ, δ, and β are the corresponding eigenvalues and they satisfy $\delta^2 = \lambda^2 - \beta^2$. The solutions are obtained as:

$$\chi_1(t) = A_1(t^{\alpha-1}E_{\alpha,\alpha}(-k\lambda^2t^\alpha)) \tag{16}$$

$$\chi_2(r) = A_2J_0(\delta r) + A_3Y_0(\delta r) \tag{17}$$

$$\chi_3(z) = A_4\cos(\beta z) + A_5\sin(\beta z) \tag{18}$$

here $E_\alpha(.)$, Mittag-Leffler function, A_i's (i=1,2,...,5) are the coefficients to be determined from the initial and boundary conditions.

The requirement of a finite temperature at the disk center given by equation (8) leads to $A_3 = 0$; besides, the boundary condition (9) produces the characteristic equation to determine δ:

$$J_1(\delta_i) + \frac{k_c}{k_t}J_0(\delta_i) = 0 \tag{19}$$

From equations (10) and (11) determine A_4 and A_5 for different β's satisfying the following relations:

$$A_4 = 0 \quad \text{for} \quad \tan\left(\frac{\beta h}{2}\right) = -\frac{k_t\beta}{k_c} \tag{20}$$

$$A_5 = 0 \quad \text{for} \quad \tan\left(\frac{\beta h}{2}\right) = \frac{k_t}{k_c \beta} \tag{21}$$

The general solution of the problem defined in equations (7–12) is obtained as:

$$\begin{aligned}\chi(t,r,z) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(J_1(\delta_i r) + \frac{k_c}{k_t}J_0(\delta_i r)\right)\big(A_{i,2j}(t^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_{i,2j}^2 t^{\alpha}))\cos(\beta_{2j}z) \\ + A_{i,2j+1}(t^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_{i,2j+1}^2 t^{\alpha}))\sin(\beta_{2j+1}z)\big)\end{aligned} \tag{22}$$

For simplicity, β_{2j}'s and β_{2j+1}'s are determined from equations (20) and (21), δ_i's are determined from equation (19), and $\lambda_{ij}^2 = \delta_i^2 + \beta_j^2$ $(i = 1, 2, ..., j = 0, 1, 2, ...)$. It is noted that A's in equations (16–18) are combined into $A_{i,2j}$'s and $A_{i,2j+1}$'s which are obtained by substituting equation (22) into equation (12) as:

$$A_{i,2j} = \frac{1}{\eta_{i,2j}}\int_0^a \int_{-h/2}^{h/2} r'\left(J_1(\delta_i r') + \frac{k_c}{k_t}J_0(\delta_i r')\right)\cos(\beta_{2j}z')(T_0(r',z') - T_\infty)dr'dz' \tag{23}$$

$$A_{i,2j+1} = \frac{1}{\eta_{i,2j+1}}\int_0^a \int_{-h/2}^{h/2} r'\left(J_1(\delta_i r') + \frac{k_c}{k_t}J_0(\delta_i r')\right)\sin(\beta_{2j+1}z')(T_0(r',z') - T_\infty)dr'dz' \tag{24}$$

From the above, the Green's function is obtained as:

$$\begin{aligned}G(t,r,z;t',r',z') = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} r'\left(J_1(\delta_i r') + \frac{k_c}{k_t}J_0(\delta_i r')\right)\left(J_1(\delta_i r) + \frac{k_c}{k_t}J_0(\delta_i r)\right) \\ \left[\frac{t^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_{i,2j}^2(t^\alpha - t'^\alpha))}{\eta_{i,2j}}\cos(\beta_{2j}z')\cos(\beta_{2j}z) + \frac{t^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_{i,2j+1}^2(t^\alpha - t'^\alpha))}{\eta_{i,2j+1}}\sin(\beta_{2j+1}z')\sin(\beta_{2j+1}z)\right]\end{aligned} \tag{25}$$

where

$\eta_{i,0} = \frac{(\beta_0 h + \sin(\beta_0 h))a^2 J_0^2(\delta_i a)}{4\beta_0}$, $\eta_{0,0} = \frac{(\beta_0 h + \sin(\beta_0 h))a^2}{4\beta_0}$, $\eta_{i,2j} = \eta_{i,2j+1} = \frac{a^2 h\left(J_1^2(\delta_i a) + \frac{k_c}{k_t}J_0^2(\delta_i a)\right)}{4}$. Hence the required temperature $T(t,r,z)$ is;

$$\begin{aligned}T(t,r,z) = T_\infty + \int_0^a \int_{-h/2}^{h/2} G(t,r,z;t',r',z')|_{t'=0}(T_0(r',z') - T_\infty)dr'dz' \\ + \frac{k}{k_t}\int_0^t \int_0^a \int_{-h/2}^{h/2} G(t,r,z;t',r',z')g(t',r',z')dr'dz'dt'\end{aligned} \tag{26}$$

For $T_\infty = T_0$ and $g(t,r,z) = U(t)\delta(r - r_c)\delta(z - z_c)$, with $U(t)$ is a unit step function, the temperature is obtain as:

$$\begin{aligned}T(t,r,z) = T_0 + \frac{r_c}{k_t}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(J_1(\delta_i r_c) + \frac{k_c}{k_t}J_0(\delta_i r_c)\right)\left(J_1(\delta_i r) + \frac{k_c}{k_t}J_0(\delta_i r)\right) \\ \left[\frac{t^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_{i,2j}^2 t^\alpha)}{\lambda_{i,2j}^2\eta_{i,2j}}\cos(\beta_{2j}z_c)\cos(\beta_{2j}z) + \frac{t^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_{i,2j+1}^2 t^\alpha)}{\lambda_{i,2j+1}^2\eta_{i,2j+1}}\sin(\beta_{2j+1}z_c)\sin(\beta_{2j+1}z)\right].\end{aligned} \tag{27}$$

4 Determination of Thermal Deflection and Thermal Stresses

The deflection function $w(r,t,z)$ and potential displacement function $\zeta(r,t,z)$ defined in [7] as:

$$\nabla^2\nabla^2 w = -\frac{1}{(1-\nu)D}\nabla^2 M_T \tag{28}$$

$$\nabla^2\zeta = (1+\nu)\alpha_t \Delta T \tag{29}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{30}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad \Delta T = T - T_0 \tag{31}$$

The term M_T is defined as:

$$M_T = \alpha_t E \int_{-h/2}^{h/2} \left(T(r,z,t) - T_0\right) z dz, \tag{32}$$

The fixed outer edges of the circular plate described as:

$$w = \zeta = 0, \qquad \frac{dw}{dr} = \frac{d\zeta}{dr} = 0 \qquad \text{at } r = a. \tag{33}$$

The stresses σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -2\mu \frac{1}{r}\frac{\partial \zeta}{\partial r} \tag{34}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 \zeta}{\partial r^2} \tag{35}$$

The axial symmetric plane-stress state are assumed:

$$\sigma_{zz} = \sigma_{rz} = \sigma_{\theta z} = \sigma_{r\theta} = 0 \tag{36}$$

The solution of equation (28) is given in following form [7]:

$$w(t,r) = \sum_{i=1}^{\infty} C_i(t) \left[\left(J_1(\delta_i r) + \frac{k_c}{k_t} J_0(\delta_i r) \right) - \left(J_1(\delta_i a) + \frac{k_c}{k_t} J_0(\delta_i a) \right) \right] \tag{37}$$

Applying the operator $w(r)$ in above equation and using the following result in [7] as:

$$\nabla^2 \left(J_1(\delta_i r) + \frac{k_c}{k_t} J_0(\delta_i r) \right) = -\delta_i^2 \left(J_1(\delta_i r) + \frac{k_c}{k_t} J_0(\delta_i r) \right) \tag{38}$$

It is obtained that:

$$\nabla^2 \nabla^2 w = \sum_{i=1}^{\infty} C_i \delta_i^4 \left(J_1(\delta_i r) + \frac{k_c}{k_t} J_0(\delta_i r) \right) \tag{39}$$

The thermal moment could be obtained by substituting equation (27) into equation (32)

$$M_T = C_M \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} K_{ij} (t^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_{i,2j+1}^2 t^{\alpha})) \left(J_1(\delta_i r) + \frac{k_c}{k_t} J_0(\delta_i r) \right) \tag{40}$$

where

$$C_M = \frac{\alpha_t r_c E(hk_c + 2k_t)}{k_t^2}, \ K_{ij} = \frac{\left(J_1(\delta_i r_c) + \frac{k_c}{k_t} J_0(\delta_i r_c) \right) \sin(\beta_{2j+1} z_c) \sin(\beta_{2j+1} h/2)}{\lambda_{i,2j+1}^2 \beta_{i,2j+1}^2 \eta_{i,2j+1}},$$

for $z_c = h/2, K_{ij} = \frac{\left(J_1(\delta_i r_c) + \frac{k_c}{k_t} J_0(\delta_i r_c) \right)}{\lambda_{i,2j+1}^2 \beta_{i,2j+1}^2 \eta_{i,2j+1}}$. Then,

$$\nabla^2 M_T = -C_M \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} K_{ij} \delta_i^2 (t^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_{i,2j+1}^2 t^{\alpha})) \left(J_1(\delta_i r) + \frac{k_c}{k_t} J_0(\delta_i r) \right) \tag{41}$$

Substituting equations (41) and (39) into equation (28) yields

$$-D(1-\nu) \sum_{i=1}^{\infty} C_i \delta_i^4 J_0(\delta_i r) = -C_M \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} K_{ij} \delta_i^2 (t^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_{i,2j+1}^2 t^{\alpha})) \times \left(J_1(\delta_i r) + \frac{k_c}{k_t} J_0(\delta_i r) \right) \tag{42}$$

where C_i is obtained as follows:

$$C_i(t) = \frac{C_M}{D(1-\nu)} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{K_{ij}}{\delta_i^2} (t^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_{i,2j+1}^2 t^{\alpha})) \tag{43}$$

and the thermal deflection is given as follows:

$$w(r,t) = \frac{C_M}{D(1-\nu)} \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{K_{ij}}{\delta_i^2}(t^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_{i,2j+1}^2 t^{\alpha})) \left[\left(J_1(\delta_i r) + \frac{k_c}{k_t}J_0(\delta_i r)\right) - \left(J_1(\delta_i a) + \frac{k_c}{k_t}J_0(\delta_i a)\right)\right] \tag{44}$$

the temperature in equation (27) is represented in the following form:

$$\triangle T = \sum_{i=1}^{\infty} F(z,t)\left(J_1(\delta_i r) + \frac{k_c}{k_t}J_0(\delta_i r)\right) \tag{45}$$

where

$$\begin{aligned} F(z,t) = \frac{r_c}{k_t}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(J_1(\delta_i r_c) + \frac{k_c}{k_t}J_0(\delta_i r_c)\right)&\left[\frac{t^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_{i,2j}^2 t^{\alpha})}{\lambda_{i,2j}^2\eta_{i,2j}}\cos(\beta_{2j}z_c)\cos(\beta_{2j}z)\right. \\ &\left. + \frac{t^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_{i,2j+1}^2 t^{\alpha})}{\lambda_{i,2j+1}^2\eta_{i,2j+1}}\sin(\beta_{2j+1}z_c)\sin(\beta_{2j+1}z)\right] \end{aligned} \tag{46}$$

Then substituting equation (45) into equation (29) we get:

$$\nabla^2\zeta = (1+\nu)\alpha_t \sum_{i=1}^{\infty} F(z,t)\left(J_1(\delta_i r) + \frac{k_c}{k_t}J_0(\delta_i r)\right) \tag{47}$$

$$\zeta = -(1+\nu)\alpha_t \sum_{i=1}^{\infty}\frac{F(z,t)}{\delta_i^2}\left[\left(J_1(\delta_i r) + \frac{k_c}{k_t}J_0(\delta_i r)\right) - \left(J_1(\delta_i a) + \frac{k_c}{k_t}J_0(\delta_i a)\right)\right] \tag{48}$$

Substituting equation (48) into (34) and (35) we obtain:

$$\sigma_{rr} = -2\mu(1+\nu)\alpha_t \sum_{i=1}^{\infty}\frac{F(z,t)}{2r\delta_i}\left(J_0(\delta_i r) - J_2(\delta_i r) - \frac{k_c}{k_t}J_1(\delta_i r)\right) \tag{49}$$

$$\sigma_{\theta\theta} = 2\mu(1+\nu)\alpha_t \sum_{i=1}^{\infty}\frac{F(z,t)}{4}\left(3J_1(\delta_i r) + J_3(\delta_i r) + \frac{k_c}{k_t}\left(J_0(\delta_i r) + J_1(\delta_i r)\right)\right) \tag{50}$$

5 Numerical Calculations and Discussion

Dimension

The constants associated with the numerical calculation are taken as
Radius of a circular plate $a = 1$ m,
Thickness of circular plate $h = 0.2$ m.

The copper material was chosen for purpose of numerical calculation for a thin circular plate. The numerical calculations and graphs are plotted with the help of computational mathematical software PTC Mathcad.

In this paper, we have obtained solutions to time fractional heat conduction equation with the Caputo time fractional derivative. The numerical calculation are carried out according to the values of parameter α reflecting the characteristic features of the solution for various order of the time-fractional derivative. There distinguishing values of the parameter α are considered, $0 < \alpha < 1$, $\alpha = 1$ and $1 < \alpha < 2$ depicting weak, normal and strong conductivity.

Figure 2-5 indicates the variation of temperature, thermal deflection and thermal stresses in radial direction $r = 1$ at instants $\alpha = 0.50$ for time $t = 100s, 200s, 300s, 400s$. Figure 2 shows the variation of temperature in radial direction for the different time parameter. Due to the axisymmetric heat source temperature will increases in the region $0 \leq r \leq 0.25$ and it decreases within the region $0.25 \leq r \leq 1$. Figure 3 represent the thermal deflection for different time parameter $t = 100s, 200s, 300s, 400s$. Due to the internal heat source thermal deflection increases and it will be maximum at $r = 0.25$ and it decreases with increasing the radius r and it will become zero towards the outer circular edge $r = 1$.

Figure 4 and Figure 5 shows the radial stress distribution and angular stress distribution for different time parameters. It is observe that the radial and angular stresses decreases within the region $0 \leq r \leq 0.3$ and it increases within the region $0.3 \leq r \leq 1$ and it will be maximum at $r = 1$.

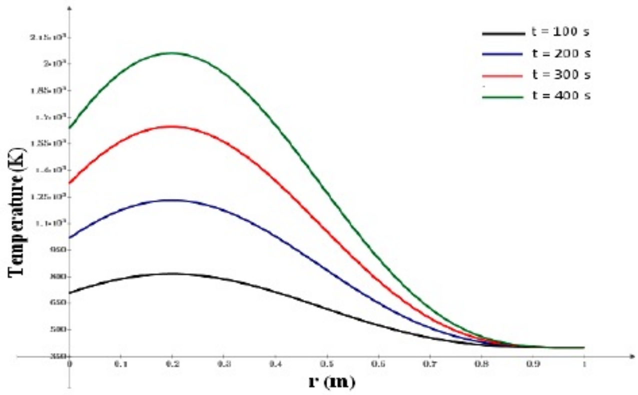

Figure 2: Temperature distribution at $\alpha = 0.5$ and different values of t.

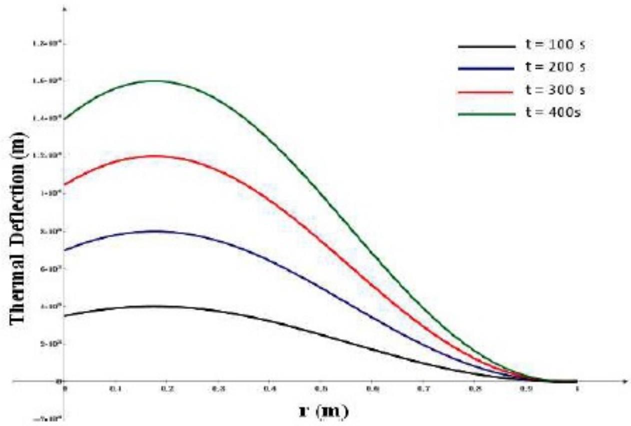

Figure 3: Thermal Deflection at $\alpha = 0.5$ and different values of t.

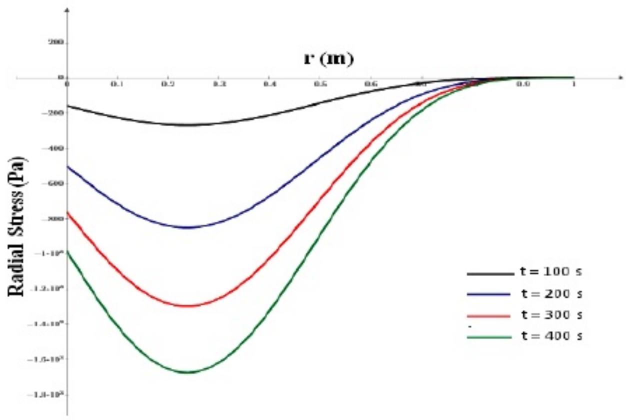

Figure 4: Radial stress distribution at $\alpha = 0.5$ and different values of t.

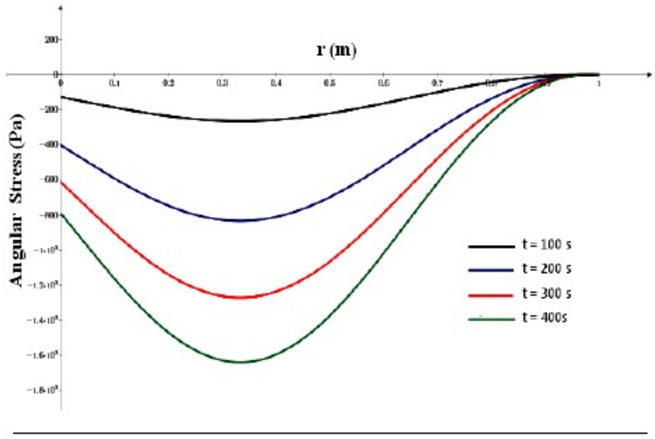

Figure 5: Angular stress distribution at $\alpha = 0.5$ and different values of t.

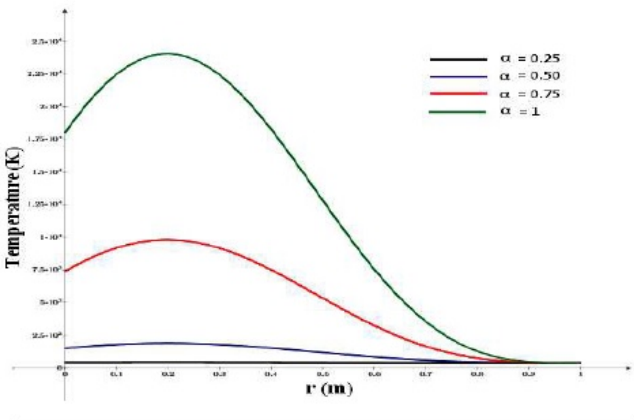

Figure 6: Temperature distribution at $t = 100$s and different values of α.

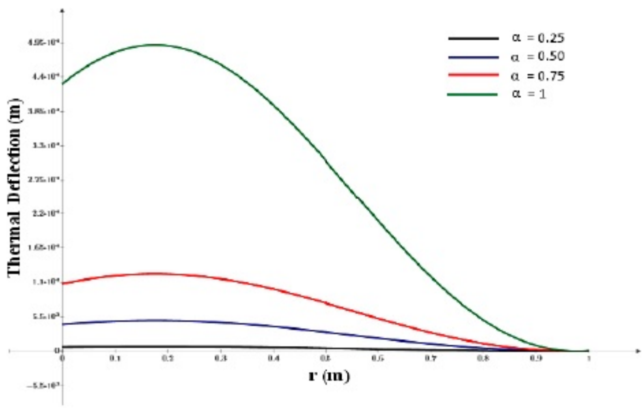

Figure 7: Thermal Deflection at $t = 100$s and different values of α.

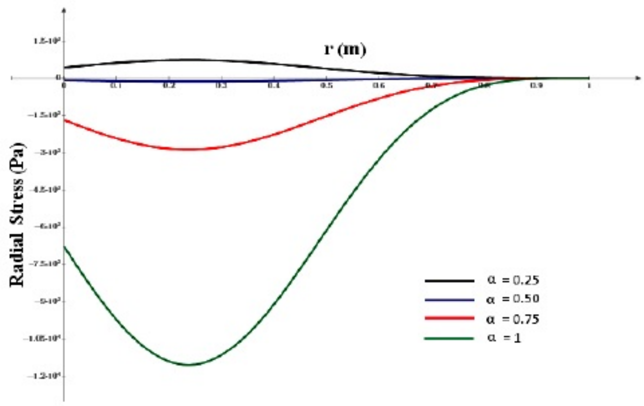

Figure 8: Radial stress distribution at t = 100s and different values of α.

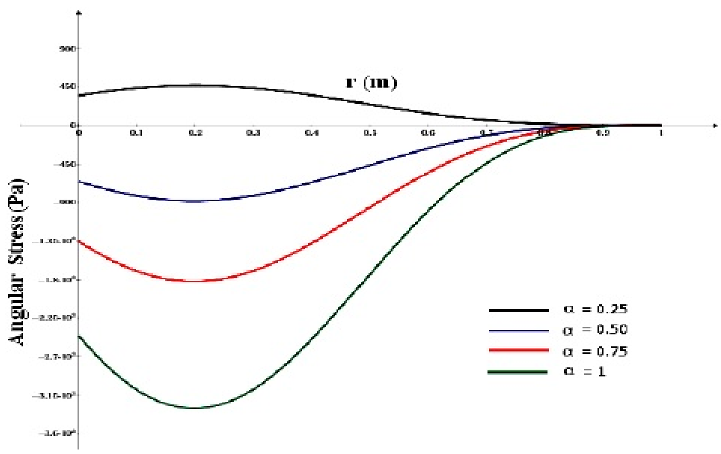

Figure 9: Angular stress distribution at t = 100s and different values of α.

Table 1: Material Constant for a thin circular plate

Physical constant	Value
Ambient Temperature (T_∞)	500 (K)
Thermal conductivity (k_t)	386 W/(m.K)
Heat transfer coefficient(k_c)	15 W/(m^2.K)
Thermal diffusivity (k)	84.18 m^2/s
Density (ρ)	8954 kg/m^3
Specific heat (c)	383.1 J/(kg.K)
Lame's constants (λ)	7.76×10^{10} kg/m.s^2
Lame's constants (μ)	3.86×10^{10} kg/m.s^2
Coefficient of linear thermal expansion (α_t)	16.5×10^{-6} /K
Young's modulus (E)	128 GPa
Poisson ratio (ν)	0.35
Instantaneous line heat source (g_1)	1×10^4 W/m^2

Figure 6–9 shows the variations of temperature, thermal deflection and thermal stresses in radial direction $r = 1$ at instants $t = 100s$ for the different values of fractional order parameter $\alpha = 0.25, 0.50, 0.75, 1$. From figure 6 indicates the temperature distribution in the radial direction. We can see that the behaviour of temperature distribution represents the normal conductivity for fractional order parameter $\alpha = 1$. It is clear that the fractional-order parameters have a strong influence on the physical quantities. It has an increasing effect (in terms of magnitude) on profiles of temperature.

Figure 7 depicts the thermal deflection in radial direction for different values of α at $t = 100$. We observe that, the thermal deflection shows the week conductivity $0 < \alpha < 1$ and normal conductivity for $\alpha = 1$. Figure 8 and Figure 9 shows the radial stress distribution and angular stress distribution for different fractional order parameter $\alpha = 0.25, 0.50, 0.75, 1.$ at $t = 100s$ in the radial direction. It is clear that the radial stress distribution and angular stress distribution represents the week conductivity $0 < \alpha < 1$ and normal conductivity for $\alpha = 1$.

6 Conclusion

In the present work, the Green's function approach and method of separation of variable have been utilized to analyze the temperature, thermal deflection, and thermal stresses for a thin circular plate under an axi-symmetric heat source using a thermoelasticity theory based on fractional order heat conduction with the Caputo time-fractional derivative. The obtained results were evaluated to investigate the convective heat transfer coefficient as well as velocity and temperature for various physical parameters.

The important findings are summarized as follows:

- The temperature, thermal deflection, and thermal stresses vary in the radial direction $r = 1$ at instants $\alpha = 0.50$ for time $t = 100s, 200s, 300s, 400s$ as shown in the Figure 2-5.
- Figures 6–9 show the significant influence of the fractional order derivative on the temperature, thermal deflection and thermal stress.
- The parameter α represents the weak, normal and strong conductivity, within the range of $0 < \alpha < 1$, $\alpha = 1$ and $1 < \alpha < 2$ respectively.
- The results presented here will be useful in studying the thermal characteristics of various bodies in real-life engineering problems, mathematical biology by considering the time fractional derivative in the field equations.

To further study for this problem, consider a thick circular plate with axi-symmetric heat supply will be suggested. Varying the boundary condition to a variable heat flux also leads to a new inspiration to continue the current work.

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