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Analytical solutions of the nonlinear Ivancevic options pricing model

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Abstract

This paper studies the nonlinear quantum-probability based Schrödinger type, Ivancevic options pricing model using the method of Lie symmetries to determine its point symmetries, invariant analytical solutions and conversation laws. In our analysis, we consider a non-zero and zero adaptive market potential model. We demonstrate that this model is invariant under a five-dimensional Lie algebra for the former, and invariant under a seven-dimensional Lie algebra for the latter case. These symmetries allow for a progressive reduction of the equation and thus facilitate a solution. We obtain reductions, exact solutions and conservation laws for both the non-zero and zero adaptive market potential models. We show that many exact solutions are expressible in terms of two transcendental functions, the Fresnel sine and cosine integrals. Graphical solutions are provided in certain cases. This analysis and solutions to such a financial derivatives pricing model are unique, providing novel insights.

Keywords: Lie symmetries, Exact solutions, Schrödinger equation, Conservation Laws
2020 MSC: 34A05, 70H33, 91G50

1 Introduction

In the 1970's, F. Black, M. Scholes and R.C. Merton introduced a mathematical method to price complex financial instruments [1], winning them a Nobel prize. This model is known as the Black-Scholes equation and is defined as follows.
$S = S(t)$ is the price function of the underlying asset at time t $(0 \leq t \leq T)$, that satisfies the stochastic differential equation describing geometric Brownian motion

$$dS = S(\mu dt + \sigma dW_t), \quad S \in [0, \infty), \tag{1.1}$$

where μ and σ are the drift parameter (in this case the rate of return of the asset price S) and volatility, respectively, and the standard Wiener process, W_t. The Black-Scholes partial differential equation associated with (1.1) for an option value $V = V(S,t)$ and risk-free interest rate r can be expressed as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1.2}$$

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with $V(0,t)=0,\ V(S,t)\to\infty$ as $S\to\infty$, $V(S,t)=max(S-E)$, E the strike price or exercise price is taken as a constant and the price function is

$$S(t)=S_0e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t},\ S_0=S(0). \tag{1.3}$$

This model is primarily used to price European call and put options [2]. The main assumption of the Black-Scholes model are that μ and σ are constant. That is, over the life of the derivative, the underlying volatility remains constant and is unaffected by changes in the underlying asset's price level. This constrains the model since it is unable to account for long-observed characteristics of the implied volatility surface such as volatility smile and skew, which indicate that implied volatility varies with strike price and expiration [3]. In relation to physics, (1.2) resembles the Fokker-Planck equation used to describe the Brownian motion of a particle in a fluid [4]. Additionally, using quantum mechanics [5] established that the Black-Scholes equation can be derived from the Schrödinger equation. Analytical solutions to this model has been extensively studied in literature. In [6], the Black-Scholes equation with a non-smooth boundary condition was transformed into the heat equation using symmetries to find an exact solution. Lie symmetry methods have been utilised in the fractional Black-Scholes equation to find invariant solutions and conservation laws [7]. Other methods to find analytical solutions to the Black-Scholes model can be found in [8, 9, 10, 11]. Because of the Black-Scholes equation's limitations as a result of its assumptions, there has been a push to develop models that loosen these assumptions and provide a more realistic picture of the markets.

The Ivancevic options pricing model (IOPM) is a nonlinear adaptive-wave model, describing controlled Brownian motion of financial markets, and a wave alternative to the standard Black-Scholes option-pricing model. The complex-valued equation, introduced by Vladimir G. Ivancevic [12], is defined by adaptive nonlinear Schrödinger (NLS) equations, in which the option-pricing wave function is expressed in terms of asset (stock) price and time. The wave function representing quantum probability amplitude, when taking its absolute square gives a probability density function. The IOPM is

$$i\frac{\partial\Psi}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2\Psi}{\partial S^2}+\beta|\Psi|^2\Psi=0,\ i^2=-1, \tag{1.4}$$

where $\Psi=\Psi(S,t)$ denotes the option pricing wave-function at time t, $|\Psi|^2=|\Psi(S,t)|^2$ represents the probability density function (PDF) for the option price in terms of stock price and time, σ represents a constant or stochastic process as the dispersion frequency volatility coefficient, and β is the Landau coefficient representing adaptive market potential. This adaptive market potential in its simplest non-adaptive form is equal to the interest rate r. For the adaptive scenario, on the other hand, $\beta=\beta(r,w)$ depends on the set of adjustable parameters $\{w_j\}$. In this adaptive case, the Landau coefficient can be linked to market temperature. We emphasize the significance of this model for understanding financial derivatives markets from a mathematical and physical standpoint.

The use of Lie symmetry methods to study partial differential equations (PDEs) is highly effective and a plethora of work can be found in the literature [13, 14, 15, 16, 17, 18, 19, 20, 21]. Conservation laws from a physical point of view states that when a physical system changes, a quantifiable amount of the system remains constant. It can be used for linearisation and analysis of solutions of a PDE. In fact, Emmy Noether discovered that symmetries and conservation laws are linked where the existence of a one implies the other. Noether symmetries have been particularly useful in cosmology [22, 23, 24, 25]. Very few works on the exact solutions to the IOPM exist in literature. In [26], the He's Frequency Amplitude Formulation was utilised, whereas in [27], authors used the projected differential transform method. The paper [28] constructed dark wave, rogue wave and perturbation solutions via the trial function method. Solutions to the time-fractional IOPM was discussed in [3]. In our analysis we consider the deterministic form of the IOPM model with a constant volatility. We present novel invariant solutions for both the adaptive and non-adaptive market potentials. Where appropriate, we provide plots of these solutions. Finally, conservation laws for both cases are determined.

2 Symmetries and invariant solutions for the IOPM with a non-zero adaptive market potential

In this section we consider a non-zero adaptive market potential i.e. $\beta\neq 0$. We start by splitting the complex equation (1.4) into its real and imaginary counterparts by taking $\Psi(S,t)=u(S,t)+iv(S,t)$, obtaining the system:

$$-\frac{\partial}{\partial t}v(S,t)+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial S^2}u(S,t)+\beta\ (u(S,t))^3+\beta\, u(S,t)\,(v(S,t))^2=0, \tag{2.1}$$

$$\frac{\partial}{\partial t}u(S,t)+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial S^2}v(S,t)+\beta\ (u(S,t))^2\,v(S,t)+\beta\ (v(S,t))^3=0. \tag{2.2}$$

The reduction of the number of independent variables in PDEs, is the subject of this paper. The reductions can be used to obtain solutions known as invariant solutions. The value of symmetries is that they allow one to connect sophisticated equations or systems to simpler ones that may potentially be solved.

The IOPM system, (2.1)-(2.2), is invariant under a five-dimensional Lie algebra consisting of the symmetry vector fields, as determined by Lie theory:

$$X_1 = \frac{\partial}{\partial S}, \tag{2.3}$$

$$X_2 = \frac{\partial}{\partial t}, \tag{2.4}$$

$$X_3 = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}, \tag{2.5}$$

$$X_4 = \sigma^2 t\frac{\partial}{\partial S} + Su\frac{\partial}{\partial v} - Sv\frac{\partial}{\partial u}, \tag{2.6}$$

$$X_5 = -S\frac{\partial}{\partial S} - 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{2.7}$$

The commutators of admitted point symmetries of the IOPM with a non-zero adaptive market potential are given in Table 1.

[,]	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	X_3	$-X_1$.
X_2	0	0	0	$\sigma^2 X_1$	$-2X_2$
X_3	0	0	0	0	0
X_4	X_3	$\sigma^2 X_1$	0	0	X_4
X_5	$-X_1$	$-2X_2$	0	X_4	0

Table 1: Commutator table for $\beta \neq 0$.

2.1 Conservation laws for $\beta \neq 0$

Now we construct conservation laws for the IOPM using the multiplier approach. Multipliers, Λ, are integrating factors, satisfying the relation

$$D_i T^i = \Lambda G, \tag{2.8}$$

where Λ is derived by equations satisfying

$$E(\Lambda G) = 0, \tag{2.9}$$

where E is the Euler operator. Now having the multipliers, they give rise to a conserved vector $T = (T^1, \ldots.., T^n)$ that satisfies the conservation relation

$$D_i T^i = 0, \tag{2.10}$$

along the solutions of a given equation. Suppose that the system (2.1)-(2.2) admits the multipliers of the form

$$\Lambda_1(S, t, u, v, v_S, u_S, v_{SS}, u_{SS}), \tag{2.11}$$

and

$$\Lambda_2(S, t, u, v, v_S, u_S, v_{SS}, u_{SS}), \tag{2.12}$$

that correspond to the conservation laws composed of T^t, the conserved density, and T^S, the conserved flux.

The system (2.1)-(2.2) with a non-zero adaptive market potential, $\beta \neq 0$, relating to the IOPM (1.4) admits the multipliers:

$$\Lambda_1 = \frac{\sigma^2 t u_S + Sv}{\sigma^2}, \tag{2.13}$$

$$\Lambda_2 = -\frac{-\sigma^2 t v_S + Su}{\sigma^2}. \tag{2.14}$$

These multipliers give rise to the conservation law vector components T_1^S and T^t:

$$\begin{aligned}
T_1^S &= \frac{1}{4}\beta u^4 t + \frac{1}{2}\beta t u^2 v^2 + \frac{1}{4}\beta t v^4 - \frac{1}{2}S u v_S + \frac{1}{4}u_S^2\sigma^2 t + \frac{1}{2}S u_S v + \frac{1}{4}\sigma^2 t v_S^2 - \frac{1}{2}t u v_t + \frac{1}{2}t u_t v,\\
T_1^t &= -\frac{1}{2}\frac{-\sigma^2 t u v_S + \sigma^2 t u_S v + S u^2 + S v^2}{\sigma^2}.
\end{aligned}$$

The above conservation laws can indeed be written in terms of the original variable Ψ:

$$\begin{aligned}
T_1^S &= \frac{1}{4}\beta t(|\Psi|^2)^2 + \frac{1}{4}\sigma^2 t\Psi_S\overline{\Psi}_S - \frac{1}{4}iS(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S) - \frac{1}{4}it(\Psi\overline{\Psi}_t - \overline{\Psi}\Psi_t),\\
T_1^t &= -\frac{1}{4\sigma^2}(ti\sigma^2(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S) + 2S|\Psi|^2),
\end{aligned}$$

where $\overline{\Psi}$ is the conjugate of Ψ.

The multipliers $\Lambda_1 = u_S$, and $\Lambda_2 = v_S$, lead to the conservation laws

$$\begin{aligned}
T_2^S &= \frac{1}{4}\beta u^4 + \frac{1}{2}\beta u^2 v^2 + \frac{1}{4}\beta v^4 + \frac{1}{4}u_S^2\sigma^2 + \frac{1}{4}v_S^2\sigma^2 - \frac{1}{2}u v_t + \frac{1}{2}v u_t\\
&= \frac{1}{4}\beta(|\Psi|^2)^2 + \frac{1}{4}\sigma^2\Psi\overline{\Psi}_S - \frac{1}{4}i(\Psi\overline{\Psi}_t - \overline{\Psi}\Psi_t),
\end{aligned}$$

and

$$T_2^t = -\frac{1}{2}u_S v + \frac{1}{2}u v_S = \frac{1}{4}i(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S).$$

Moreover, the admitted integrating factors that satisfy (2.8) and (2.9) are

$$\Lambda_1 = -\frac{2\beta u^2 v + 2\beta v^3 + \sigma^2 v_{SS}}{\sigma^2}, \tag{2.15}$$

$$\Lambda_2 = \frac{2\beta u^3 + 2\beta u v^2 + \sigma^2 u_{SS}}{\sigma^2}, \tag{2.16}$$

with corresponding conservation laws

$$\begin{aligned}
T_3^S &= -\frac{1}{2}v v_{St} + \frac{1}{2}v_S v_t - \frac{1}{2}u u_{St} + \frac{1}{2}u_S u_t = -\frac{1}{4}(\Psi\overline{\Psi}_{St} + \overline{\Psi}\Psi_{St}) + \frac{1}{4}(\Psi_S\overline{\Psi}_t + \overline{\Psi}_S\Psi_t),\\
T_3^t &= \frac{1}{2}\frac{\beta u^4 + 2\beta u^2 v^2 + \beta v^4 + \sigma^2 u u_{SS} + \sigma^2 v v_{SS}}{\sigma^2} = \frac{1}{4\sigma^2}(\sigma^2(\Psi\overline{\Psi}_{SS} + \overline{\Psi}\Psi_{SS}) + 2\beta(|\Psi|^2)^2).
\end{aligned}$$

The last pair of multipliers for our system with $\beta \neq 0$ are $\Lambda_1 = -v$,$\Lambda_2 = u$. As a result, we obtain the conservation laws

$$T_4^S = \frac{1}{2}u v_S\sigma^2 - \frac{1}{2}\sigma^2 u_S v = \frac{1}{4}\sigma^2 i(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S), \qquad T_4^t = \frac{1}{2}v^2 + \frac{1}{2}u^2 = \frac{1}{2}(|\Psi|^2).$$

We now construct some exact solutions to the investigated system using the acquired symmetries. The terms C_i denote arbitrary constants. Additionally, since $\Psi(S,t) = u(S,t) + iv(S,t)$, one may easily recover the solution to the IOPM equation (1.4), from the solutions of $u(S,t)$ and $v(S,t)$ obtained.

2.2 Reductions and invariant solutions for $\beta \neq 0$

A reduction with X_1 leads to the reduce system

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}K(t) + \beta\,(K(t))^2 W(t) + \beta\,(W(t))^3 = 0,\\
&-\frac{\mathrm{d}}{\mathrm{d}t}W(t) + \beta\,(K(t))^3 + \beta K(t)(W(t))^2 = 0,
\end{aligned}$$

where $u(S,t) = K(t)$ and $v(S,t) = W(t)$. The above system solves to give us a trivial (constant) solution.

A reduction using X_2 gives us a non-trivial solution of

$$\begin{aligned} u\left(S,t\right) &= 0, \\ v\left(S,t\right) &= C_2\, JacobiSN\left(\left(\sqrt{\beta}S + C_1\right)C_2, i\right), \end{aligned} \tag{2.17}$$

where the $JacobiSN$ is the inverse of elliptic integrals and doubly periodic elliptic functions. We plot the above solution in Figure 1 for different values of β.

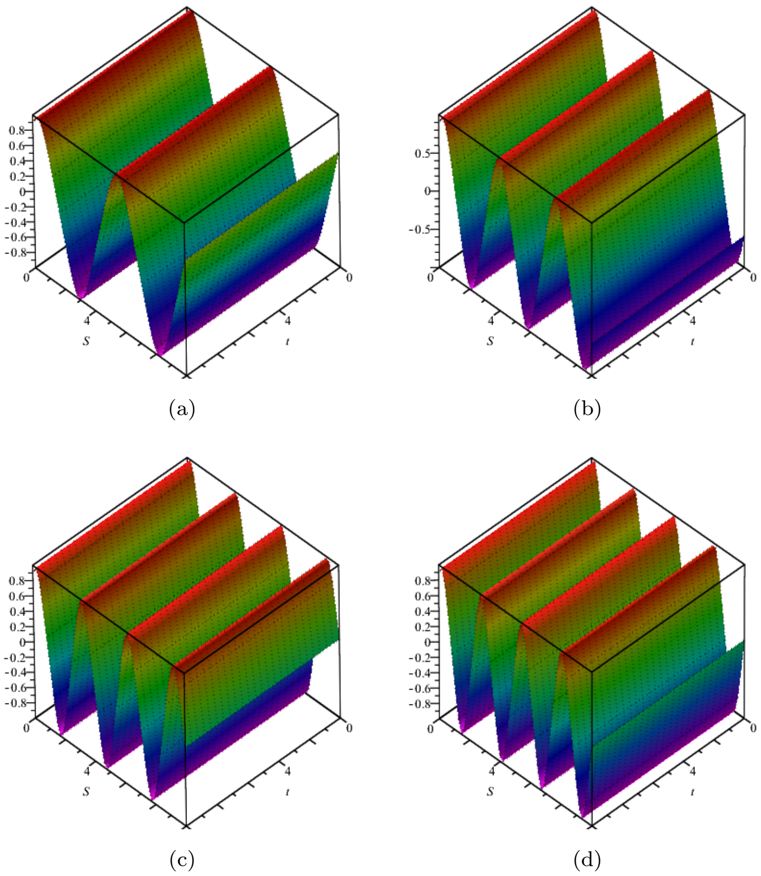


Figure 1: Graphical illustration of the analytical solutions are depicted, we select the parameter values $C_1 = C_2 = 1$ and the range $0 \leq S \leq 10$: (a) solution of (2.17) where $\beta = 1.0$; (b) solution of (2.17) where $\beta = 2.0$; (c) solution of (2.17) where $\beta = 3.0$; and, (d) solution of (2.17) where $\beta = 4.0$.

Lastly, reductions with X_5 yields

$$2\sigma^2\left(\frac{d^2}{d\frac{t}{S^2}^2}\kappa\left(\frac{t}{S^2}\right)\right)\frac{t}{S^2}^2 + \beta\,\kappa\left(\frac{t}{S^2}\right)\left(\Omega\left(\frac{t}{S^2}\right)\right)^2 + \beta\,\left(\kappa\left(\frac{t}{S^2}\right)\right)^3 + 5\,\sigma^2\left(\frac{d}{d\frac{t}{S^2}}\kappa\left(\frac{t}{S^2}\right)\right)\frac{t}{S^2} + \sigma^2\kappa\left(\frac{t}{S^2}\right) - \frac{d}{d\frac{t}{S^2}}\Omega\left(\frac{t}{S^2}\right) = 0,$$

$$2\sigma^2\left(\frac{d^2}{d\frac{t}{S^2}^2}\Omega\left(\frac{t}{S^2}\right)\right)\frac{t}{S^2}^2 + \beta\,\left(\Omega\left(\frac{t}{S^2}\right)\right)^3 + \beta\,\left(\kappa\left(\frac{t}{S^2}\right)\right)^2\Omega\left(\frac{t}{S^2}\right) + 5\,\sigma^2\left(\frac{d}{d\frac{t}{S^2}}\Omega\left(\frac{t}{S^2}\right)\right)\frac{t}{S^2} + \sigma^2\Omega\left(\frac{t}{S^2}\right) + \frac{d}{d\frac{t}{S^2}}\kappa\left(\frac{t}{S^2}\right) = 0.$$

In the above system $u(S,t) = \kappa(\frac{t}{S^2})$ and $v(S,t) = \Omega(\frac{t}{S^2})$, where unfortunately an analytical solution is not found for this reduced system.

3 Symmetries and invariant solutions for the IOPM with a zero adaptive market potential

In the special case where we do not consider the adaptive market potential i.e. $\beta = 0$, thus (1.4) becomes linear.

We use Lie's theory and find that under the action of the following seven-dimensional point transformation with generators given below, the IOPM system with $\beta = 0$ is invariant:

$$X_1, X_2, X_3, X_4, \tag{3.1}$$
$$Y_1 = \frac{1}{2}S\frac{\partial}{\partial S} + t\frac{\partial}{\partial t}, \tag{3.2}$$
$$Y_2 = 2\sigma^2 St\frac{\partial}{\partial S} + 2\sigma^2 t^2\frac{\partial}{\partial t} + (-\sigma^2 tu - S^2 v)\frac{\partial}{\partial u} + (-\sigma^2 tv + S^2 u)\frac{\partial}{\partial v}, \tag{3.3}$$
$$Y_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{3.4}$$

The commutators of the admitted Lie point symmetries of system (2.1)-(2.2), given a zero adaptive market potential are presented in Table 2.

[,]	X_1	X_2	X_3	X_4
Y_1	$\frac{1}{2}X_1$	X_2	0	$-\frac{1}{2}X_4$
Y_2	$2X_4$	$2\sigma^2 SX_1 + 4\sigma^2 tX_2 - \sigma^2 Y_3$	0	0
Y_3	0	0	0	0

Table 2: Commutator table for $\beta = 0$.

3.1 Conservation laws for $\beta = 0$

Employing the same mathematical method as detailed in the preceding section, we derive the conservation laws for our system (2.1)-(2.2), now using $\beta = 0$. This case is rich in conservation laws, possessing eight of them.

The first two multipliers derived are

$$\Lambda_1 = \frac{1}{2}v_{SS}\sigma^4 t^2 - \sigma^2 tSu_S - \frac{1}{2}\sigma^2 tu - \frac{1}{2}vS^2, \tag{3.5}$$
$$\Lambda_2 = -\frac{1}{2}u_{SS}\sigma^4 t^2 - \sigma^2 tSv_S - \frac{1}{2}\sigma^2 tv + \frac{1}{2}uS^2, \tag{3.6}$$

from which the conservation laws include

$$\begin{aligned} T_1^S =& \frac{1}{2}utS\sigma^2 v_t - \frac{1}{2}\sigma^2 vtSu_t - \frac{1}{4}v_S\sigma^4 t^2 v_t + \frac{1}{4}u\sigma^4 t^2 u_{St} - \frac{1}{4}u_S\sigma^4 t^2 u_t - \frac{1}{4}S\sigma^4 tv_S^2 + \frac{1}{4}u\sigma^4 tu_S + \frac{1}{4}uS^2\sigma^2 v_S \\ & -\frac{1}{4}u_S^2\sigma^4 tS - \frac{1}{4}u_S\sigma^2 vS^2 + \frac{1}{4}v\sigma^4 tv_S + \frac{1}{4}v\sigma^4 t^2 v_{St} \\ =& \frac{1}{4}S\sigma^2 it(\Psi\overline{\Psi}_t - \overline{\Psi}\Psi_t) - \frac{1}{4}\sigma^4 t^2\frac{1}{2}(\Psi_S\overline{\Psi}_t + \overline{\Psi}_S\Psi_t) + \frac{1}{4}\sigma^4 t^2\frac{1}{2}(\Psi\overline{\Psi}_{St} + \overline{\Psi}\Psi_{St}) + \frac{1}{4}\sigma^4 t\frac{1}{2}(\Psi\overline{\Psi}_S + \overline{\Psi}\Psi_S) \\ & -\frac{1}{4}\sigma^4 tS\Psi_S\overline{\Psi}_S + \frac{1}{4}\sigma^2 S^2\frac{1}{2}i(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S), \end{aligned}$$

and

$$\begin{aligned} T_1^t =& -\frac{1}{4}vv_{SS}\sigma^4 t^2 + \frac{1}{2}v\sigma^2 tSu_S + \frac{1}{4}v^2 S^2 - \frac{1}{4}\sigma^4 t^2 uu_{SS} - \frac{1}{2}S\sigma^2 tuv_S + \frac{1}{4}u^2 S^2 \\ =& -\frac{1}{4}\sigma^4 t^2(\frac{1}{2}(\Psi\overline{\Psi}_{SS} + \overline{\Psi}\Psi_{SS}) - \frac{1}{4}S\sigma^2 ti(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S) + \frac{1}{4}2S^2|\Psi|^2. \end{aligned}$$

Similarly, we obtained the multipliers

$$\Lambda_1 = -\sigma^2 tu_S - Sv, \tag{3.7}$$
$$\Lambda_2 = -\sigma^2 tv_S + Su, \tag{3.8}$$

with conservation laws

$$\begin{aligned}
T_2^S &= \frac{1}{2}uS\sigma^2 v_S - \frac{1}{4}\sigma^4 t u_S^2 - \frac{1}{2}vS\sigma^2 u_S - \frac{1}{4}\sigma^4 t v_S^2 + \frac{1}{2}\sigma^2 tuv_t - \frac{1}{2}\sigma^2 tvu_t \\
&= -\frac{1}{4}\sigma^4 t\Psi_S\overline{\Psi}_S + \frac{1}{4}iS\sigma^2(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S) + \frac{1}{4}\sigma^2 it(\Psi\overline{\Psi}_t - \overline{\Psi}\Psi_t), \\
T_2^t &= \frac{1}{2}\sigma^2 tvu_S + \frac{1}{2}Sv^2 - \frac{1}{2}\sigma^2 tuv_S + \frac{1}{2}Su^2 = -\frac{1}{4}\sigma^2 ti(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S) + \frac{1}{2}S|\Psi|^2.
\end{aligned}$$

Further, we apply the multipliers $\Lambda_1 = -v$, $\Lambda_2 = u$, to acquire the conservation laws

$$T_3^S = \frac{1}{2}uv_S\sigma^2 - \frac{1}{2}\sigma^2 u_S v = \frac{1}{4}\sigma^2 i(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S), \qquad T_3^t = \frac{1}{2}v^2 + \frac{1}{2}u^2 = \frac{1}{2}(|\Psi|^2).$$

We deduce the multipliers

$$\Lambda_1 = -2\sigma^2 tv_{SS} + 2Su_S + u, \tag{3.9}$$
$$\Lambda_2 = 2\sigma^2 tu_{SS} + 2Sv_S + v, \tag{3.10}$$

from which we derive the conservation laws and express them in terms of Ψ:

$$\begin{aligned}
T_4^S &= -\frac{1}{2}uu_S\sigma^2 + \frac{1}{2}S\sigma^2 u_S^2 - \frac{1}{2}vv_S\sigma^2 + \frac{1}{2}S\sigma^2 v_S^2 - uSv_t - \sigma^2 tvv_{St} + tv_S\sigma^2 v_t - \sigma^2 tuu_{St} + tu_S\sigma^2 u_t + vSu_t \\
&= -\frac{1}{4}\sigma^2(\Psi\overline{\Psi}_S + \overline{\Psi}\Psi_S) + \frac{1}{2}S\sigma^2\Psi\overline{\Psi}_S - \frac{1}{2}\sigma^2 t(\Psi\overline{\Psi}_{St} + \overline{\Psi}\Psi_{St}) - \frac{1}{2}iS(\Psi\overline{\Psi}_t - \overline{\Psi}\Psi_t) + \frac{1}{2}\sigma^2 t(\Psi_S\overline{\Psi}_t + \overline{\Psi}_S\Psi_t),
\end{aligned}$$

and

$$T_4^t = \sigma^2 tuu_{SS} + \sigma^2 tvv_{SS} + Suv_S - Su_S v = \frac{1}{2}\sigma^2 t(\Psi\overline{\Psi}_{SS} + \overline{\Psi}\Psi_{SS}) + \frac{1}{2}Si(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S).$$

The admitted multipliers also include $\Lambda_1 = u_S$, $\Lambda_2 = v_S$, leading to the corresponding conservation law

$$\begin{aligned}
T_5^S &= \frac{1}{4}u_S^2\sigma^2 + \frac{1}{4}\sigma^2 v_S^2 - \frac{1}{2}uv_t + \frac{1}{2}vu_t = \frac{1}{4}\sigma^4\Psi_S\overline{\Psi}_S - \frac{1}{4}i(\Psi\overline{\Psi}_t - \overline{\Psi}\Psi_t), \\
T_5^t &= -\frac{1}{2}vu_S + \frac{1}{2}uv_S = \frac{1}{4}i(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S),
\end{aligned}$$

followed by $\Lambda_1 = -v_{SS}$, and $\Lambda_2 = u_{SS}$, to get the conservation law

$$\begin{aligned}
T_6^S &= -\frac{1}{2}vv_{St} + \frac{1}{2}v_S v_t - \frac{1}{2}uu_{St} + \frac{1}{2}u_S u_t = -\frac{1}{4}(\Psi\overline{\Psi}_{St} + \overline{\Psi}\Psi_{St}) + \frac{1}{4}(\Psi_S\overline{\Psi}_t + \overline{\Psi}_S\Psi_t), \\
T_6^t &= \frac{1}{2}vv_{SS} + \frac{1}{2}uu_{SS} = \frac{1}{4}(\Psi\overline{\Psi}_{SS} + \overline{\Psi}\Psi_{SS}).
\end{aligned}$$

The multipliers $\Lambda_1 = S$, $\Lambda_2 = 0$, yield conservation laws

$$T_7^S = -\frac{1}{2}u\sigma^2 + \frac{1}{2}S\sigma^2 u_S = -\frac{1}{4}\sigma^2(\Psi + \overline{\Psi}) + \frac{1}{4}S\sigma^2(\Psi_S + \overline{\Psi}_S), \qquad T_7^t = -vS = \frac{1}{2}S(i\Psi - i\overline{\Psi}),$$

and the multipliers $\Lambda_1 = 1$, $\Lambda_2 = 0$, lead to the conservation law

$$T_8^S = \frac{1}{2}u_S\sigma^2 = \frac{1}{4}\sigma^2(\Psi_S + \overline{\Psi}_S), \qquad T_8^t = -v = \frac{1}{2}(i\Psi - i\overline{\Psi}).$$

3.2 Reductions and invariant solutions for $\beta = 0$

3.2.1 Case 1: Reduction for Y_1

We obtain the following exact solution by applying Lie symmetry vector Y_1:

$$u(S,t) = C_2 + C_3\, FresnelS\left(\frac{S}{\sigma\sqrt{\pi t}}\right) + C_4\, FresnelC\left(\frac{S}{\sigma\sqrt{\pi t}}\right), \tag{3.11}$$

$$v(S,t) = -FresnelC\left(\frac{S}{\sigma\sqrt{\pi t}}\right) C_3 + FresnelS\left(\frac{S}{\sigma\sqrt{\pi t}}\right) C_4 + C_1, \tag{3.12}$$

where $FresnelS$ and $FresnelC$ are the Fresnel sine integral and the Fresnel cosine integral respectively. In Figure 2 we show the graphical solutions (3.11) and (3.12) for Case 1.

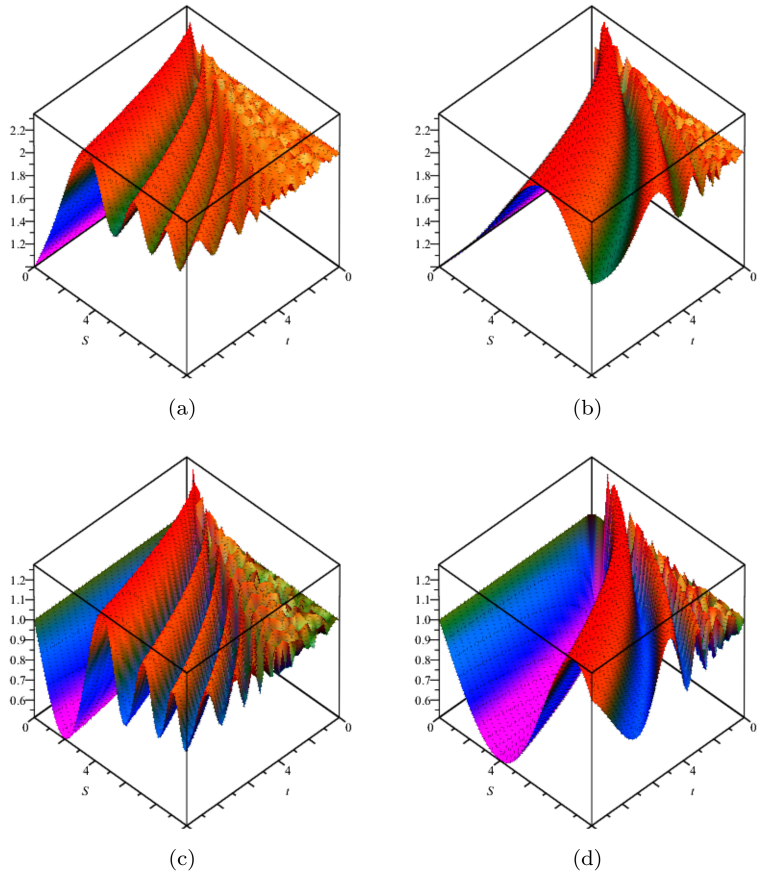


Figure 2: Graphical illustration of the analytical solutions are depicted, we select the parameter values $C_1 = C_2 = C_3 = C_4 = 1$ and the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$: (a) solution of (3.11) where $\sigma = 0.5$; (b) solution of (3.11) where $\sigma = 1.0$; (c) solution of (3.12) where $\sigma = 0.5$; and, (d) solution of (3.12) where $\sigma = 1.0$.

3.2.2 Case 2: Reduction via the linear combination $c_1X_1 + c_2X_2$

Reduction with X_1 and X_2, individually gives the solution for $\Psi(S,t)$ as a trivial (constant) solution. Instead, under this case we use the linear combinations of symmetries to get the analytical solutions

$$u(S,t) = C_2 + C_3 \sin\left(2\,\frac{c_1(Sc_2 - tc_1)}{{c_2}^2\sigma^2}\right) + C_4 \cos\left(2\,\frac{c_1(Sc_2 - tc_1)}{{c_2}^2\sigma^2}\right), \tag{3.13}$$

$$v(S,t) = \cos\left(2\,\frac{c_1(Sc_2 - tc_1)}{{c_2}^2\sigma^2}\right) C_3 - \sin\left(2\,\frac{c_1(Sc_2 - tc_1)}{{c_2}^2\sigma^2}\right) C_4 + C_1. \tag{3.14}$$

In Figure 3 we plot the solutions (3.13) and (3.14).

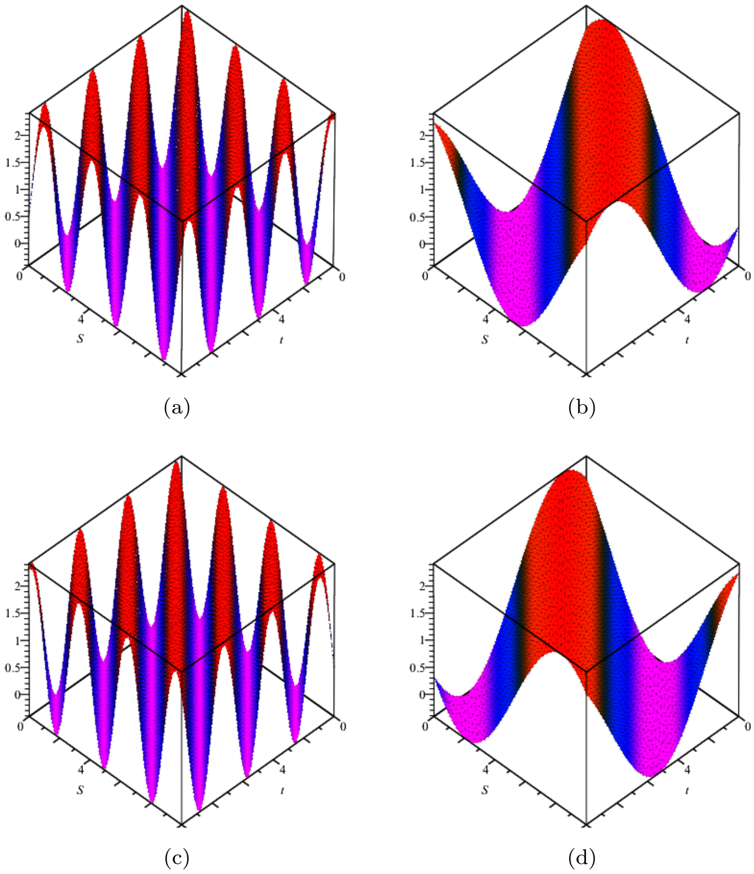


Figure 3: Graphical illustration of the analytical solutions are depicted, we select the parameter values $c_1 = c_2 = C_1 = C_2 = C_3 = C_4 = 1$ and the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$: (a) solution of (3.13) where $\sigma = 1.0$; (b) solution of (3.13) where $\sigma = 2.0$; (c) solution of (3.14) where $\sigma = 1.0$; and, (d) solution of (3.14) where $\sigma = 2.0$.

3.2.3 Case 3: Reductions with $c_1X_1 + c_2Y_1$

Similar to Case 2 , we take a linear combination $c_1X_1 + c_2Y1$ which yields the invariant solutions below:

$$u(S,t) = C_2 + C_3\, FresnelS\left(\frac{Sc_2 + 2\,c_1}{\sqrt{\pi t c_2 \sigma}}\right) + C_4\, FresnelC\left(\frac{Sc_2 + 2\,c_1}{\sqrt{\pi t c_2 \sigma}}\right), \tag{3.15}$$

$$v(S,t) = C_1 + FresnelS\left(\frac{Sc_2 + 2\,c_1}{\sqrt{\pi t c_2 \sigma}}\right) C_4 - FresnelC\left(\frac{Sc_2 + 2\,c_1}{\sqrt{\pi t c_2 \sigma}}\right) C_3. \tag{3.16}$$

In Figure 4 we plot the solutions (3.15) and (3.16).

3.2.4 Case 4: A reduction with $c_1X_2 + c_2Y_1$

We consider the linear combination $c_1X_2 + c_2Y1$ gives solutions:

$$u(S,t) = C_2 + C_3\, FresnelS\left(\frac{S\sqrt{c_2}}{\sigma\,\sqrt{\pi\,(tc_2 + c_1)}}\right) + C_4\, FresnelC\left(\frac{S\sqrt{c_2}}{\sigma\,\sqrt{\pi\,(tc_2 + c_1)}}\right), \tag{3.17}$$

$$v(S,t) = C_1 - FresnelC\left(\frac{S\sqrt{c_2}}{\sigma\,\sqrt{\pi\,(tc_2 + c_1)}}\right) C_3 + FresnelS\left(\frac{S\sqrt{c_2}}{\sigma\,\sqrt{\pi\,(tc_2 + c_1)}}\right) C_4. \tag{3.18}$$

In Figure 5 we plot the solutions (3.17) and (3.18).

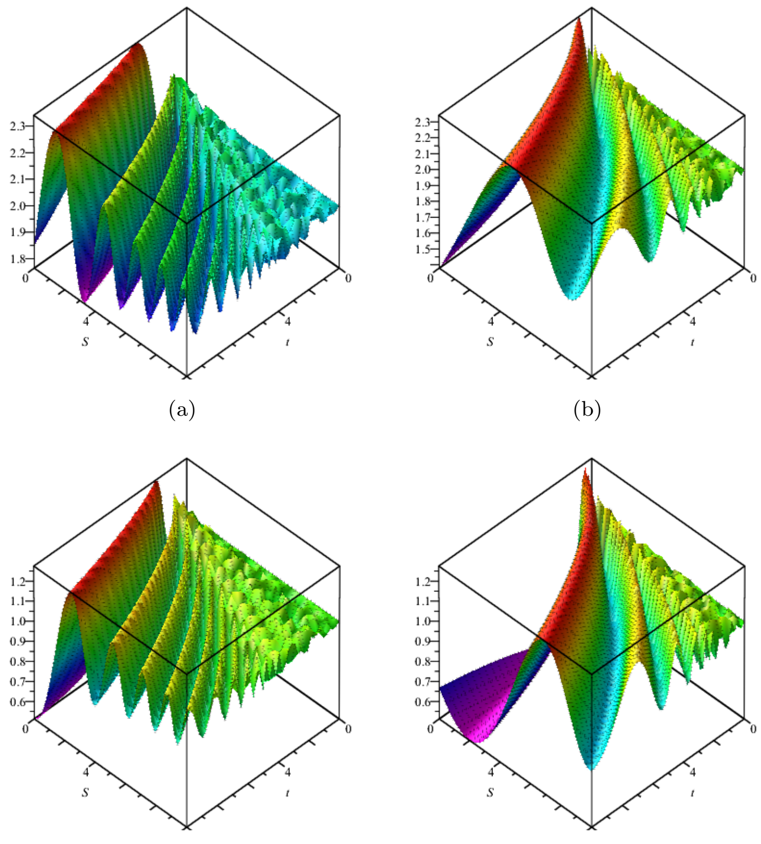


Figure 4: Graphical illustration of the analytical solutions are depicted, we select the parameter values $c_1 = c_2 = C_1 = C_2 = C_3 = C_4 = 1$ and the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$: (a) solution of (3.15) where $\sigma = 0.5$; (b) solution of (3.15) where $\sigma = 1.0$; (c) solution of (3.16) where $\sigma = 0.5$; and, (d) solution of (3.16) where $\sigma = 1.0$.

3.2.5 Case 5: Reduction via $c_1Y_3 + c_2X_1$

Application of the linear combinations $c_1Y_3 + c_2X_1$ provides the following analytical solutions below:

$$u(S,t) = e^{\frac{c_1 S}{c_2}} \left(C_1 \sin\left(\frac{1}{2}\frac{{c_1}^2\sigma^2 t}{{c_2}^2}\right) + C_2 \cos\left(\frac{1}{2}\frac{{c_1}^2\sigma^2 t}{{c_2}^2}\right)\right), \tag{3.19}$$

$$v(S,t) = e^{\frac{c_1 S}{c_2}} \left(-\cos\left(\frac{1}{2}\frac{{c_1}^2\sigma^2 t}{{c_2}^2}\right) C_1 + \sin\left(\frac{1}{2}\frac{{c_1}^2\sigma^2 t}{{c_2}^2}\right) C_2\right). \tag{3.20}$$

In Figure 6 we plot the solutions (3.19) and (3.20).

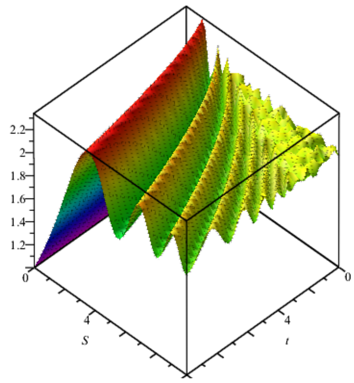
3.2.6 Case 6: $c_1Y_3 + c_2X_2$

Under this case we use the linear combinations of symmetries $c_1Y_3 + c_2X_2$ which yield the exact solutions below:

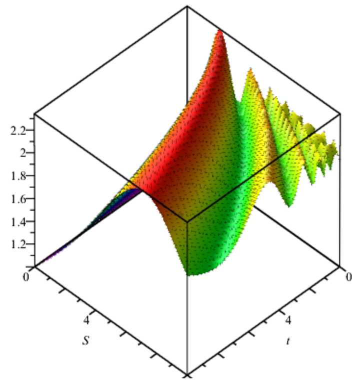
$$u(S,t) = e^{\frac{c_1 t}{c_2}} \left(C_1 e^{-i\gamma} + C_2 e^{-\gamma} + C_3 e^{i\gamma} + C_4 e^{\gamma}\right), \tag{3.21}$$

$$v(S,t) = e^{\frac{c_1 t}{c_2}} \left(\frac{\sqrt{-{c_1}^2{c_2}^2}}{c_2 c_1} \left(C_4 e^{\gamma} - C_1 e^{-i\gamma} + C_2 e^{-\gamma} - C_3 e^{i\gamma}\right)\right), \tag{3.22}$$

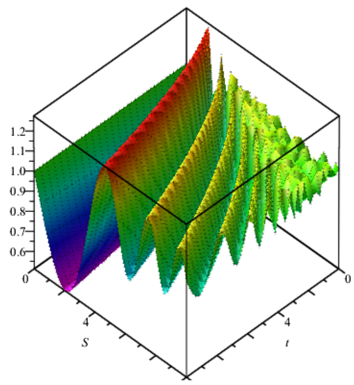
where $\gamma = \frac{\sqrt{2}\sqrt[4]{-{c_1}^2{c_2}^2}S}{c_2\sigma}$. In Figure 7 we plot the solutions (3.21) and (3.22).



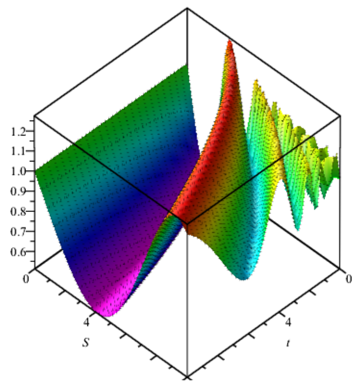
(a)



(b)

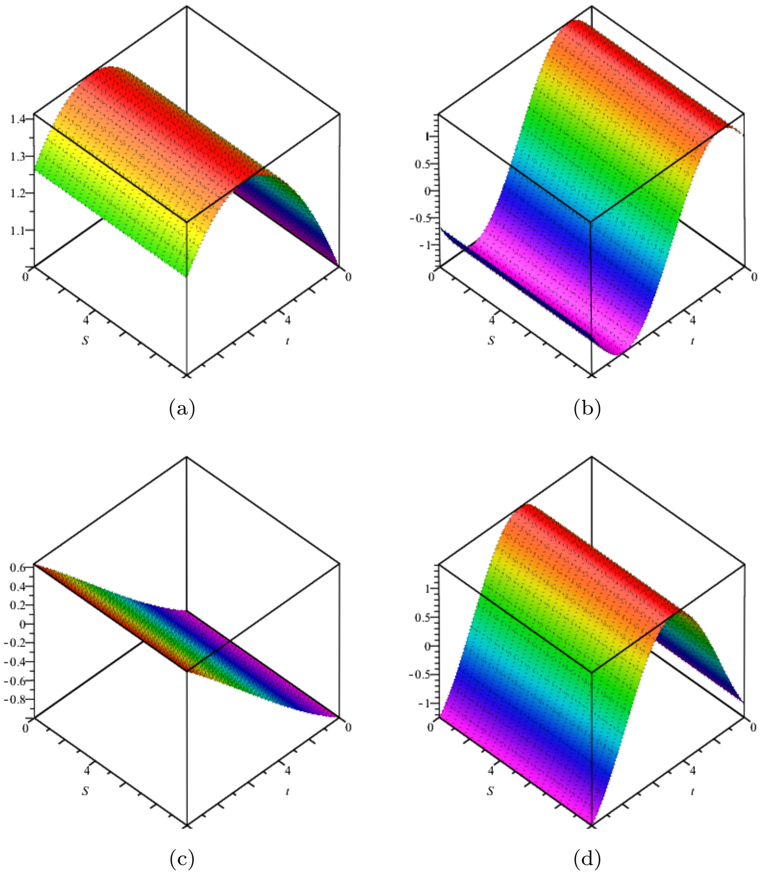


(c)



(d)

Figure 5: Graphical illustration of the analytical solutions are depicted, we select the parameter values $c_1 = c_2 = C_1 = C_2 = C_3 = C_4 = 1$ and the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$: (a) solution of (3.17) where $\sigma = 0.5$; (b) solution of (3.17) where $\sigma = 1.0$; (c) solution of (3.18) where $\sigma = 0.5$; and, (d) solution of (3.18) where $\sigma = 1.0$.



(a) (b) (c) (d)

Figure 6: Graphical illustration of the analytical solutions are depicted, we select the parameter values $c_1 = c_2 = C_1 = C_2 = 1$ the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$: (a) solution of (3.19) where $\sigma = 0.5$; (b) solution of (3.19) where $\sigma = 1.0$; (c) solution of (3.20) where $\sigma = 0.5$; and, (d) solution of (3.20) where $\sigma = 1.0$.

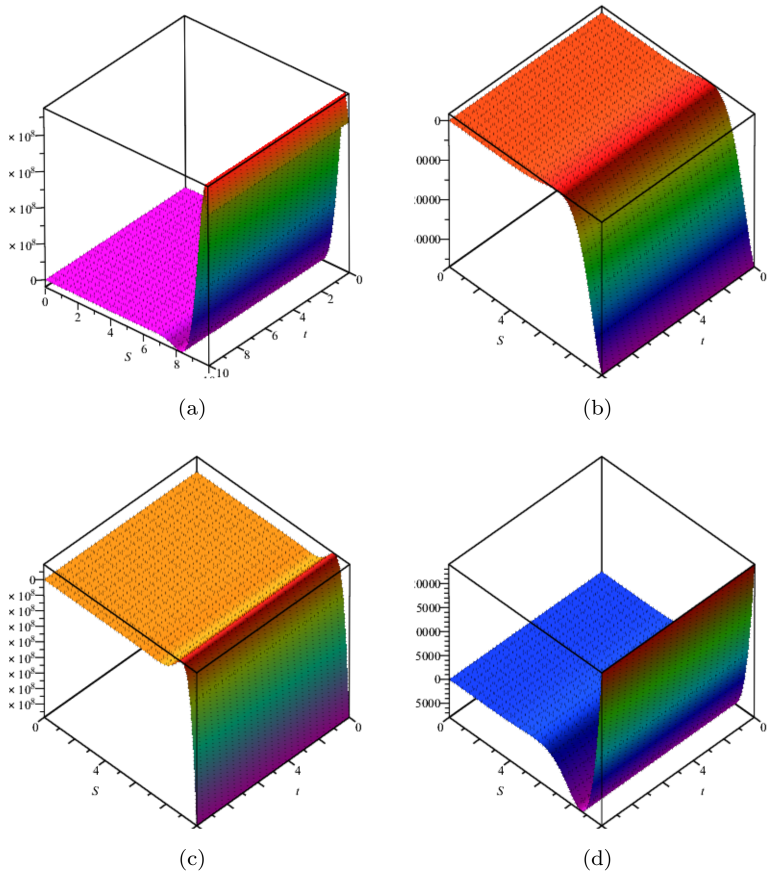


Figure 7: Graphical illustration of the analytical solutions are depicted, we select the parameter values $c_1 = c_2 = C_1 = C_2 = C_3 = C_4 = 1$ the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$: (a) solution of (3.21) where $\sigma = 0.5$; (b) solution of (3.21) where $\sigma = 1.0$; (c) solution of (3.22) where $\sigma = 0.5$; and, (d) solution of (3.22) where $\sigma = 1.0$.

4 Conclusion

Econophysics, as the name suggests is an interdisciplinary field, that deviates from mainstream economics taking concepts and models from the domain of physics to explain economic phenomena. Subsets of the field include quantum finance, thermoeconomics etc. In this paper we have successfully performed a full symmetry analysis for the IOPM considering both $\beta \neq 0$ and $\beta = 0$, the Landau coefficient representing the adaptive market potential. We discovered that the IOPM admits several invariant solutions as well as conservation laws for the zero and nonzero adaptive market potential. We provided several plots for our solutions given different levels of volatility. These solutions are novel and haven't appeared elsewhere in literature. We also note this is the first time symmetry analysis has been done on the model and thus we give original insights on the options pricing model's symmetries, conservation laws and solutions. As can be observed from the analysis above, point symmetries that correspond to the IOPM equation are necessary in order to solve for it. The order of the equation can be gradually reduced using these symmetries. If any of the remaining symmetries of the Lie algebra are inherited by reduced equations, the order may be lowered once again. Since the IOPM equation in our situation is a two-variable PDE, at least one symmetry is necessary to convert it to an ordinary differential equation (ODE). In one case of a zero adaptive market potential, the IOPM wave function solution exhibits sinusoidal characteristics, with wave frequency positively correlated to the Landau coefficient.

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