

Existence, asymptotic stability and blow-up results for a variable-exponent viscoelastic double-Kirchhoff-type wave equation

Mohammad Shahrouzi^{a,*}, Jorge Ferreira^b, Erhan Pişkin^c

^aDepartment of Mathematics, Jahrom University, P.O.Box: 74137-66171, Jahrom, Iran

^bFederal Fluminense University, Department of Exact Sciences, Volta Redonda, RJ, Brazil

^cDicle University, Department of Mathematics, Diyarbakir, Turkey

(Communicated by Javad Damirchi)

Abstract

The present study examined the behavior of solutions for a viscoelastic double-Kirchhoff-type wave equation with nonlocal degenerate damping term and variable exponent nonlinearities. Under appropriate conditions for the data and exponents, we prove the global existence, asymptotic stability, and blow up of solutions with arbitrary initial energy.

Keywords: Global existence, asymptotic stability, blow-up, Kirchhoff-type equation, nonlocal damping, variable exponent

2010 MSC: Primary 35A01; Secondary 35B35, 35B44, 74D10

1 Introduction

In this work, we consider the following boundary value problem

$$u_{tt} - M(\|\nabla u\|^2)\Delta u - \left(a + b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \Delta_{p(x)} u + (g * \Delta u)(x, t) + \sigma(\|\nabla u\|^2)h(u_t) = \phi(u), \quad (x, t) \in \Omega \times (0, \infty) \quad (1.1)$$

$$u(x, t) = \frac{\partial u}{\partial n}(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, \infty) \quad (1.2)$$

$$u(x, 0) = u_0, \quad u_t(x, 0) = u_1(x), \quad x \in \Omega. \quad (1.3)$$

*Corresponding author

Email addresses: mshahrouzi@jahromu.ac.ir (Mohammad Shahrouzi), ferreirajorge2012@gmail.com (Jorge Ferreira), episkin@dicle.edu.tr (Erhan Pişkin)

Here $a, b > 0$ and Ω is a bounded domain of $R^n (n \geq 1)$ with a smooth boundary $\partial\Omega$. Also, $\Delta_{p(x)}$ is called $p(x)$ -Laplacian operator which defined as

$$\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u).$$

Moreover, suppose that r is a positive constant in which

$$M(s) = 1 + \sigma(s), \quad \sigma(s) = s^r,$$

and the viscoelastic term is defined as

$$(g * \Delta u)(x, t) = \int_0^t g(t-s)\Delta u(s)ds,$$

where the kernel of the memory satisfies the following condition:

(A1) g is a nonincreasing and nonnegative function satisfying

$$\begin{aligned} g(t) &\geq 0, \quad 1 - \int_0^\infty g(t)dt = l > 0, \\ g'(t) &\leq -\xi(t)g(t), \quad \int_0^{+\infty} \xi(t)dt = +\infty, \end{aligned}$$

and ξ is a positive differentiable function.

The variable-exponent nonlinearity terms, i.e., $h(\cdot)$ and $\phi(\cdot)$ are defined as:

$$\begin{aligned} h(v) &= |v|^{m(x)-2}u_t, \\ \phi(v) &= |v|^{q(x)-2}u. \end{aligned}$$

In this study, the following additional condition is considered on the variable exponents:

(A2) the exponents $p(\cdot)$, $m(\cdot)$ and $q(\cdot)$ are given measurable functions on $\bar{\Omega}$ such that:

$$\begin{aligned} 2 \leq p^- \leq p(x) \leq p^+ < \infty, \\ 2 \leq m^- \leq m(x) \leq m^+ < \infty, \\ 2 \leq q^- \leq q(x) \leq q^+ < \infty, \end{aligned}$$

with

$$\begin{aligned} p^- &:= \operatorname{ess\,inf}_{x \in \bar{\Omega}} p(x), \quad p^+ := \operatorname{ess\,sup}_{x \in \bar{\Omega}} p(x), \\ m^- &:= \operatorname{ess\,inf}_{x \in \bar{\Omega}} m(x), \quad m^+ := \operatorname{ess\,sup}_{x \in \bar{\Omega}} m(x), \\ q^- &:= \operatorname{ess\,inf}_{x \in \bar{\Omega}} q(x), \quad q^+ := \operatorname{ess\,sup}_{x \in \bar{\Omega}} q(x). \end{aligned}$$

Before going any further, some previous works in the literature about the Kirchhoff-type models with nonlocal damping terms and constant-exponent nonlinearities are reviewed. Ikehata and Matsuyama [12] studied the associated mixed problem for the following Kirchhoff-type equation

$$u_{tt} - M(\|\nabla u\|^2)\Delta u + \delta|u_t|^{p-1}u_t = \mu|u_t|^{q-1}u.$$

Under some restrictions to the initial data, they proved the existence of global solutions and decay of the associated energy. In another study, Lazo [14], proved the existence of a global solution for the Kirchhoff-type equation of the form

$$u_{tt} + M(|A^{\frac{1}{2}}u|^2)Au + N(|A^\alpha u|^2)A^\alpha u_t = f,$$

where A is a positive self-adjoint operator which defined in a real Hilbert space H , and $0 < \alpha \leq 1$. Also, it is assumed that there exist constants $m_0, n_0 > 0$ such that the functions M and N satisfy the following conditions

$$\begin{aligned} M &\in C^0((0, +\infty); R), \quad M(s) \geq m_0 \text{ for all } s \geq 0. \\ N &\in C^0((0, +\infty); R), \quad N(s) \geq n_0 \text{ for all } s \geq 0. \end{aligned}$$

In [6], Chueshov examined the following Kirchhoff wave model with nonlocal and nonlinear strong damping

$$u_{tt} - \Phi(\|\nabla u(t)\|_2^2)\Delta u - \sigma(\|\nabla u(t)\|_2^2)\Delta u_t + f(u) = h.$$

He demonstrated the existence and uniqueness of weak solutions and studied their properties for a wide class of nonlinearities which covers the case of possible degeneration (or even negativity) of the stiffness coefficient and the case of a supercritical source term. He proved that in the natural energy space endowed with a partially strong topology there exists a global finite-dimensional attractor. In a recent study, Narciso [19] investigated well-posedness and long-time behavior of the solutions for the following quasi-linear Kirchhoff-type wave model with nonlocal nonlinear damping

$$u_{tt} - \Phi(\|\nabla u(t)\|_2^2)\Delta u - \sigma(\|\nabla u(t)\|_2^2)g(u_t) + f(u) = h,$$

The existence of a local attractor was also proved under appropriate conditions on data.

Zhang et. al. [30] considered a wave equation with nonlocal nonlinear damping and source terms of the form

$$u_{tt} - \Delta u + M(\|\nabla u\|^2)g(u_t) = f(u),$$

in a bounded domain. They proved a general energy decay property for the solutions by constructing a stable set and using the multiplier technique. In this regard, see also Zhang et. al. [31]. Note that, numerous researchers have studied the different kinds of generalizations of the Kirchhoff-type wave models. Thus, due to the extensive literature on the topic, we selectively refer to [5, 7, 13, 18, 20, 28, 29] and the references therein. Nonetheless, the stability and blow-up of solutions of viscoelastic Kirchhoff-type wave equations with variable-exponent nonlinearities is a less investigated area. For example, Pişkin [21], proved the blow-up of solutions for the following Kirchhoff-type equation with variable exponents nonlinearities:

$$u_{tt} - M(\|\nabla u\|^2)\Delta u + |u_t|^{p(x)-2}u_t = |u|^{q(x)-2}u.$$

Afterward, Shahrouzi and Kargarfard [27] considered the following Kirchhoff-type equation with $m(x)$ -Laplacian operator

$$u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta_{m(x)}u + h(x, t, u, \nabla u) + \beta u_t = \phi_{p(x)}(u),$$

where $\phi_{p(x)}(u) = |u|^{p(x)}u$ and $h(x, t, u, \nabla u)$ is a function that satisfies

$$|h(x, t, u, \nabla u)| \leq M_1|u| + M_2|\nabla u|,$$

for some positive M_1, M_2 . They proved the blow up of solutions with positive initial energy and suitable conditions on datum and variable exponents.

Recently, Antontsev et. al [2] looked into the following nonlinear Timoshenko equation with variable exponents:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|_{L^2(\Omega)}^2)\Delta u + |u_t|^{p(x)-2}u_t = |u|^{q(x)-2}u,$$

and demonstrated the local existence of the solution under suitable conditions. Moreover, the nonexistence of solutions was proved with negative initial energy.

Dai and Hao [8] studied the following equation

$$-M\left(\int_{\Omega} \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x, u).$$

Using a direct variational approach and the theory of the variable-exponent Sobolev spaces, they established conditions through which the existence and multiplicity of solutions for the problem were verified. In another study, Hamdani et. al. [11] investigated the following nonlocal $p(x)$ -Kirchhoff-type equation

$$-(a - b \int_{\Omega} \frac{1}{p(x)}|\nabla u|^{p(x)}dx)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{p(x)-2}u + g(x, u).$$

They obtained a nontrivial weak solution by using the Mountain Pass theorem. The relevant equations with variable-exponent nonlinearities have also been studied in [3, 10, 16, 17, 22, 23, 24, 25].

What distinguishes the current research from the other studies is the investigation of a new viscoelastic wave model with double-Kirchhoff-type terms as well as the inclusion of a product of $\sigma(\|\nabla u\|_2^2)$ and variable-exponent nonlinearity term $h(u_t)$. Our study extends and improves the results in the literature as obtained in [19, 26, 30, 31] to viscoelastic double-Kirchhoff-type equation with degenerate nonlocal damping and variable-exponent nonlinearities.

The rest of the paper is organized as follows. In section 2 we present some definitions and Lemmas about the variable-exponent Lebesgue space, $L^{p(\cdot)}(\Omega)$, the Sobolev space, $W^{1,p(\cdot)}(\Omega)$, that we use for main results. Section 3 proves the global existence of solutions for the problem (1.1)-(1.3). In section 4 the asymptotic stability of solutions for appropriate initial data has been proved. Finally, the blow-up of solutions is proved with arbitrary initial energy and suitable conditions on datum, in section 5.

2 Preliminaries

In order to study problem (1.1)-(1.3), some theories about Lebesgue and Sobolev spaces with variable-exponents are required (for more details see [4, 9]).

Let $p : \Omega \rightarrow [1, \infty]$ be a measurable function, where Ω is a domain of R^n . we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u \mid u \text{ is measurable in } \Omega \text{ and } \int_{\Omega} |\lambda u(x)|^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\}.$$

We equip the Lebesgue space with a variable exponent, $L^{p(\cdot)}(\Omega)$, with the following Luxembourg-type norm

$$\|u\|_{p(x)} := \inf \left\{ \lambda > 0 \mid \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Lemma 2.1. [9] Let Ω be a bounded domain in R^n . The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a Banach space, and its conjugate space is $L^{q(\cdot)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, the generalized Hölder inequality holds

$$\left| \int_{\Omega} f g dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)} \leq 2 \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}.$$

The relation between the modular $\int_{\Omega} |f|^{p(x)} dx$ and the norm follows from

$$\min(\|f\|_{p(\cdot)}^-, \|f\|_{p(\cdot)}^+) \leq \int_{\Omega} |f|^{p(x)} dx \leq \max(\|f\|_{p(\cdot)}^+, \|f\|_{p(\cdot)}^-).$$

The variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

This space is a Banach space with respect to the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

Furthermore, let $W_0^{1,p(\cdot)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)}.$$

Let the variable exponent $p(\cdot)$ satisfy the log-Hölder continuity condition

$$|p(x) - p(y)| \leq \frac{A}{\log \frac{1}{|x-y|}}, \text{ for all } x, y \in \Omega \text{ with } |x - y| < \delta,$$

where $A > 0$ and $0 < \delta < 1$.

Lemma 2.2. (The Poincare inequality) Assume that Ω is a bounded domain of R^n and $p(\cdot)$ satisfies log-Hölder condition, then

$$\|u\|_{p(x)} \leq C_* \|\nabla u\|_{p(x)}, \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega), \tag{2.1}$$

where $C_* = C_*(p^-, p^+, |\Omega|) > 0$.

Lemma 2.3. Let $p(\cdot) \in C(\overline{\Omega})$ and $q : \Omega \rightarrow [1, \infty)$ be a measurable function that satisfy

$$\text{ess inf}_{x \in \overline{\Omega}} (p^*(x) - q(x)) > 0.$$

Then, the Sobolev embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact. Where

$$p^*(x) = \begin{cases} \frac{np^-}{n-p^-}, & \text{if } p^- < n, \\ \infty, & \text{if } p^- \geq n. \end{cases}$$

If in addition $p(\cdot)$ satisfies log-Hölder condition, then

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & \text{if } p(x) < n, \\ \infty, & \text{if } p(x) \geq n. \end{cases}$$

We sometimes use the Young's inequality

$$ab \leq \delta a^{q(x)} + C_\delta(x) b^{q'(x)}, \quad a, b \geq 0, \quad \delta > 0, \quad \frac{1}{q(x)} + \frac{1}{q'(x)} = 1, \tag{2.2}$$

where $C_\delta(x) = \frac{1}{q'(x)} (\delta q(x))^{-\frac{q'(x)}{q(x)}}$. In special case when $\theta = \frac{1}{q(x)}$, we have from (2.2)

$$ab \leq \frac{a^{q(x)}}{q(x)} + \frac{b^{q'(x)}}{q'(x)}. \tag{2.3}$$

We set

$$V = \{v \in W^{1,p(x)} \cap H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}.$$

The energy function is defined as

$$\begin{aligned} E(t) &= \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla u\|^2 + \frac{1}{2} (g \odot \nabla u)(t) + \frac{1}{2(r+1)} \|\nabla u\|^{2(r+1)} \\ &+ a \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \frac{b}{2} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx, \end{aligned} \tag{2.4}$$

where

$$(g \odot \nabla u)(t) = \int_0^t g(t-s) \int_\Omega (\nabla u(s) - \nabla u)^2 dx ds.$$

Now, we first state a well-known lemma that will be needed later to prove the asymptotic stability result for problem (1.1)-(1.3).

Lemma 2.4. [15] Let $E : R^+ \rightarrow R^+$ be a nonincreasing function and assume that there are two constants $\gamma \geq 1$ and $A > 0$ such that

$$\int_0^{+\infty} E^{(\gamma+1)/2}(t) dt \leq AE(S), \quad 0 \leq S < +\infty,$$

then

$$\begin{aligned} E(t) &\leq cE(0)(1+t)^{2/(\gamma-1)}, \quad t \geq 0, \text{ if } \gamma > 1, \\ E(t) &\leq cE(0)e^{-\omega t}, \quad t \geq 0, \text{ if } \gamma = 1, \end{aligned}$$

where c and ω are positive constants independent of the initial energy $E(0)$.

For the sake of completeness, the local existence result for the problem (1.1)-(1.3) is stated without providing the proof. Indeed, by using the Faedo-Galerkin method and a combination of the works [1, 2, 19], one could prove this Theorem.

Theorem 2.5. (Local existence and monotonicity of energy) Let us assume that (A1) and (A2) hold. If $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, then problem (1.1)-(1.3) possesses a unique weak solution in the case

$$u \in L^\infty((0, T), H^2(\Omega) \cap V), \quad u_t \in L^\infty((0, T), L^{m(\cdot)}(\Omega)),$$

$$u_{tt} \in L^\infty((0, T), L^2(\Omega) \cap W^{-1, m'(\cdot)}(\Omega)),$$

where $\frac{1}{m(x)} + \frac{1}{m'(x)} = 1$. Moreover, for any solution of problem (1.1)-(1.3) the energy functional along the solution satisfies

$$E'(t) \leq \frac{1}{2}(g' \odot \nabla u)(t) - \|\nabla u\|^{2r} \int_{\Omega} |u_t|^{m(x)} dx \leq 0. \quad (2.5)$$

3 Global existence

In this section, we are going to prove the global existence of solutions for the problem (1.1)-(1.3). For this goal, we define the following functionals to obtain the potential well.

$$I(u(t)) = (1 - \int_0^t g(s) ds) \|\nabla u\|^2 + (g \odot \nabla u)(t) + \|\nabla u\|^{2(r+1)} + a \int_{\Omega} |\nabla u|^{p(x)} dx + b \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^2 - \int_{\Omega} |u|^{q(x)} dx. \quad (3.1)$$

$$J(u(t)) = \frac{1}{2}(1 - \int_0^t g(s) ds) \|\nabla u\|^2 + \frac{1}{2}(g \odot \nabla u)(t) + \frac{1}{2(r+1)} \|\nabla u\|^{2(r+1)}$$

$$+ a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx. \quad (3.2)$$

From the definitions (3.1) and (3.2), we have

$$E(t) = J(u(t)) + \frac{1}{2} \|u_t\|^2. \quad (3.3)$$

Lemma 3.1. Under the assumptions of Theorem 2.5, we assume that $I(u(0)) > 0$, $q^- > \max\{2(r+1), 2(p^+)^2\}$ and moreover,

$$\gamma_0 := \max\left\{ C_q^{q^-} \left(\frac{E(0)}{\beta l} \right)^{\frac{q^- - 2}{2}}, C_q^{q^+} \left(\frac{E(0)}{\beta l} \right)^{\frac{q^+ - 2}{2}} \right\} < 1, \quad (3.4)$$

where C_q is the best embedding constant of $H_0^1(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ and

$$\beta := \min\left\{ \frac{q^- - 2}{2q^-}, \frac{q^- - 2(r+1)}{2q^-(r+1)}, \frac{a(q^- - p^+)}{p^+ q^-}, \frac{b(q^- - 2(p^+)^2)}{2(p^+)^2 q^-} \right\}.$$

Then we have

$$I(u(t)) > 0, \text{ for all } t \in [0, T].$$

Proof . By using the continuity of $u(t)$ and since $I(u(0)) > 0$, thus there exists a time $T^* < T$ such that

$$I(u(t)) \geq 0, \text{ for all } t \in [0, T^*].$$

Now, under (A1) and from the definition of $J(u(t))$ in (3.2), we deduce that

$$\begin{aligned} J(u(t)) &\geq \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla u\|^2 + \frac{1}{2} (g \odot \nabla u)(t) + \frac{1}{2(r+1)} \|\nabla u\|^{2(r+1)} \\ &\quad + \frac{a}{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx + \frac{b}{2(p^+)^2} \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^2 - \frac{1}{q^-} \int_{\Omega} |u|^{q(x)} dx. \end{aligned}$$

From definition of $I(u(t))$, we obtain

$$\begin{aligned} J(u(t)) &\geq \frac{q^- - 2}{2q^-} \left(1 - \int_0^t g(s) ds\right) \|\nabla u\|^2 + \frac{q^- - 2}{2q^-} (g \odot \nabla u)(t) + \frac{q^- - 2(r+1)}{2q^-(r+1)} \|\nabla u\|^{2(r+1)} + \frac{a(q^- - p^+)}{p^+ q^-} \int_{\Omega} |\nabla u|^{p(x)} dx \\ &\quad + \frac{b(q^- - 2(p^+)^2)}{2(p^+)^2 q^-} \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^2 + \frac{1}{q^-} I(u(t)). \end{aligned}$$

Thanks to the assumptions of Lemma, since $q^- > \max\{2(r+1), 2(p^+)^2\}$, therefore we get

$$\begin{aligned} J(u(t)) &\geq \beta \left(\left(1 - \int_0^t g(s) ds\right) \|\nabla u\|^2 + (g \odot \nabla u)(t) + \|\nabla u\|^{2(r+1)} + \int_{\Omega} |\nabla u|^{p(x)} dx + \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^2 \right) + \frac{1}{q^-} I(u(t)) \\ &\geq \beta \left(l \|\nabla u\|^2 + (g \odot \nabla u)(t) + \|\nabla u\|^{2(r+1)} + \int_{\Omega} |\nabla u|^{p(x)} dx + \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^2 \right), \end{aligned} \quad (3.5)$$

where (A2) has been used and

$$\beta := \min \left\{ \frac{q^- - 2}{2q^-}, \frac{q^- - 2(r+1)}{2q^-(r+1)}, \frac{a(q^- - p^+)}{p^+ q^-}, \frac{b(q^- - 2(p^+)^2)}{2(p^+)^2 q^-} \right\}.$$

Inequality (3.5) yields

$$l \|\nabla u\|^2 \leq \beta^{-1} J(u(t)) \leq \beta^{-1} E(t) \leq \beta^{-1} E(0),$$

and therefore we have

$$\|\nabla u\|^2 \leq \frac{E(0)}{\beta l}. \quad (3.6)$$

At this point, we shall prove that $I(u(t)) > 0$, $\forall t \in [0, T^*]$. For this purpose, suppose that C_q is the best embedding constant of $H_0^1(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} |u|^{q(x)} dx &\leq \max \left\{ \left(\int_{\Omega} |u|^{q(x)} dx \right)^{q^-}, \left(\int_{\Omega} |u|^{q(x)} dx \right)^{q^+} \right\} \\ &\leq \max \{ C_q^{q^-} \|\nabla u\|^{q^-}, C_q^{q^+} \|\nabla u\|^{q^+} \} \\ &= \max \{ C_q^{q^-} \|\nabla u\|^{q^- - 2}, C_q^{q^+} \|\nabla u\|^{q^+ - 2} \} \|\nabla u\|^2 \\ &\leq \max \left\{ C_q^{q^-} \left(\frac{E(0)}{\beta l} \right)^{\frac{q^- - 2}{2}}, C_q^{q^+} \left(\frac{E(0)}{\beta l} \right)^{\frac{q^+ - 2}{2}} \right\} \|\nabla u\|^2, \end{aligned}$$

which implies

$$\int_{\Omega} |u|^{q(x)} dx \leq \gamma_0 \|\nabla u\|^2, \quad (3.7)$$

and therefore by virtue of (3.1) and (3.4), we deduce

$$I(u(t)) > 0, \quad \forall t \in [0, T^*].$$

By repeating this procedure T^* extended to T and proof of Lemma 3.1 is completed. \square

Theorem 3.2. Let $u(x, t)$ be the local solution of (1.1)-(1.3). Under the assumption of Lemma 3.1, $u(x, t)$ is global.

Proof . By virtue of (3.3) and (3.5), we obtain

$$\begin{aligned} E(t) &= \frac{1}{2} \|u_t\|^2 + J(u(t)) \\ &\geq \frac{1}{2} \|u_t\|^2 + \beta \left(l \|\nabla u\|^2 + (g \odot \nabla u)(t) + \|\nabla u\|^{2(r+1)} + \int_{\Omega} |\nabla u|^{p(x)} dx + \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^2 \right). \end{aligned}$$

From the fact that $\beta \leq \frac{q^- - 2}{2q^-} < \frac{1}{2}$, we get

$$\|u_t\|^2 + \|\nabla u\|^2 + (g \odot \nabla u)(t) + \|\nabla u\|^{2(r+1)} + \int_{\Omega} |\nabla u|^{p(x)} dx + \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^2 \leq \frac{E(t)}{\beta l}. \quad (3.8)$$

Hence by attention to the nonincreasingness of $E(t)$, we have

$$\|u_t\|^2 + \|\nabla u\|^2 + (g \odot \nabla u)(t) + \|\nabla u\|^{2(r+1)} + \int_{\Omega} |\nabla u|^{p(x)} dx + \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^2 \leq \frac{E(0)}{\beta l},$$

and this shows that the local solution $u(x, t)$ of (1.1)-(1.3) is global and bounded. \square

4 Asymptotic stability result

This section aims at proving the asymptotic stability of the global solutions of (1.1)-(1.3). To prove this result, we establish the following Lemma:

Lemma 4.1. Assume that assumptions of Lemma 3.1 hold. Then for any solution of (1.1)-(1.3), there exists $\gamma_1 > 0$ such that

$$\int_{\Omega} |u|^{m(x)} dx \leq \gamma_1 \|\nabla u\|^2. \quad (4.1)$$

Proof . By using (A2), (3.6) and embedding $H_0^1(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$ with the constant C_m , we have

$$\begin{aligned} \int_{\Omega} |u|^{m(x)} dx &\leq \max \left\{ \left(\int_{\Omega} |u|^{m(x)} dx \right)^{m^-}, \left(\int_{\Omega} |u|^{m(x)} dx \right)^{m^+} \right\} \\ &\leq \max \{ C_m^{m^-} \|\nabla u\|^{m^-}, C_m^{m^+} \|\nabla u\|^{m^+} \} \\ &= \max \{ C_m^{m^-} \|\nabla u\|^{m^- - 2}, C_m^{m^+} \|\nabla u\|^{m^+ - 2} \} \|\nabla u\|^2 \\ &\leq \max \left\{ C_m^{m^-} \left(\frac{E(0)}{\beta l} \right)^{\frac{m^- - 2}{2}}, C_m^{m^+} \left(\frac{E(0)}{\beta l} \right)^{\frac{m^+ - 2}{2}} \right\} \|\nabla u\|^2 \\ &\leq \gamma_1 \|\nabla u\|^2, \end{aligned}$$

and therefore the proof is completed. \square

We are in a position that states and prove the main result:

Theorem 4.2. Suppose that the assumptions of Lemma 3.1 hold. Moreover, we assume that r is sufficiently small and $b > p^+(1 - \gamma_0)$ where γ_0 (c.f. (3.4)) satisfies $\gamma_0 < \frac{2\beta l}{2\beta l + 1}$. Then the solutions of problem (1.1)-(1.3) are asymptotically stable.

Proof . Multiplying equation (1.1) by $E(t)^\alpha u$ and integrating it over $\Omega \times (S, T)$, we obtain

$$\begin{aligned} &\left[E(t)^\alpha \int_{\Omega} uu_t dx \right] \Big|_S^T - \alpha \int_S^T E(t)^{\alpha-1} E'(t) \int_{\Omega} uu_t dx dt - \int_S^T E(t)^\alpha \|u_t\|^2 dt \\ &+ \int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)} dt + \int_S^T E(t)^\alpha \left(1 - \int_0^t g(s) ds \right) \|\nabla u\|^2 dt \\ &+ a \int_S^T E(t)^\alpha \int_{\Omega} |\nabla u|^{p(x)} dx dt - \int_S^T E(t)^\alpha \int_{\Omega} |u|^{q(x)} dx dt \end{aligned}$$

$$\begin{aligned}
 &+ b \int_S^T E(t)^\alpha \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \left(\int_\Omega |\nabla u|^{p(x)} dx \right) dt \\
 &- \int_S^T E(t)^\alpha \int_0^t g(t-s) \int_\Omega \nabla u (\nabla u(s) - \nabla u) dx ds dt \\
 &+ \int_S^T E(t)^\alpha \|\nabla u\|^{2r} \int_\Omega u |u_t|^{m(x)-2} u_t dx dt = 0.
 \end{aligned} \tag{4.2}$$

Now, we add and subtract the following term in (4.2)

$$\begin{aligned}
 &\int_S^T E(t)^\alpha \left((\gamma_0 + 1) \|u_t\|^2 + \gamma_0 \left[\left(1 - \int_0^t g(s) ds \right) \|\nabla u\|^2 + (g \odot \nabla u)(t) \right. \right. \\
 &\quad \left. \left. + \|\nabla u\|^{2(r+1)} + a \int_\Omega |\nabla u|^{p(x)} dx + \left(\int_\Omega |\nabla u|^{p(x)} dx \right)^2 \right] \right) dt,
 \end{aligned}$$

then, by using (3.1), we obtain

$$\begin{aligned}
 &(1 - \gamma_0) \int_S^T E(t)^\alpha \left(\|u_t\|^2 + \int_\Omega |u|^{q(x)} dx + I(u(t)) \right) dt \\
 &= (2 - \gamma_0) \int_S^T E(t)^\alpha \|u_t\|^2 dt - \gamma_0 \int_S^T E(t)^\alpha \left(1 - \int_0^t g(s) ds \right) \|\nabla u\|^2 dt \\
 &\quad - \gamma_0 \int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)} dt + (1 - \gamma_0) \int_S^T E(t)^\alpha (g \odot \nabla u)(t) dt \\
 &\quad - a \gamma_0 \int_S^T E(t)^\alpha \int_\Omega |\nabla u|^{p(x)} dx dt + \int_S^T E(t)^\alpha \int_\Omega |u|^{q(x)} dx dt \\
 &\quad + (1 - \gamma_0) \int_S^T E(t)^\alpha \left(\int_\Omega |\nabla u|^{p(x)} dx \right)^2 dt \\
 &\quad - \left[E(t)^\alpha \int_\Omega uu_t dx \right]_S^T + \alpha \int_S^T E(t)^{\alpha-1} E'(t) \int_\Omega uu_t dx dt \\
 &\quad - b \int_S^T E(t)^\alpha \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \left(\int_\Omega |\nabla u|^{p(x)} dx \right) dt \\
 &\quad - \int_S^T E(t)^\alpha \|\nabla u\|^{2r} \int_\Omega u |u_t|^{m(x)-2} u_t dx dt \\
 &\quad + \int_S^T E(t)^\alpha \int_0^t g(t-s) \int_\Omega \nabla u (\nabla u(s) - \nabla u) dx ds dt.
 \end{aligned} \tag{4.3}$$

Thanks to the assumptions (A1) and (A2), we deduce that

$$\begin{aligned}
 (1 - \gamma_0) \int_S^T E(t)^\alpha \left(\|u_t\|^2 + \int_\Omega |u|^{q(x)} dx + I(u(t)) \right) dt &\leq (2 - \gamma_0) \int_S^T E(t)^\alpha \|u_t\|^2 dt - \gamma_0 \int_S^T E(t)^\alpha \|\nabla u\|^2 dt \\
 &\quad - \gamma_0 \int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)} dt + (1 - \gamma_0) \int_S^T E(t)^\alpha (g \odot \nabla u)(t) dt \\
 &\quad - a \gamma_0 \int_S^T E(t)^\alpha \int_\Omega |\nabla u|^{p(x)} dx dt + \int_S^T E(t)^\alpha \int_\Omega |u|^{q(x)} dx dt \\
 &\quad + \left(1 - \gamma_0 - \frac{b}{p^+} \right) \int_S^T E(t)^\alpha \left(\int_\Omega |\nabla u|^{p(x)} dx \right)^2 dt \\
 &\quad - \left[E(t)^\alpha \int_\Omega uu_t dx \right]_S^T + \alpha \int_S^T E(t)^{\alpha-1} E'(t) \int_\Omega uu_t dx dt \\
 &\quad - \int_S^T E(t)^\alpha \|\nabla u\|^{2r} \int_\Omega u |u_t|^{m(x)-2} u_t dx dt \\
 &\quad + \int_S^T E(t)^\alpha \int_0^t g(t-s) \int_\Omega \nabla u (\nabla u(s) - \nabla u) dx ds dt.
 \end{aligned} \tag{4.4}$$

By using the assumption of Theorem 4.2, i.e $b > p^+(1 - \gamma_0)$ and inequality (3.7), we get

$$\begin{aligned}
& (1 - \gamma_0) \int_S^T E(t)^\alpha \left(\|u_t\|^2 + \int_\Omega |u|^{q(x)} dx + I(u(t)) \right) dt \\
& \leq (2 - \gamma_0) \int_S^T E(t)^\alpha \|u_t\|^2 dt + \gamma_0(1 - l) \int_S^T E(t)^\alpha \|\nabla u\|^2 dt \\
& \quad - \gamma_0 \int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)} dt + (1 - \gamma_0) \int_S^T E(t)^\alpha (g \odot \nabla u)(t) dt \\
& \quad - \left[E(t)^\alpha \int_\Omega uu_t dx \right] \Big|_S^T + \alpha \int_S^T E(t)^{\alpha-1} E'(t) \int_\Omega uu_t dx dt \\
& \quad - \int_S^T E(t)^\alpha \|\nabla u\|^{2r} \int_\Omega u |u_t|^{m(x)-2} u_t dx dt \\
& \quad + \int_S^T E(t)^\alpha \int_0^t g(t-s) \int_\Omega \nabla u(\nabla u(s) - \nabla u) dx ds dt. \tag{4.5}
\end{aligned}$$

At this point, we estimate the last four terms on the right-hand side of (4.5). Firstly, by using Hölder, Poincaré and Young inequalities, and since $E(t)$ is a nonincreasing and nonnegative function, we obtain

$$\begin{aligned}
| - \left[E(t)^\alpha \int_\Omega uu_t dx \right] \Big|_S^T | &= | E(S)^\alpha \int_\Omega u(x, S) u_t(x, S) dx - E(T)^\alpha \int_\Omega u(x, T) u_t(x, T) dx | \\
&\leq E(S)^\alpha \left| \int_\Omega u(x, S) u_t(x, S) dx - \int_\Omega u(x, T) u_t(x, T) dx \right| \\
&\leq E(S)^\alpha \left[\|u(x, S)\| \|u_t(x, S)\| + \|u(x, T)\| \|u_t(x, T)\| \right] \\
&\leq C_1 E(S)^{\alpha+1} \\
&\leq C_1 E(0)^\alpha E(S) \\
&\leq C_2 E(S), \tag{4.6}
\end{aligned}$$

where (3.8) has been used. Also, we have

$$\begin{aligned}
\left| \alpha \int_S^T E(t)^{\alpha-1} E'(t) \int_\Omega uu_t dx dt \right| &\leq \frac{\alpha}{2} \int_S^T |E(t)^{\alpha-1} E'(t)| (\|u\|^2 + \|u_t\|^2) dt \\
&\leq C_3 \int_S^T E(t)^\alpha (-E'(t)) dt \\
&= -\frac{C_3}{\alpha+1} \int_S^T d(E(t)^{\alpha+1}) \\
&= \frac{C_3}{\alpha+1} (E(S)^{\alpha+1} - E(T)^{\alpha+1}) \\
&\leq C_3 E(S)^{\alpha+1} \\
&\leq C_3 E(0)^\alpha E(S) \\
&\leq C_4 E(S). \tag{4.7}
\end{aligned}$$

Moreover, by using Hölder, Poincaré and Young inequalities, we get

$$\begin{aligned}
\left| \int_S^T E(t)^\alpha \int_0^t g(t-s) \int_\Omega \nabla u(\nabla u(s) - \nabla u) dx ds dt \right| &\leq \int_S^T E(t)^\alpha \int_0^t g(t-s) \int_\Omega \nabla u(\nabla u(s) - \nabla u) dx ds dt \\
&\leq \int_S^T E(t)^\alpha (\gamma_0 l \|\nabla u\|^2 + \frac{1-l}{2\gamma_0 l} (g \odot \nabla u)(t)) dt \\
&= \gamma_0 l \int_S^T E(t)^\alpha \|\nabla u\|^2 dt + \frac{1-l}{2\gamma_0 l} \int_S^T E(t)^\alpha (g \odot \nabla u)(t) dt. \tag{4.8}
\end{aligned}$$

On the other hand, by using Hölder, Poincaré, Young inequalities, Lemma 4.1 and (2.5), we obtain for any $\delta_0 > 0$:

$$\begin{aligned}
 \left| \int_S^T E(t)^\alpha \|\nabla u\|^{2r} \int_\Omega |u_t|^{m(x)-2} u_t dx dt \right| &\leq \int_S^T E(t)^\alpha \|\nabla u\|^{2r} \left| \int_\Omega |u_t|^{m(x)-2} u_t dx \right| dt & (4.9) \\
 &\leq \int_S^T E(t)^\alpha \|\nabla u\|^{2r} \left(\delta_0 \int_\Omega |u_t|^{m(x)} dx + \int_\Omega C_{\delta_0}(x) |u_t|^{m(x)} dx \right) dt \\
 &\leq \delta_0 \gamma_1 \int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)} dt + \overline{C}_{\delta_0} \int_S^T E(t)^\alpha \|\nabla u\|^{2r} \int_\Omega |u_t|^{m(x)} dx dt \\
 &\leq \delta_0 \gamma_1 \int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)} dt + \overline{C}_{\delta_0} \int_S^T E(t)^\alpha (-E'(t)) dt \\
 &= \delta_0 \gamma_1 \int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)} dt + \frac{\overline{C}_{\delta_0}}{\alpha + 1} \int_T^S d(E(t)^{\alpha+1}) \\
 &\leq \delta_0 \gamma_1 \int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)} dt + C_5 E(S)^{\alpha+1} \\
 &\leq \delta_0 \gamma_1 \int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)} dt + C_5 E(0)^\alpha E(S) \\
 &\leq \delta_0 \gamma_1 \int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)} dt + C_6 E(S), & (4.10)
 \end{aligned}$$

where \overline{C}_{δ_0} is an upper bound of $C_{\delta_0}(x) = \frac{m(x)-1}{m(x)} (\delta_0 m(x))^{-\frac{1}{m(x)-1}}$. Utilizing (4.6)-(4.9) into (4.5) and setting $\delta_0 := \frac{\gamma_0}{\gamma_1}$, then we have

$$\begin{aligned}
 (1 - \gamma_0) \int_S^T E(t)^\alpha \left(\|u_t\|^2 + \int_\Omega |u_t|^{q(x)} dx + I(u(t)) \right) dt &\leq (2 - \gamma_0) \int_S^T E(t)^\alpha \|u_t\|^2 dt + \gamma_0 \int_S^T E(t)^\alpha \|\nabla u\|^2 dt \\
 &\quad + \left(1 + \frac{1-l}{2\gamma_0 l} - \gamma_0 \right) \int_S^T E(t)^\alpha (g \odot \nabla u)(t) dt + C_7 E(S), & (4.11)
 \end{aligned}$$

where inequality (3.7) and the assumptions of Theorem 4.2 about γ_0 and p^+ have been used. Therefore, by using the definition of $E(t)$, inequality (3.8) and since $\xi(t)$ is a positive differentiable function, we deduce

$$\left[2(1 - \gamma_0) - \frac{\gamma_0}{\beta l} \right] \int_S^T E(t)^{\alpha+1} dt \leq (2 - \gamma_0) \int_S^T E(t)^\alpha \|u_t\|^2 dt + \left(1 + \frac{1-l}{2\gamma_0 l} - \gamma_0 \right) \int_S^T E(t)^\alpha \xi(t) (g \odot \nabla u)(t) dt + C_7 E(S). \tag{4.12}$$

At this point, we estimate the first term on the right hand side of (4.12). For this goal, by using the inequality (3.6) we have

$$\|\nabla u\|^{2r} \leq \left(\frac{E(0)}{\beta l} \right)^r, \tag{4.13}$$

thus, by using (2.5), (4.13) and for sufficiently small r ($r < \frac{m^-}{2}$), we obtain

$$\begin{aligned}
 (2 - \gamma_0) \int_S^T E(t)^\alpha \|u_t\|^2 dt &\leq (2 - \gamma_0) \int_S^T E(t)^\alpha \left(\int_{\Omega_-} |u_t|^2 dx + \int_{\Omega_+} |u_t|^2 dx \right) dt \\
 &\leq C_8 \int_S^T E(t)^\alpha \left[\left(\int_{\Omega_-} |u_t|^{m^+} dx \right)^{\frac{2}{m^+}} + \left(\int_{\Omega_+} |u_t|^{m^-} dx \right)^{\frac{2}{m^-}} \right] dt \\
 &\leq C_8 \int_S^T E(t)^\alpha \left[\left(\int_{\Omega_-} |u_t|^{m(x)} dx \right)^{\frac{2}{m^+}} + \left(\int_{\Omega_+} |u_t|^{m(x)} dx \right)^{\frac{2}{m^-}} \right] dt \\
 &\leq C_8 \int_S^T E(t)^\alpha \left[\left(\frac{\|\nabla u\|^{2r}}{\left(\frac{E(0)}{\beta l} \right)^r} \int_{\Omega} |u_t|^{m(x)} dx \right)^{\frac{2}{m^+}} + \left(\frac{\|\nabla u\|^{2r}}{\left(\frac{E(0)}{\beta l} \right)^r} \int_{\Omega} |u_t|^{m(x)} dx \right)^{\frac{2}{m^-}} \right] dt \\
 &\leq C_9 \int_S^T E(t)^\alpha (-E'(t))^{\frac{2}{m^+}} dt + C_9 \int_S^T E(t)^\alpha (-E'(t))^{\frac{2}{m^-}} dt. & (4.14)
 \end{aligned}$$

Thanks to the Young’s inequality, we get for any $\varepsilon > 0$

$$\int_S^T E(t)^\alpha (-E'(t))^{\frac{2}{m^+}} dt \leq \frac{\alpha\varepsilon}{\alpha + 1} \int_S^T E(t)^{\alpha+1} dt + \frac{1}{(\alpha + 1)\varepsilon^\alpha} \int_S^T (-E'(t))^{\frac{2(\alpha+1)}{m^+}} dt.$$

We take $\alpha = \frac{m^+-2}{2}$ to find

$$\begin{aligned} \int_S^T E(t)^\alpha (-E'(t))^{\frac{2}{m^+}} dt &\leq \varepsilon C_{10} \int_S^T E(t)^{\alpha+1} dt + C_{\varepsilon,m^+} \int_S^T (-E'(t)) dt \\ &\leq \varepsilon C_{10} \int_S^T E(t)^{\alpha+1} dt + C_{\varepsilon,m^+} E(S). \end{aligned} \tag{4.15}$$

On the other hand, if $m^- = 2$, then we have

$$\int_S^T E(t)^\alpha (-E'(t))^{\frac{2}{m^-}} dt = \int_S^T E(t)^\alpha (-E'(t)) dt \leq C_\varepsilon E(S).$$

Moreover, if $m^- > 2$, then by using the Young’s inequality, we obtain

$$\begin{aligned} \int_S^T E(t)^\alpha (-E'(t))^{\frac{2}{m^-}} dt &\leq \frac{\varepsilon(m^- - 2)}{m^-} \int_S^T E(t)^{\frac{\alpha m^-}{m^- - 2}} dt + \frac{2\varepsilon^{-\frac{m^- - 2}{2}}}{m^-} \int_S^T (-E'(t)) dt \\ &\leq \varepsilon C_{11} \int_S^T E(t)^{\frac{\alpha m^-}{m^- - 2}} dt + C_{\varepsilon,m^-} E(S), \end{aligned} \tag{4.16}$$

it is easy to see that $\frac{\alpha m^-}{m^- - 2} = \alpha + 1 + \frac{m^+ - m^-}{m^- - 2}$ (note that we had $\alpha = \frac{m^+ - 2}{2}$), thus the inequality (4.16) can be rewrite as

$$\begin{aligned} \int_S^T E(t)^\alpha (-E'(t))^{\frac{2}{m^-}} dt &\leq \varepsilon C_{11} \int_S^T E(t)^{\alpha+1} E(t)^{\frac{m^+ - m^-}{m^- - 2}} dt + C_{\varepsilon,m^-} E(S) \\ &\leq \varepsilon C_{11} E(S)^{\frac{m^+ - m^-}{m^- - 2}} \int_S^T E(t)^{\alpha+1} dt + C_{\varepsilon,m^-} E(S) \\ &\leq \varepsilon C_{12} \int_S^T E(t)^{\alpha+1} dt + C_{\varepsilon,m^-} E(S). \end{aligned} \tag{4.17}$$

By combining (4.15) and (4.17) with (4.14), we obtain

$$(2 - \gamma_0) \int_S^T E(t)^\alpha \|u_t\|^2 dt \leq \varepsilon C_{13} \int_S^T E(t)^{\alpha+1} dt + C_\varepsilon E(S), \tag{4.18}$$

where $C_\varepsilon = \max\{(2 - \gamma_0)C_{\varepsilon,m^-}, (2 - \gamma_0)C_{\varepsilon,m^+}\}$. Now, by using the condition (A1), we have

$$\begin{aligned} \int_S^T E(t)^\alpha \xi(t) (g \odot \nabla u)(t) dt &\leq \int_S^T E(t)^\alpha (-g' \odot \nabla u)(t) dt \\ &\leq \int_S^T E(t)^\alpha (-E'(t)) dt \\ &\leq E(S)^{\alpha+1} \\ &\leq E(0)^\alpha E(S) \\ &\leq C_{14} E(S). \end{aligned} \tag{4.19}$$

Finally, suppose that ε is sufficiently small and since $\gamma_0 < \frac{2\beta l}{2\beta l + 1}$, thus by utilizing (4.18) and (4.19) into the inequality (4.12), we get

$$\int_S^T E(t)^{\alpha+1} dt \leq C_{15} E(S). \tag{4.20}$$

Now, if we choose $\alpha = 0$, then we have $\gamma = 1$ and $A = C_{15}$ in Lemma 2.4 and therefore, by letting T goes to infinity, we get the result.

Moreover, by choosing $\alpha = \frac{m^+ - 2}{2} = \frac{\gamma - 1}{2}$ ($m^+ > 2$), then with $\gamma > 1$ hypotheses of Lemma 2.4 hold and the proof of Theorem 4.2 is completed. \square

5 Blow-up

In this section, we shall prove the blow-up of solutions with positive as well as negative initial energy. Under an appropriate range of initial data and variable exponents, we show that there exists a finite time t^* such that any local solution of the problem (1.1)-(1.3) blows up at this time, that is

$$\lim_{t \rightarrow t^*} u(x, t) = +\infty.$$

The following Lemma will be used in the proof of blow-up results:

Lemma 5.1. Assume that assumptions of Theorem 2.5 hold. Moreover suppose that for sufficiently large q^- , we have $I(u(t)) < 0$. Then for any weak solution of (1.1)-(1.3), there exists a constant $\gamma_2 > 1$ such that

$$\int_{\Omega} |u|^{m(x)} dx \leq \gamma_2 \|\nabla u\|^2. \tag{5.1}$$

Proof . By using (A2) and embedding $H_0^1(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$, one could easily get the inequality (5.1) with positive as well as negative initial energy similar to Lemma 4.1. \square

5.1 Blow-up with positive initial energy

Our blow-up result for problem (1.1)-(1.3) with positive initial energy $E(0) \geq 0$ reads as follows:

Theorem 5.2. Let $u(x, t)$ is a local solution of problem (1.1)-(1.3) and the conditions of Theorem 2.5 hold. Moreover, suppose that $E(0) > 0$ (maybe sufficiently large) and

(B1)

$$\max\left\{\frac{\sqrt{2}}{l}, \frac{(r+1)(m^- + \gamma_2)}{m^-}, \frac{(p^+)^2}{p^-}\right\} \leq m^+ \leq \frac{q^-}{2},$$

$$\int_{\Omega} u_0(x)u_1(x)dx > m^+ C_* E(0) > 0,$$

Then there exists a finite time which the solutions of problem (1.1)-(1.3) blow up at this time.

Proof . Suppose that u is a global solution of the problem (1.1)-(1.3). Let define

$$A(t) = \int_{\Omega} uu_t dx. \tag{5.2}$$

Differentiate (5.2), we get

$$\begin{aligned} A'(t) &= \|u_t\|^2 + \int_{\Omega} uu_{tt} dx \\ &= \|u_t\|^2 - (1 - \int_0^t g(s) ds) \|\nabla u\|^2 - \|\nabla u\|^{2(r+1)} - a \int_{\Omega} |\nabla u|^{p(x)} dx \\ &\quad - b \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right) + \int_0^t g(t-s) \int_{\Omega} \nabla u(\nabla u(s) - \nabla u) dx ds \\ &\quad - \|\nabla u\|^{2r} \int_{\Omega} u|u_t|^{m(x)-2} u_t dx + \int_{\Omega} |u|^{q(x)} dx. \end{aligned} \tag{5.3}$$

By using definitions of $A(t)$ and $E(t)$, we have for any $\delta, \varepsilon > 0$:

$$\begin{aligned}
 A'(t) \geq & \delta(A(t) - \varepsilon E(t)) - \delta \int_{\Omega} uu_t dx + (1 + \frac{\delta\varepsilon}{2})\|u_t\|^2 \\
 & + (\frac{\delta\varepsilon}{2} - 1)(1 - \int_0^t g(s)ds)\|\nabla u\|^2 + (\frac{\delta\varepsilon}{2(r+1)} - 1)\|\nabla u\|^{2(r+1)} \\
 & + a(\frac{\delta\varepsilon}{2p^+} - 1) \int_{\Omega} |\nabla u|^{p(x)} dx + b(\frac{\delta\varepsilon}{2(p^+)^2} - \frac{1}{p^-}) (\int_{\Omega} |\nabla u|^{p(x)} dx)^2 \\
 & + (1 - \frac{\delta\varepsilon}{q^-}) \int_{\Omega} |u|^{q(x)} dx + \int_0^t g(t-s) \int_{\Omega} \nabla u(\nabla u(s) - \nabla u) dx ds \\
 & + \frac{\delta\varepsilon}{2} (g \odot \nabla u)(t) - \|\nabla u\|^{2r} \int_{\Omega} u|u_t|^{m(x)-2} u_t dx,
 \end{aligned} \tag{5.4}$$

where condition (A2) has been used. At this point, by using the Hölder and Young inequalities, we estimate the terms on the right hand side of (5.4) as follows

$$\int_0^t g(t-s) \left| \int_{\Omega} \nabla u(\nabla u(s) - \nabla u) dx \right| ds \leq (1-l)\|\nabla u\|^2 + \frac{1}{4}(g \odot \nabla u)(t). \tag{5.5}$$

Also, by using (5.1) and (2.5), we have

$$\begin{aligned}
 \|\nabla u\|^{2r} \left| \int_{\Omega} u|u_t|^{m(x)-2} u_t dx \right| & \leq \|\nabla u\|^{2r} \left(\int_{\Omega} \frac{1}{m(x)} |u|^{m(x)} dx + \int_{\Omega} \frac{m(x)-1}{m(x)} |u_t|^{m(x)} dx \right) \\
 & \leq \frac{\gamma_2}{m^-} \|\nabla u\|^{2(r+1)} + \frac{m^+ - 1}{m^+} \|\nabla u\|^{2r} \int_{\Omega} |u_t|^{m(x)} dx \\
 & \leq \frac{\gamma_2}{m^-} \|\nabla u\|^{2(r+1)} - \frac{m^+ - 1}{m^+} E'(t).
 \end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6) with (5.4), we deduce

$$\begin{aligned}
 A'(t) \geq & \delta(A(t) - \varepsilon E(t)) - \delta \int_{\Omega} uu_t dx + (1 + \frac{\delta\varepsilon}{2})\|u_t\|^2 + (\frac{\delta\varepsilon l}{2} - 1)\|\nabla u\|^2 \\
 & + (\frac{\delta\varepsilon}{2(r+1)} - \frac{\gamma_2}{m^-} - 1)\|\nabla u\|^{2(r+1)} + a(\frac{\delta\varepsilon}{2p^+} - 1) \int_{\Omega} |\nabla u|^{p(x)} dx \\
 & + b(\frac{\delta\varepsilon}{2(p^+)^2} - \frac{1}{p^-}) (\int_{\Omega} |\nabla u|^{p(x)} dx)^2 + (1 - \frac{\delta\varepsilon}{q^-}) \int_{\Omega} |u|^{q(x)} dx \\
 & + \int_0^t g(t-s) \int_{\Omega} \nabla u(\nabla u(s) - \nabla u) dx ds \\
 & + (\frac{\delta\varepsilon}{2} - \frac{1}{4})(g \odot \nabla u)(t) + \frac{m^+ - 1}{m^+} E'(t),
 \end{aligned} \tag{5.7}$$

where the fact that $1 - \int_0^t g(s)ds > 1 - \int_0^\infty g(s)ds = l$, has been used. Now if we choose $\delta\varepsilon := 2m^+$, then we obtain

$$A'(t) \geq \frac{2m^+}{\varepsilon} (A(t) - \varepsilon E(t)) - \frac{2m^+}{\varepsilon} \int_{\Omega} uu_t dx + (m^+ + 1)\|u_t\|^2 + (lm^+ - 1)\|\nabla u\|^2 + \frac{m^+ - 1}{m^+} E'(t), \tag{5.8}$$

where (B1) has been used. On the other hand, by using Cauchy-Schwartz and Young inequalities, we have

$$\left| \int_{\Omega} uu_t dx \right| \leq \frac{\varepsilon(lm^+ - 1)}{2m^+ C_*^2} \|u\|^2 + \frac{m^+ C_*^2}{2\varepsilon(lm^+ - 1)} \|u_t\|^2,$$

Thus by using Poincaré inequality, we have

$$\frac{2m^+}{\varepsilon} \left| \int_{\Omega} uu_t dx \right| \leq (lm^+ - 1)\|\nabla u\|^2 + \frac{(m^+)^2 C_*^2}{\varepsilon^2 (lm^+ - 1)} \|u_t\|^2. \tag{5.9}$$

Now we choose $\varepsilon := m^+C_*$, then we get

$$\begin{aligned} \frac{2m^+}{\varepsilon} \left| \int_{\Omega} uu_t dx \right| &\leq (lm^+ - 1) \|\nabla u\|^2 + \frac{1}{lm^+ - 1} \|u_t\|^2 \\ &\leq (lm^+ - 1) \|\nabla u\|^2 + (m^+ + 1) \|u_t\|^2, \end{aligned} \tag{5.10}$$

where we used this fact that if $m^+ > \frac{\sqrt{2}}{l}$ then $(lm^+ - 1)^{-1} < lm^+ + 1 < m^+ + 1$. Applying (5.10) into (5.8), we obtain

$$A'(t) \geq \frac{2}{C_*} (A(t) - m^+C_*E(t)) + \frac{m^+ - 1}{m^+} E'(t),$$

Therefore it is easy to see that

$$\begin{aligned} A'(t) - m^+C_*E'(t) &\geq -(m^+C_* - \frac{m^+ - 1}{m^+})E'(t) + \frac{2}{C_*} (A(t) - m^+C_*E(t)) \\ &\geq \frac{2}{C_*} (A(t) - m^+C_*E(t)), \end{aligned}$$

Now if we set $H(t) = A(t) - m^+C_*E(t)$ then we have

$$H'(t) \geq \frac{2}{C_*} H(t). \tag{5.11}$$

By (B1), it holds that $H(0) = A(0) - m^+C_*E(0) > 0$. Therefore, by (5.11), we conclude that

$$H(t) \geq e^{2C_*^{-1}t} H(0), \quad \forall t \geq 0,$$

this shows that if time goes to infinity then $H(t)$ grows exponentially. From definitions of $H(t)$ and $A(t)$, it is easy to see that

$$H(t) = A(t) - m^+C_*E(t) \leq \int_{\Omega} uu_t dx \leq \frac{C_*^2}{2} \|\nabla u\|^2 + \frac{1}{2} \|u_t\|^2. \tag{5.12}$$

Thus inequality (5.12) shows that since $H(t)$ grows exponentially then must $\|\nabla u\|$ or $\|u_t\|$ grow exponentially. Monotonicity of energy yields

$$E(0) = E(t) - \frac{1}{2} \int_0^t (g' \odot \nabla u)(s) ds + \frac{1}{2} \int_0^t g(s) \|\nabla u(s)\|^2 ds + \int_0^t \|\nabla u\|^{2r} \int_{\Omega} |u_t(s)|^{m(x)} dx ds, \tag{5.13}$$

thus by (A1) and assumption of Theorem 5.2 that $0 < E(t) \leq E(0)$, therefore we obtain from (5.13)

$$\int_0^t \|\nabla u\|^{2r} \int_{\Omega} |u_t(s)|^{m(x)} dx ds \leq E(0). \tag{5.14}$$

Finally, since $m(x) > 2$, we have

$$\int_0^t \|\nabla u\|^{2r} \int_{\Omega} |u_t|^2 dx ds \leq C_8 E(0), \tag{5.15}$$

which contradicts the previous result that $\|\nabla u\|$ or $\|u_t\|$ are exponentially growing. Therefore there exists a finite time T^* such that solutions of problem (1.1)-(1.3) blow up and proof of Theorem 5.2 is completed. \square

5.2 Blow-up with negative initial energy

At this point, we shall prove the blow-up of solutions for problem (1.1)-(1.3) when the initial energy is negative. For this goal, we make the following condition on variable exponents:

(B2)

$$\max\left\{\frac{2(p^+)^2}{p^-}, 2(r+1)\left(1 + \frac{\gamma_2}{m^-}\right)\right\} < m^+ < q^-.$$

Now, we are in a position to state and prove our blow-up result.

Theorem 5.3. Let the assumptions of Theorem 2.5 and (B2) be satisfied and assume that $E(0) < 0$. Then the solution to the problem (1.1)-(1.3) blows up in finite time T^* such that for a positive constant η

$$T^* \leq \frac{1 - \theta}{\eta \theta \psi^{\frac{\theta}{1-\theta}}(0)},$$

where $0 < \theta < 1$, and ε is sufficiently small such that

$$\psi(t) = (-E(t))^{1-\theta} + \varepsilon \int_{\Omega} uu_t dx.$$

Proof . Define $H(t) = -E(t)$ and thus by using (2.5), we arrive at energy relation

$$H'(t) = -E'(t) = \|\nabla u\|^{2l} \int_{\Omega} |u_t|^{m(x)} dx \geq 0. \quad (5.16)$$

Since $E(0) < 0$, then (5.16) gives $H(t) \geq H(0) > 0$. Also

$$H(t) \leq \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx \leq \frac{1}{q^-} \int_{\Omega} |u|^{q(x)} dx. \quad (5.17)$$

Define

$$\psi(t) = H^{1-\theta}(t) + \varepsilon \int_{\Omega} uu_t dx, \quad (5.18)$$

where $0 < \theta < 1$ and ε is sufficiently small. By taking a derivation of (5.18) and using equation (1.1), we obtain

$$\begin{aligned} \psi'(t) &= (1 - \theta)H^{-\theta}(t)H'(t) + \varepsilon \|u_t\|^2 + \varepsilon \int_{\Omega} uu_{tt} dx \\ &= (1 - \theta)H^{-\theta}(t)H'(t) + \varepsilon \|u_t\|^2 - \varepsilon M(\|\nabla u\|^2) \|\nabla u\|^2 - \varepsilon \left(a + b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \int_{\Omega} |\nabla u|^{p(x)} dx \\ &\quad + \varepsilon \int_0^t g(t-s) \int_{\Omega} \nabla u \nabla u(s) dx ds + \varepsilon \int_{\Omega} |u|^{q(x)} dx - \varepsilon \sigma(\|\nabla u\|^2) \int_{\Omega} u |u_t|^{m(x)-2} u_t dx. \end{aligned}$$

Utilizing (A1) gives

$$\begin{aligned} \psi'(t) &\geq (1 - \theta)H^{-\theta}(t)H'(t) + \varepsilon \|u_t\|^2 - \varepsilon(1 + \|\nabla u\|^{2r}) \|\nabla u\|^2 - \varepsilon a \int_{\Omega} |\nabla u|^{p(x)} dx - \frac{\varepsilon b}{p^-} \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^2 + \varepsilon \int_{\Omega} |u|^{q(x)} dx \\ &\quad + \varepsilon \int_0^t g(t-s) \int_{\Omega} \nabla u \nabla u(s) dx ds - \varepsilon \sigma(\|\nabla u\|^2) \int_{\Omega} u |u_t|^{m(x)-2} u_t dx. \end{aligned} \quad (5.19)$$

Using the definition of $H(t)$, it follows

$$\begin{aligned} -\varepsilon m^+ H(t) &= \frac{\varepsilon m^+}{2} \|u_t\|^2 + \frac{\varepsilon m^+}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla u\|^2 + \frac{\varepsilon m^+}{2} (g \odot \nabla u)(t) \\ &\quad + \varepsilon a m^+ \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \frac{\varepsilon m^+ b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 \\ &\quad - \varepsilon m^+ \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx + \frac{\varepsilon m^+}{2(r+1)} \|\nabla u\|^{2(r+1)} \\ &\geq \frac{\varepsilon m^+}{2} \|u_t\|^2 + \frac{\varepsilon m^+}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla u\|^2 + \frac{\varepsilon m^+}{2} (g \odot \nabla u)(t) \\ &\quad + \frac{\varepsilon m^+ a}{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx + \frac{\varepsilon m^+ b}{2(p^+)^2} \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^2 \\ &\quad - \frac{\varepsilon m^+}{q^-} \int_{\Omega} |u|^{q(x)} dx + \frac{\varepsilon m^+}{2(r+1)} \|\nabla u\|^{2(r+1)}, \end{aligned} \quad (5.20)$$

where the condition (A1) has been used. By combining (5.20) and (5.19), we arrive at

$$\begin{aligned}
\psi'(t) &\geq \varepsilon m^+ H(t) + (1 - \theta) H^{-\theta}(t) H'(t) + \frac{\varepsilon(m^+ + 2)}{2} \|u_t\|^2 + \left(\frac{\varepsilon m^+ l}{2} - \varepsilon \right) \|\nabla u\|^2 + \varepsilon \left(\frac{m^+}{2(r+1)} - 1 \right) \|\nabla u\|^{2(r+1)} \\
&\quad + \varepsilon a \left(\frac{m^+}{p^+} - 1 \right) \int_{\Omega} |\nabla u|^{p(x)} dx + \frac{\varepsilon m^+}{2} (g \odot \nabla u)(t) \\
&\quad + \varepsilon b \left(\frac{m^+}{2(p^+)^2} - \frac{1}{p^-} \right) \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^2 + \varepsilon \left(1 - \frac{m^+}{q^-} \right) \int_{\Omega} |u|^{q(x)} dx \\
&\quad + \varepsilon \int_0^t g(t-s) \int_{\Omega} \nabla u \nabla u(s) dx ds - \varepsilon \sigma(\|\nabla u\|^2) \int_{\Omega} u |u_t|^{m(x)-2} u_t dx.
\end{aligned} \tag{5.21}$$

To estimate the last two terms in (5.21), by applying Hölder's inequality, Young's inequality, (5.1), (A1) and (5.16), we obtain

$$\begin{aligned}
\varepsilon \int_0^t g(t-s) \int_{\Omega} \nabla u \nabla u(s) dx ds &= \varepsilon \int_0^t g(t-s) \int_{\Omega} \nabla u (\nabla u(s) - \nabla u) dx ds + \varepsilon \int_0^t g(s) ds \|\nabla u\|^2 \\
&\leq \varepsilon l \|\nabla u\|^2 + \frac{\varepsilon(1-l)}{4l} (g \odot \nabla u)(t) + \varepsilon \int_0^{+\infty} g(s) ds \|\nabla u\|^2 \\
&= \varepsilon \|\nabla u\|^2 + \frac{\varepsilon(1-l)}{4l} (g \odot \nabla u)(t).
\end{aligned} \tag{5.22}$$

and

$$\begin{aligned}
\sigma(\|\nabla u\|^2) \int_{\Omega} u |u_t|^{m(x)-2} u_t dx &\leq \|\nabla u\|^{2r} \left(\int_{\Omega} \frac{1}{m(x)} |u|^{m(x)} dx + \int_{\Omega} \frac{m(x)-1}{m(x)} |u_t|^{m(x)} dx \right) \\
&\leq \frac{\gamma_2}{m^-} \|\nabla u\|^{2(r+1)} + \frac{m^+ - 1}{m^+} \|\nabla u\|^{2r} \int_{\Omega} |u_t|^{m(x)} dx \\
&= \frac{\gamma_2}{m^-} \|\nabla u\|^{2(r+1)} + \frac{m^+ - 1}{m^+} H'(t).
\end{aligned} \tag{5.23}$$

Utilizing (5.22) and (5.23) into (5.21), we get

$$\begin{aligned}
\psi'(t) &\geq \varepsilon m^+ H(t) + (1 - \theta) H^{-\theta}(t) H'(t) - \frac{\varepsilon(m^+ - 1)}{m^+} H'(t) + \frac{\varepsilon(m^+ + 2)}{2} \|u_t\|^2 \\
&\quad + \varepsilon \left(\frac{m^+ l}{2} - 2 \right) \|\nabla u\|^2 + \varepsilon \left(\frac{m^+}{2(r+1)} - \frac{\gamma_2}{m^-} - 1 \right) \|\nabla u\|^{2(r+1)} \\
&\quad + \frac{\varepsilon}{2} \left(m^+ - \frac{1-l}{2l} \right) (g \odot \nabla u)(t) + \varepsilon a \left(\frac{m^+}{p^+} - 1 \right) \int_{\Omega} |\nabla u|^{p(x)} dx \\
&\quad + \varepsilon b \left(\frac{m^+}{2(p^+)^2} - \frac{1}{p^-} \right) \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^2 + \varepsilon \left(1 - \frac{m^+}{q^-} \right) \int_{\Omega} |u|^{q(x)} dx.
\end{aligned} \tag{5.24}$$

At this point, applying (B2) into (5.24) to obtain

$$\psi'(t) \geq \varepsilon m^+ H(t) + (1 - \theta) H^{-\theta}(t) H'(t) - \frac{\varepsilon(m^+ - 1)}{m^+} H'(t). \tag{5.25}$$

Thanks to (5.16) and the definition of $H(t)$, there exists a positive constant K (specified later) such that

$$H'(t) = -E'(t) \leq K H^{-\theta}(t) H'(t),$$

substituting this inequality into (5.25) and since $H(t) \geq 0$, we arrive at

$$\psi'(t) \geq \left[1 - \theta - \frac{\varepsilon(m^+ - 1)}{m^+} K \right] H^{-\theta}(t) H'(t). \tag{5.26}$$

At this moment, by choosing K large enough and picking ε sufficiently small such that $1 - \theta - \frac{\varepsilon(m^+ - 1)}{m^+}K > 0$, we consequently get

$$\psi(t) \geq \psi(0) = H^{1-\theta}(0) + \varepsilon \int_{\Omega} u_0 u_1 dx > 0, \quad \forall t \geq 0.$$

Suppose that C is a generic constant. By using the Hölder and Young inequalities, we have

$$\left| \int_{\Omega} uu_t dx \right|^{\frac{1}{1-\theta}} \leq C(\|u\|_q^{\frac{1}{1-\theta}} \|u_t\|_q^{\frac{1}{1-\theta}}) \leq C(\|u\|_q^{q^-} + \|u_t\|^2 + H(t)). \tag{5.27}$$

Combining (5.27) with (5.18), we obtain for some $\eta > 0$

$$\begin{aligned} \psi^{\frac{1}{1-\theta}}(t) &= \left[H^{1-\theta}(t) + \varepsilon \int_{\Omega} uu_t dx \right]^{\frac{1}{1-\theta}} \\ &\leq 2^{\frac{1}{1-\theta}} \left(H(t) + \varepsilon^{\frac{1}{1-\theta}} \left| \int_{\Omega} uu_t dx \right|^{\frac{1}{1-\theta}} \right) \\ &\leq C(\|u\|_q^{q^-} + \|u_t\|^2 + H(t)) \leq \eta^{-1} \psi'(t), \end{aligned}$$

therefore

$$\psi'(t) \geq \eta \psi^{\frac{1}{1-\theta}}(t). \tag{5.28}$$

Integrating (5.28) from 0 to t , we deduce

$$\psi^{\frac{\theta}{1-\theta}}(t) \geq \frac{1}{\psi^{-\frac{\theta}{1-\theta}}(0) - \frac{\eta \theta t}{1-\theta}}.$$

This shows that solutions blow up in finite time $t^* = \frac{1-\theta}{\eta \theta \psi^{\frac{\theta}{1-\theta}}(0)}$, and proof of Theorem 5.3 has been completed. \square

Acknowledgments

The authors would like to thank the referee for his/her valuable comments and corrections which improved this work.

References

- [1] S. Antontsev, J. Ferreira and E. Pişkin, *Existence and blow up of solutions for a strongly damped petrovsky equation with variable-exponent nonlinearities*, Electron. J. Differ. Eq. **2021** (2021), no. 6, 1–18.
- [2] S. Antontsev, J. Ferreira, E. Pişkin and M.S. Cordeiro, *Existence and non-existence of solutions for Timoshenko-type equations with variable exponents*, Nonlinear Anal. Real World Appl. **61** (2021), 10.1016/j.nonrwa.2021.103341.
- [3] S. Antontsev, J. Ferreira, E. Pişkin, H. Yükksekaya and M. Shahrouzi, *Blow up and asymptotic behavior of solutions for a $p(x)$ -Laplacian equation with delay term and variable exponents*, Electron. J. Differ. Eq. **2021** (2021), no. 84, 1–20.
- [4] S. Antontsev and S. Shmarev, *Evolution PDEs with Nonstandard Growth Conditions: Existence, Uniqueness, Localization, Blow-Up. in: Atlantis Studies in Differential Equations*, Atlantis Press, Paris, 2015.
- [5] I. Chueshov, *Global attractors for a class of Kirchhoff wave models with a structural nonlinear damping*, J. Abstr. Differ. Equ. Appl. **1** (2010), 86–106.
- [6] I. Chueshov, *Long-time dynamics of Kirchhoff wave models with strong nonlinear damping*, J. Differ. Equ. **252** (2012), 1229–1262.
- [7] A.T. Cousin, C.L. Frota and N.A. Larkin, *Global solvability and decay of the energy for the nonhomogeneous Kirchhoff equation*, Differ. Integral Equ. **15** (2002), 1219–1236.

- [8] G. Dai and R. Hao, *Existence of solutions for a $p(x)$ -Kirchhoff-type equation*, J. Math. Anal. Appl. **359** (2009), 275–284.
- [9] L. Diening, P. Hasto, P. Harjulehto and M.M. Ruzicka, *Lebesgue and Sobolev Spapce with Variable Exponents*, Springer-Verlag, 2011.
- [10] S. Ghegal, I. Hamchi and S.A. Messaoudi, *Global existence and stability of a nonlinear wave equation with variable-exponent nonlinearities*, Appl. Anal. **99** (2020), no. 8, 1333–1343.
- [11] M.K. Hamdani, A. Harrabi, F. Mtiri and D.D. Repovš, *Existence and multiplicity results for a new $p(x)$ -Kirchhoff problem*, Nonlinear Anal. **190** (2020), 10.1016/j.na.2019.111598.
- [12] R. Ikehata and A.T. Matsuyama, *On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms*, J. Math. Anal. Appl. **204** (1996), 729–753.
- [13] M.A. Jorge Silva and V. Narciso, *Long-time dynamics for a class of extensible beams with nonlocal nonlinear damping*, Evol. Equ. Control Theory. **6** (2017), 437–470.
- [14] P.P.D. Lazo, *Global solutions for a nonlinear wave equation*, Appl. Math. Comput. **2008** (2008), 596–601.
- [15] P. Martinez, *A new method to obtain decay rate estimates for dissipative systems*, ESAIM Control. Optim. Calc. Var. **4** (1994), 419–444.
- [16] S.A. Messaoudi, *On the decay of solutions of a damped quasilinear wave equation with variable-exponent nonlinearities*, Math. Meth. Appl. Sci. **43** (2020), no. 8, 5114–5126.
- [17] S.A. Messaoudi and A.A. Talahmeh, *A blow-up result for a nonlinear wave equation with variable-exponent nonlinearities*, Appl. Anal. **96** (2017), no. 9, 1509–1515.
- [18] N. Nakao, *An attractor for a nonlinear dissipative wave equation of Kirchhoff type*, J. Math. Anal. Appl. **353** (2009), 652–659.
- [19] V. Narciso, *On a Kirchhoff wave model with nonlocal nonlinear damping*, Evol. Equ. Control. Theory. **9** (2020), no. 2, 478–508.
- [20] K. Ono, *Global existence, decay, and blow up of solution for some mildly degenerate nonlinear Kirchhoff strings*, J. Differ. Equ. **137** (1997), 273–301.
- [21] E. Pişkin, *Finite time blow up of solutions of the Kirchhoff-type equation with variable exponents*, Int. J. Nonlinear Anal. Appl. **11** (2020), no. 1, 37–45.
- [22] M. Shahrouzi, *Blow up of solutions for a $r(x)$ -Laplacian Lamé equation with variable-exponent nonlinearities and arbitrary initial energy level*, Int. J. Nonlinear Anal. Appl. **13** (2022), no. 1, 441–450.
- [23] M. Shahrouzi, *General decay and blow up of solutions for a class of inverse problem with elasticity term and variable-exponent nonlinearities*, Math. Meth. Appl. Sci. **45** (2022), no. 4, 1864–1878.
- [24] M. Shahrouzi, *Global nonexistence of solutions for a class of viscoelastic Lamé system*, Indian J. Pure Appl. Math. **51** (2020), no. 4, 1383–1397.
- [25] M. Shahrouzi, *On behaviour of solutions for a nonlinear viscoelastic equation with variable-exponent nonlinearities*, Comput. Math. Appl. **75** (2018), no. 11, 3946–3956.
- [26] M. Shahrouzi, J. Ferreira and E. Pişkin, *Stability result for a variable-exponent viscoelastic double-Kirchhoff type inverse source problem with nonlocal degenerate damping term*, Ricerche Mat. (2022). <https://doi.org/10.1007/s11587-022-00713-5>
- [27] M. Shahrouzi and F. Kargarfard, *Blow-up of solutions for a Kirchhoff type equation with variable-exponent nonlinearities*, J. Appl. Anal. **27** (2021), no. 1, 97–105.
- [28] M.T. Taniguchi, *Existence and asymptotic behaviour of solutions to weakly damped wave equations of Kirchhoff type with nonlinear damping and source terms*, J. Math. Anal. Appl. **361** (2010), 566–578.
- [29] Z.J. Yang, *Long-time behavior of the Kirchhoff type equation with strong damping in R^N* , J. Differ. Equ. **242** (2007), 269–286.
- [30] H. Zhang, D. Li and Q. Hu, *General energy decay of solutions for a wave equation with nonlocal nonlinear damping*

and source terms, Chin. Quart. J. Math. **35** (2020), no. 3, 302–310.

- [31] H. Zhang, D. Li, W. Zhang and Q. Hu, *Asymptotic stability and blow-up for the wave equation with degenerate nonlocal nonlinear damping and source terms*, Appl. Anal. **101** (2020), no. 9, 3170-3181.