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Certain new type of fixed point results in partially ordered M-metric space

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Abstract

The most recent performances in fixed point theory related to the fixed point, coincidence point, and coupled coincidence point involving mappings in ordered metric spaces are the result of concentrated overwork ordered metric space. Its conclusion was expansive and generalized to well-known oral literature results. A few fixed point outcomes were discovered to be sophisticated self-mappings. Anything that satisfies a generalized week contraction was partially ordered as m-metric space (mms). The specific results also include two self-mappings for coupling coincidence points, coupled common fixed points, and coincidence points in the same qualification. An example is offered to support the findings.

Keywords: M-metric, (ϕ, ψ)-contraction mapping, fixed point
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1 Introduction and Preliminaries

Based on functional analysis, fixed point theory is an useful and important category. It offers a pioneering framework for resolving issues in many mathematical analysis extensions. It's indeed uniqueness, integral solution, and differential equations agreement. Mathematician Banach [7] permits the contraction in 1922, establishing it one of the most well-known and significant examples of practical mathematics. There are essentially two main approaches to employ the generalized Banach contraction principle : either to modify the metric space or to revise the contraction condition. Several contractions in fixed theory, defined in metric spaces, include Boyd and Wong's nonlinear contraction principle [8], Suzuki Contraction [27], Kannan Contraction [13], Ćirić Generalized Contraction [10], Ciric's Quasi Contraction [11], Weak_Contraction [23], Chatterjea Contraction [9], Zamfirescu Contraction [29] and many more [6, 20].

Wardowski [28] introduced innovative contraction being real-valued mappings in 2012, defined positive real numbers, and obtained fixed point theory. In various metric space, authors have worked on F-contraction mapping. Piri and Kumam [21] worked in 2014 and Minak et al. [18] expanded results and applied weaker condition self mappings. For generalized F-contractions counting Ciric type generic F-contraction and almost F-contraction in complete metric

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space, Piri et al. [22] achieved results in 2014. In order to extend several results, Piri and Kumam tried to introduced F-contraction setting in the context of complete asymmetric metric space in 2017. Regarding F-contraction in b-metric space, Kadelburg and Radenovic [16] achieved results in 2018. In 2019 saw the introduction of fixed point theory for F-contraction in partial metric space by Luambano [17], and he attained a number of findings that are good examples, see [25, 24].

Asadi et al. [1] provided m-metric Space in 2014, extending the p-metric space presented in [2] and demonstrating the Banach contraction principle. In metric space, several authors were engaged [2, 19]. According to Shukla [26], p-metric space is a generalization of rectangular metric space. In 2019, Asim et al. [3, 4, 5] extended rectangular metric space and generalized rectangular m-metric space as rectangular Mb-metric space. See [12, 14, 15] for more related results.

In this article, we introduce generalized weak contraction condition watching over altering distance functions $\phi \in \Phi$, $\psi \in \Psi$ below acquire fixed point of mapping $A : \hat{W} \to \hat{W}$ in $M-$metric space.

$$(1)\ \phi\left(M(A, A\kappa_2)\right) \leq \phi(\text{Ç}(\kappa_1, \kappa_2)) - \psi(\text{D}(\kappa_1, \kappa_2)) \text{ for any } \kappa_1, \kappa_2 \in \hat{W},$$

with $\kappa_1 \leq \kappa_2$ and where

$$\text{Ç}(\kappa_1, \kappa_2) = \max \left\{ \begin{array}{c} \frac{M(\kappa_2, A\kappa_2)[1+M(\kappa_1, A\kappa_1)]}{1+M(\kappa_1,\kappa_2)}, \frac{M(\kappa_1, A\kappa_1)M(\kappa_2, A\kappa_2)}{1+M(\kappa_1,\kappa_2)}, \\ M(\kappa_2, A\kappa_2), M(\kappa_1, A\kappa_1), M(\kappa_1, \kappa_2) \end{array} \right\},$$

and

$$\text{D}(\kappa_1, \kappa_2) = \max \left\{ \frac{M(\kappa_2, A\kappa_2)[1 + M(\kappa_1, A\kappa_1)]}{1 + M(\kappa_1, \kappa_2)}, M(\kappa_1, \kappa_2) \right\}.$$

We utilize the following conditions for adjusting distance functions: $\Phi = \{\phi/\phi$ continuous, non-decreasing self mapping on $[0, +\infty)\}$ $\phi(\upsilon) = 0$ if and only if $\upsilon = 0$, for $\upsilon \in [0, +\infty)\}$ and $\Psi = \{\psi/\psi$ lower semi-continuous self mapping on $[0, +\infty)$ such that $\psi(\upsilon) = 0$ if and only if $\upsilon = 0$, where $\upsilon \in [0, +\infty)\}$.

Notation: $m_{x,y} := \min\{m(x, x), m(y, y)\}$; $M_{x,y} := \max\{m(x, x), m(y, y)\}$

Definition 1.1. [28] For a given non empty set $\hat{W}$, we say that a function $M : \hat{W} \times \hat{W} \to [0, 1)$ is an $M-$metric if

(m1) $M(\kappa_1, \kappa_1) = M(\kappa_2, \kappa_2) = M(\kappa_1, \kappa_2)$ if and only if $\kappa_1 = \kappa_2$,

(m2) $m_{\kappa_1,\kappa_2} \leq M(\kappa_1, \kappa_2)$,

(m3) $M(\kappa_1, \kappa_2) = M(\kappa_2, \kappa_1)$,

(m4) $(M(\kappa_1, \kappa_2) - M_{\kappa_1,\kappa_2}) \leq (M(\kappa_1, z) - M_{\kappa_1,z}) + (M(z, \kappa_2) - M_{z,\kappa_2})$.

then, pair (κ_1, M) is called $M-$metric space.

Example 1.2. [28] Let M be $M-$metric space. Put

(1) $\varpi_m^{\check{z}}(\kappa_1, \kappa_2) = M(\kappa_1, \kappa_2) - 2m_{\kappa_1,\kappa_2} + M_{\kappa_1,\kappa_2}$,

(2) $\varpi_m^{\hat{s}}(\kappa_1, \kappa_2) = M(\kappa_1, \kappa_2) - m_{\kappa_1,\kappa_2}$ if $\kappa_1 \neq \kappa_2$, and $\varpi_m^{\hat{s}}(\kappa_1, \kappa_2) = 0$ if $\kappa_1 = \kappa_2$.

Then $\varpi_m^{\check{z}}$ and $\varpi_m^{\hat{s}}$ ordinary metrics.

As mentioned [28], each M-metric on $\hat{W}$ generates T_o topology τ_M on $\hat{W}$. Then set

$$\{B_M(\kappa_1, \varepsilon) : \kappa_1 \in \hat{W}, \varepsilon > 0\}$$

where

$$B_M(\kappa_1, \varepsilon) : \{\kappa_2 \in \hat{W}, M(\kappa_1, \kappa_2) < \varpi_{m\kappa_1,\kappa_2} + \varepsilon\}$$

for all $\kappa_1 \in \hat{W}$ and $\varepsilon > 0$, forms basis of τ_M.

Remark 1.3. [28] For every $\kappa_1, \kappa_2 \in \hat{W}$

1. $0 \le M_{\kappa_1,\kappa_2} + m_{\kappa_1,\kappa_2} = M(\kappa_1,\kappa_1) + M(\kappa_2,\kappa_2)$,
2. $0 \le M_{\kappa_1,\kappa_2} - m_{\kappa_1,\kappa_2} = |M(\kappa_1,\kappa_1) + M(\kappa_2,\kappa_2)|$,
3. $M_{\kappa_1,\kappa_2} - m_{\kappa_1,\kappa_2} \le (M_{\kappa_1,z} - m_{z,\kappa_2}) + (M_{z,\kappa_2} + m_{z,\kappa_2})$.

Definition 1.4. [28] Let $(\hat{W}, M)$ be M-metric space. Then:

1. A sequence $\{\kappa_{1n}\}$ in M-metric space $(\hat{W}, M)$ converges to $\kappa_1 \in \hat{W}$ if and only if

$$\lim_{n\to\infty}(M(\kappa_{1n},\kappa_1) - M_{\kappa_{1n},\kappa_1}) = 0.$$

2. A $\{\kappa_{1n}\}$ in M-metric space $(\hat{W}, M)$ is called M-Cauchy sequence if

$$\lim_{n\to\infty}(M(\kappa_{1n},\kappa_{1m}) - M_{\kappa_{1n},\kappa_{1m}}), \qquad \lim_{n\to\infty}(M_{\kappa_{1n},\kappa_{1m}} - m_{\kappa_{1n},\kappa_{1m}}),$$

exist (and are finite).

3. An M-metric space $(\hat{W}, M)$ is said complete if every M-Cauchy sequence $\{\kappa_{1n}\}$ in $\hat{W}$ converges, with respect to τ_M, to $\kappa_1 \in \hat{W}$ such that

$$\lim_{n\to\infty}(M(\kappa_{1n},\kappa_1) - M_{\kappa_{1n},\kappa_1}) = 0 \quad \text{and} \quad \lim_{n\to\infty}(M_{\kappa_{1n},\kappa_1} - m_{\kappa_{1n},\kappa_1}) = 0.$$

Lemma 1. [21] If $\{\kappa_{1n}\}_{n\in N}$ and $\{\kappa_{2n}\}_{n\in N}$ are two sequences such that $\kappa_{1n} \to \kappa_1$ and $\kappa_{2n} \to \kappa_2$ as $n \to \infty$ in $M-$metric space $(\hat{W}, M)$, then

$$\lim_{n\to\infty}(M(\kappa_{1n},\kappa_{2n}) - m_{\kappa_{1n},k_{2n}}) = M(\kappa_1,\kappa_2) - m_{\kappa_1,\kappa_2}).$$

Lemma 2. [21] If $\{\kappa_{1n}\}_{n\in N}$ is sequence such that $\kappa_{1n} \to \kappa_1$ as $n \to \infty$ in $M-$metric space $(\hat{W}, M)$, then

$$\lim_{n\to\infty}(M(\kappa_{1n},\kappa_2) - m_{\kappa_{1n},\kappa_2}) = M(\kappa_1,\kappa_2) - m_{\kappa_1,\kappa_2}),$$

for all $\kappa_1, \kappa_2 \in \hat{W}$. Moreover, $M(\kappa_1,\kappa_2) = m_{\kappa_1,\kappa_2}$. Further if $M(\kappa_1,\kappa_1) = M(\kappa_2,\kappa_2)$, then $\kappa_1 = \kappa_2$.

Lemma 3. [21] Let $(\hat{W}, M)$ be M-metric space & self mapping A, consider sequence $\{\kappa_{1n}\}_{n\in N}$ defined by $\kappa_{1n+1} = A\kappa_{1n}$. If $\kappa_{1n} \to \tilde{u}$ as $n \to \infty$, then $A\kappa_{1n} \to A\tilde{u}$ as $n \to \infty$.

Lemma 4. [28] Let $\{\kappa_{1n}\}$ be sequence in M-metric space $(\hat{W}, M)$ such that there exist $r \in [0, 1)$ and $M(\kappa_{1n+1},\kappa_{1n}) \le rM(\kappa_{1n},\kappa_{1n-1})$, for all $n \in N$. Then

A1) $\lim_{n\to\infty} M(\kappa_{1n},\kappa_{1n-1}) = 0$,

A2) $\lim_{n\to\infty} M(\kappa_{1n},\kappa_{1n}) = 0$,

A3) $\lim_{\varpi_m,n\to\infty} m_{\kappa m,\kappa_{1n}} = 0$,

A4) $\{\kappa_{1n}\}$ is an M-cauchy sequence.

Definition 2.1 [24] Let f and g two self$-$mappings on $\hat{W}$. If $\omega = f\kappa_1 = g\kappa_1$ for $\kappa_1 \in \hat{W}$, then κ_1 is called coincidence point of f and g, where ω is called point of coincidence of f & g.

Definition 2.2 [24] Let f and g two self$-$mappings defined on $\hat{W}$. Then f and g said to weakly compatible if they commute every coincidence point, i.e., if $f\kappa_1 = g\kappa_1$ for some $\kappa_1 \in \hat{W}$, then $fg\kappa_1 = gf\kappa_1$.

Let denote set of $\varphi : [\hat{W}, +\infty) \to [\hat{W}, +\infty)$ satisfying:

(1) φ continuous and non$-$decreasing;

(2) $\varphi(\upsilon) = 0$ if and only if $\upsilon = 0$.

2 Main results

In this section, we derived several fixed point result in the context of complete partially ordered M-metric space.

Theorem 2.1. Suppose $(\hat{W}, M, \preceq)$ be complete partially ordered $M-$metric space. Consider continuous $A : \hat{W} \to \hat{W}$ non$-$decreasing with respect to $\preceq$ and satisfies (1). If some $\kappa_1 \in \hat{W}$ such that $\kappa_1 \preceq A\kappa_1$, then A has a fixed point in $\hat{W}$.

Proof . Consider $A\kappa_0 = \kappa_0$, for some $\kappa_0 \in \hat{W}$. assume not then $\kappa_0 \prec A\kappa_0$. Now sequence $\{\kappa_{1n}\} \subset \hat{W}$ defined by $\kappa_{1n+1} = A\kappa_{1n}$, for $n \geq 0$. Since A non$-$decreasing,

$$(2)\ \kappa_0 \prec A\kappa_0 = \kappa_{11} \preceq \cdots \preceq \kappa_{1n} \preceq A\kappa_{1n} = \kappa_{1n+1} \preceq \cdots .$$

If as some $n_0 \in N$, $\kappa_{1n_0} = \kappa_{1n_0+1}$,then from (4.2), A has fixed point κ_{1n_0}. Assume $\kappa_{1n} \neq \kappa_{1n+1}$ $\forall$ $n \geq 1$. Since $\kappa_{1n} > \kappa_{1n-1}$ for all $n \geq 1$, then (1),

$$\begin{aligned}(3)\ \phi(M(\kappa_{1n}, \kappa_{1n+1})) &= \phi(M(A\kappa_{1n-1}, A\kappa_{1n})) \\ &\leq \phi(\text{Ç}(\kappa_{1n-1}, \kappa_{1n})) - \psi(\text{D}(\kappa_{1n-1}, \kappa_{1n})).\end{aligned}$$

Thus from (3),

$$(4)\ M(\kappa_{1n}, \kappa_{1n+1}) = M(A\kappa_{1n-1}, A\kappa_{1n}) \leq \text{Ç}(\kappa_{1n-1}, \kappa_{1n}),$$

where

$$\begin{aligned}\text{Ç}(\kappa_{1n-1}, \kappa_{1n}) &= \max\left\{\begin{array}{c}\frac{M(\kappa_{1n}, A\kappa_{1n})[1+M(\kappa_{1n-1}, A\kappa_{1n-1})]}{1+M(\kappa_{1n-1}, \kappa_{1n})}, \frac{M(\kappa_{1n}, A\kappa_{1n})M(\kappa_{1n-1}, A\kappa_{1n-1})}{M(\kappa_{1n-1}, \kappa_{1n})} \\ , M(\kappa_{1n}, A\kappa_{1n}), M(\kappa_{1n-1}, A\kappa_{1n-1}), M(\kappa_{1n-1}, \kappa_{1n})\end{array}\right\} \\ &= \max\left\{\begin{array}{c}\frac{M(\kappa_{1n}, \kappa_{1n+1})[1+M(\kappa_{1n-1}, \kappa_{1n})]}{1+M(\kappa_{1n-1}, \kappa_{1n})}, \frac{M(\kappa_{1n}, \kappa_{1n+1})M(\kappa_{1n-1}, \kappa_{1n})}{M(\kappa_{1n-1}, \kappa_{1n})} \\ , M(\kappa_{1n}, \kappa_{1n+1}), M(\kappa_{1n-1}, \kappa_{1n}), M(\kappa_{1n-1}, \kappa_{1n})\end{array}\right\} \\ &= \max\left\{M(\kappa_{1n}, \kappa_{1n+1}), \frac{M(\kappa_{1n}, \kappa_{1n+1})M(\kappa_{1n-1}, \kappa_{1n})}{1+M(\kappa_{1n-1}, \kappa_{1n})}, M(\kappa_{1n-1}, \kappa_{1n})\right\} \\ &\leq \max\left\{M(\kappa_{1n}, \kappa_{1n+1}), M(\kappa_{1n-1}, \kappa_{1n})\right\}.\end{aligned}$$

If $\max\{M(\kappa_{1n}, \kappa_{1n+1}), M(\kappa_{1n-1}, \kappa_{1n})\} = M(\kappa_{1n}, \kappa_{1n+1})$ for some $n \geq 1$, from (4) we have $M(\kappa_{1n}, \kappa_{1n+1}) \leq M(\kappa_{1n}, \kappa_{1n+1})$ this a contradiction. Hence,

$$\max\{M(\kappa_{1n}, \kappa_{1n+1}), M(\kappa_{1n-1}, \kappa_{1n})\} = M(\kappa_{1n-1}, \kappa_{1n}) \text{ for all } n \geq 1.$$

Thus (4)

$$(5)\ M(\kappa_{1n}, \kappa_{1n+1}) \leq M(\kappa_{1n-1}, \kappa_{1n}).$$

Now, we prove that $\{\kappa_{1n}\}$ is an $M-$cauchy sequence $\hat{W}$. By (M4),

$$\begin{aligned}M(\kappa_{2n+1}, \kappa_{2n+3}) - m_{\kappa_{2n+1}, \kappa_{2n+3}} &\leq \left(M(\kappa_{12n+1}, \kappa_{12n+2}) - m_{\kappa_{12n+1}, \kappa_{12n+2}}\right) + \left(M(\kappa_{12n+2}, \kappa_{12n+3}) - m_{\kappa_{12n+2}, \kappa_{12n+3}}\right) \\ &\leq M(\kappa_{12n+1}, \kappa_{12n+2}) + M(\kappa_{12n+2}, \kappa_{12n+3}).\end{aligned}$$

Similarly

$$\begin{aligned}M(\kappa_{12n+1}, \kappa_{12n+4}) - m_{\kappa_{12n+1}, \kappa_{12n+4}} &\leq \left(M(\kappa_{12n+1}, \kappa_{12n+2}) - m_{\kappa_{12n+1}, \kappa_{12n+2}}\right) + \left(M(\kappa_{12n+2}, \kappa_{12n+4}) - m_{\kappa_{12n+2}, \kappa_{12n+4}}\right) \\ &\leq M(\kappa_{12n+1}, \kappa_{12n+2}) + M(\kappa_{12n+2}, \kappa_{12n+3}) + M(\kappa_{12n+3}, \kappa_{12n+4}).\end{aligned}$$

In general, for all $q > p > n_1$ with $p = 2n + 1$, we obtain

$$M(\kappa_{1p}, \kappa_{1q}) - m_{\kappa_{1p}, \kappa_{1q}} \leq \sum_{i=p}^{q-1} M(\kappa_{1i}, \kappa_{1i+1}) \leq \sum_{i=p}^{q-1} \mu_i.$$

The convergence of series $\sum_{i=p}^{\infty} \mu_i$ leads

$$(6)\ \lim_{p,q \to \infty} \left(M(\kappa_{1p}, \kappa_{1q}) - m_{\kappa_{1p}, \kappa_{1q}}\right) = 0.$$

By same way,

$$\begin{aligned} m_{\kappa_{1p},\kappa_{1q}} - m_{\kappa_{1p},\kappa_{1q}} &\leq \sum_{i=p}^{q-1} \left(m_{\kappa_{1i},\kappa_{1i+1}} - m_{\kappa_{1i},\kappa_{1i+1}}\right) \leq \sum_{i=p}^{q-1} m_{m\kappa_{1i},\kappa_{1i+1}}, \\ &\leq \sum_{i=p}^{q-1} M(\kappa_{1i},\kappa_{1i}) \leq \sum_{i=p}^{q-1} M(\kappa_{1i},\kappa_{1i+1}), \\ &\leq \sum_{i=p}^{q-1} \mu_i. \end{aligned}$$

The convergence of series $\sum_{i=p}^{q-1} \mu_i$

$$\lim_{p,q\to\infty} \left(M_{\kappa_{1p},\kappa_{1q}} - m_{\kappa_{1p},\kappa_{1q}}\right) = 0$$

Therefore, $\{\kappa_{1n}\}$ is $M-$cauchy sequence in $\hat{W}$. Since $\hat{W}$ is M-complete, then there exist $\kappa_1 \in \hat{W}$ so $\kappa_{1n} \to \kappa_1$ as $n \to \infty$, implies that $\kappa_{12n+1} \to \kappa_1$ and $\kappa_{12n+1} \to \kappa_1$ as $n \to \infty$. Also,completeness of $\hat{W}$ implies that $\kappa_{1n} \to t$ for $t \in \hat{W}$. Furthermore, continuity of A, then

$$A(t) = A(\lim_{n\to\infty} \kappa_{1n}) = \lim_{n\to\infty} A(\kappa_{1n}) = \lim_{n\to\infty} \kappa_{1n+1} = t,$$

which shows that A has a fixed point $t \in \hat{W}$. □

Example 2.2. Let $\hat{W} = [0,1]$ and $M : \hat{W} \times \hat{W} \to \hat{W}$ be a M-metric space defined by

$$M(\kappa_1,\kappa_2) = |\kappa_1 - \kappa_2|,$$

with $\preceq$ and $A : \hat{W} \to \hat{W}$ defined

$$A(\kappa_1) = \frac{\kappa_1}{2},$$

and $\phi(t) = \frac{t}{3}$ and $\psi = \log t$ where $t \in [0,\infty)$.

Solution: It is clear that $M(\kappa_1,\kappa_2) = |\kappa_1 - \kappa_2|$ is a M-metric space on $\hat{W}$.

$$\phi(M(A\hat{s}, A\hat{c})) \leq \phi(\text{Ç}(\hat{s},\hat{c})) - \psi(\text{D}(\hat{s},\hat{c})).$$

Therefore all condition of result (2.1) satisfied. Hence, A has 0 as unique fixed point in $\hat{W}$.

We have the following result in which the mapping A is not continuous, still is valid to have a faxed point.

Theorem 2.3. By Result 2.1, non$-$continuous $A : \hat{W} \to \hat{W}$ has fixed point if $\hat{W}$ meets conditions, non$-$decreasing $\{\kappa_{1n}\} \subseteq \hat{W}$ such that $\kappa_{1n} \to \theta \in \hat{W}$ then $\kappa_{1n} \preceq \theta$ for $\theta \in N$, that is, $\theta = \sup k_{1n}$.

Proof . As from Result 2.1, a non$-$decreasing cauchy $\{\kappa_{1n}\} \subseteq \hat{W}$ exist such that $\kappa_{1n} \to \theta \in \hat{W}$. Hence condition, $\kappa_{1n} \preceq \theta \ \forall\ n$, $\theta = \sup \kappa_{1n}$. Next to show that θ is fixed point of A in $\hat{W}$. Assume that $A\theta = \theta$.

$$\text{Ç}(\kappa_{1n},\theta) = \max \left\{ \begin{array}{c} \frac{M(\theta,A\theta)[1+M(\kappa_{1n},A\kappa_{1n})]}{1+M(\kappa_{1n},\theta)}, \frac{M(\theta,A\theta)M(\kappa_{1n},A\kappa_{1n})}{1+M(\kappa_{1n},\theta)} \\ , M(\theta,A\theta), M(\kappa_{1n},A\kappa_{1n}), M(\kappa_{1n},\theta) \end{array} \right\},$$

and

$$\text{D}(\kappa_{1n},\theta) = \max \left\{ \frac{M(\theta,A\theta)[1+M(\kappa_{1n},A\kappa_{1n})]}{1+M(\kappa_{1n},\theta)}, M(\kappa_{1n},\theta) \right\}.$$

As $n \to +\infty$ and since $\{\kappa_{1n}\}$ is cauchy sequence in M-metric space then M-metric space complete there exist $\kappa_1 \in \hat{W}$ so $\kappa_{1n} \to \kappa_1$ as $n \to \infty$,which implies that $\kappa_{1n+1} \to \kappa_1$ and $\kappa_{1n+2} \to \kappa_1$ as $n \to \infty$.

$$\lim_{n\to\infty} (M(\kappa_{1n},\kappa_1) - m_{\kappa_{1n},\kappa_1}) = 0.$$

Since

$$\lim_{n\to\infty}(M(\kappa_{1n},\kappa_1))=0,$$

we have,

$$(7)\ \lim_{n\to\infty} Ç(\kappa_{1n},\theta)=\max\{M(\theta,A\theta),0\}=M(\theta,A\theta),$$

and

$$(8)\ \mathrm{D}\ (\kappa_{1n},\theta)=\max\{M(\theta,A\theta),0\}=M(\theta,A\theta).$$

Since $\kappa_{1n}\preceq\theta$ for any n, then from (1),

$$\begin{aligned}(9)\ \phi(M(\kappa_{1n+1},A\theta)) &= \phi(M(A\kappa_{1n},A\theta)),\\ &\leq \phi(Ç(\kappa_{1n},\theta))-\psi(\mathrm{D}(\kappa_{1n},\theta)).\end{aligned}$$

Taking $n\to\infty$ (9) and from (7) and (8),

$$\phi(M(\theta,A\theta))\leq\phi(M(\theta,A\theta))-\psi(M(\theta,A\theta))<\phi(M(\theta,A\theta)),$$

contradiction. Hence, $A\theta=\theta$, i.e., A has a fixed point $\theta\in\hat{W}$. □

Theorem 2.4. If every two elements of $\hat{W}$ are comparable then A has a unique fixed point in Theorems 2.1 and 2.3.

Proof . Consider $\hat{s}\neq\hat{c}$ be two fixed points of A in $\hat{W}$, then from (1), we have

$$\phi(M(A\hat{s},A\hat{c}))\leq\phi(Ç(\hat{s},\hat{c}))-\psi(\mathrm{D}(\hat{s},\hat{c})).$$

As a result,

$$(10)\ M(\hat{s},\hat{c})=M(A\hat{s},A\hat{c})\leq Ç(\hat{s},\hat{c}),$$

where

$$\begin{aligned}Ç(\hat{s},\hat{c}) &= \max\left\{\begin{array}{c}\frac{M(\hat{c},A\hat{c})[1+M(\hat{s},A\hat{s})]}{1+M(\hat{s},\hat{c})},\frac{M(\hat{c},A\hat{c})M(\hat{s},A\hat{s})}{1+M(\hat{s},\hat{c})}\\ ,M(\hat{c},A\hat{c}),M(\hat{s},A\hat{s}),M(\hat{s},\hat{c})\end{array}\right\},\\ &= \max\left\{\begin{array}{c}\frac{M(\hat{c},\hat{c})[1+M(\hat{s},\hat{s})]}{1+M(\hat{s},\hat{c})},\frac{M(\hat{c},\hat{c})M(\hat{s},\hat{s})}{1+M(\hat{s},\hat{c})}\\ ,M(\hat{c},\hat{c}),M(\hat{s},\hat{s}),M(\hat{s},\hat{c})\end{array}\right\},\\ &= \max\{M(\hat{s},\hat{c}),M(\hat{s},\hat{s}),M(\hat{c},\hat{c})\}.\end{aligned}$$

Therefore from (10), we have$M(\hat{s},\hat{c})<M(\hat{s},\hat{c})$, which leads contradiction to $\hat{s}\neq\hat{c}$. Thus, $\hat{s}=\hat{c}$. □

Following consequences by Theorems 2.1, 2.3 and 2.3.

Corollary 2.5. In place of D (κ_1,κ_2) by $Ç(\kappa_1,\kappa_2)$ in (1), same conclusions from Theorems 2.1, 2.3 and 2.3.

Corollary 2.6. Taking $\phi(\kappa_1)=\kappa_1$ and $\psi(\kappa_1)=(1-k)\kappa_1$ Corollary 2.5, then contraction condition,

$$M(A\hat{s},A\hat{c})\leq\max\left\{\begin{array}{c}\frac{M(\hat{c},A\hat{c})[1+M(\hat{s},A\hat{s})]}{1+M(\hat{s},\hat{c})},\frac{M(\hat{c},A\hat{c})M(\hat{s},A\hat{s})}{1+M(\hat{s},\hat{c})}\\ ,M(\hat{c},A\hat{c}),M(\hat{s},A\hat{s}),M(\hat{s},\hat{c})\end{array}\right\}.$$

Then can arrive same conclusions as Theorems 2.1, 2.3 and 2.3. A $A:\hat{W}\to\hat{W}$ with respect to $B:\hat{W}\to\hat{W}$ generalized contraction mapping, if it satisfies following condition for all $\hat{s},\hat{c}\in\hat{W}$ with $B\hat{s}\leq B\hat{c}$, $\phi\in\Phi$ & $\psi\in\Psi$:

$$(11)\ \phi(M(A\hat{s},A\hat{c}))\leq\phi(Ç_B(\hat{s},\hat{c}))-\psi(\mathrm{D}_B(\hat{s},\hat{c})),$$

where

$$Ç_B(\hat{s},\hat{c})=\max\left\{\begin{array}{c}\frac{M(B\hat{c},A\hat{c})[1+M(B\hat{s},A\hat{s})]}{1+M(B\hat{s},B\hat{c})},\frac{M(B\hat{c},A\hat{c})M(B\hat{s},A\hat{s})}{1+M(B\hat{s},B\hat{c})}\\ ,M(B\hat{c},A\hat{c}),M(B\hat{s},A\hat{s}),M(B\hat{s},B\hat{c})\end{array}\right\},$$

and

$$\mathrm{D}_B(\hat{s},\hat{c})=\max\left\{\frac{M(B\hat{c},B\hat{c})[1+M(B\hat{s},A\hat{s})]}{1+M(B\hat{s},B\hat{c})},M(B\hat{s},B\hat{c})\right\}.$$

Theorem 2.7. Two continuous self-mappings $A,\, B : \hat{W} \to \hat{W}$ coincidence point, if following conditions holds:

1. A monotone $B-$non$-$decreasing,
2. $A\hat{W} \subseteq B\hat{W}$ and a pair (A, B) are compatible,
3. $B\kappa_0 \preceq A\kappa_0$ for some $\kappa_0 \in \hat{W}$,
4. Satisfies condition (11) complete partially ordered $M-$metric space $(\hat{W}, M, \preceq)$.

Proof . Result 2.2 of [10], have sequences $\{\hat{s}_n\}, \{\hat{c}_n\} \subseteq \hat{W}$ with

$$(14)\ \hat{c}_n = A\hat{s}_n = A\hat{s}_{n+1} \text{ for all } n \geq 0,$$

which

$$(15)\ B\hat{s}_0 \preceq B\hat{s}_1 \preceq \cdots \preceq B\hat{s}_n \preceq B\hat{s}_{n+1} \preceq \cdots,$$

now from, we have to show that

$$(16)\ M(A\hat{c}_n, A\hat{c}_{n+1}) \leq \kappa M(\hat{c}_n, \hat{c}_{n+1}) \text{ for all } n \geq 1 \text{ and where } \kappa \in \{0, 1).$$

From (11), (14) and (15),

$$\begin{aligned}(17)\ \phi(M(A\hat{c}_n, A\hat{c}_{n+1})) &= \phi(M(\hat{s}_n, \hat{s}_{n+1})) \\ &\leq \phi(\text{Ç}_B(\hat{s}_n, \hat{s}_{n+1})) - \psi(\text{D}_B(\hat{s}_n, \hat{s}_{n+1})).\end{aligned}$$

where

$$\begin{aligned}\text{Ç}_B(\hat{s}_n, \hat{s}_{n+1}) &= \max\left\{\begin{array}{c}\frac{M(B\hat{s}_{n+1}, A\hat{s}_{n+1})[1+M(B\hat{s}_n, A\hat{s}_n)]}{1+M(B\hat{s}_n, B\hat{s}_{n+1})}, \frac{M(B\hat{s}_{n+1}, A\hat{s}_{n+1})M(B\hat{s}_n, A\hat{s}_n)}{1+M(B\hat{s}_n, B\hat{s}_{n+1})} \\ , M(B\hat{s}_{n+1}, A\hat{s}_{n+1}), M(B\hat{s}_n, A\hat{s}_n), M(B\hat{s}_n, B\hat{s}_{n+1})\end{array}\right\}, \\ &= \max\left\{\begin{array}{c}\frac{M(\hat{c}_n, \hat{c}_{n+1})[1+M(\hat{c}_{n-1}, \hat{c}_n)]}{1+M(\hat{c}_{n-1}, \hat{c}_n)}, \frac{M(\hat{c}_{n-1}, \hat{c}_n)M(\hat{c}_n, \hat{c}_{n+1})}{1+M(\hat{c}_{n-1}, \hat{c}_n)} \\ , M(\hat{c}_{n-1}, \hat{c}_n), M(\hat{c}_n, \hat{c}_{n+1}), M(\hat{c}_{n-1}, \hat{c}_n)\end{array}\right\}, \\ &\leq \max\{M(\hat{c}_n, \hat{c}_{n+1}), M(\hat{c}_{n-1}, \hat{c}_n)\}.\end{aligned}$$

and

$$\begin{aligned}\text{D}_B(\hat{s}_n, \hat{s}_{n+1}) &= \max\left\{\frac{M(B\hat{s}_{n+1}, A\hat{s}_{n+1})[1 + M(B\hat{s}_n, A\hat{s}_n)]}{1 + M(B\hat{s}_n, B\hat{s}_{n+1})}, M(B\hat{s}_n, B\hat{s}_{n+1})\right\}, \\ &= \max\left\{\frac{M(\hat{c}_n, \hat{c}_{n+1})[1 + M(\hat{c}_{n-1}, \hat{c}_n)]}{1 + M(\hat{c}_{n-1}, \hat{c}_n)}, M(\hat{c}_{n-1}, \hat{c}_n)\right\}, \\ &= \max\{M(\hat{c}_n, \hat{c}_{n+1}), M(\hat{c}_{n-1}, \hat{c}_n)\}\end{aligned}$$

From (17),

$$(18)\ \phi(M(A\hat{c}_n, A\hat{c}_{n+1})) \leq \phi(\max\{M(\hat{c}_n, \hat{c}_{n+1}), M(\hat{c}_{n-1}, \hat{c}_n)\}), -\psi(\max\{M(\hat{c}_n, \hat{c}_{n+1}), M(\hat{c}_{n-1}, \hat{c}_n)\}).$$

If $M(\hat{c}_{n-1}, \hat{c}_n) \leq M(\hat{c}_n, \hat{c}_{n+1})$ some n, Eq. (18),

$$(19)\ \phi(M(A\hat{c}_n, A\hat{c}_{n+1})) \leq \phi(M(\hat{c}_n, \hat{c}_{n+1})) - \psi(M(\hat{c}_n, \hat{c}_{n+1})) \leq \phi(M(\hat{c}_n, \hat{c}_{n+1})),$$

or equivalently

$$(20)\ \phi(M(A\hat{c}_n, A\hat{c}_{n+1})) \leq \phi(M(\hat{c}_n, \hat{c}_{n+1})),$$

a contradiction. Therefore, from Eq. (18) we have

$$(21)\ \phi(M(\hat{c}_n, \hat{c}_{n+1})) \leq \phi(M(\hat{c}_{n-1}, \hat{c}_n)).$$

Hence, $\kappa \in [0, 1)$. By Lemma 3.1 of [22] and from eq. (16),

$$\lim_{n\to+\infty} A\hat{s}_n = \lim_{n\to+\infty} BA\hat{s}_{n+1} = \tilde{u},\ \text{for } \tilde{u} \in \hat{W},$$

From condition (b), we have

$$\lim_{n\to+\infty} M(BA(\hat{s}_n), AB(\hat{s}_n)) = 0,$$

continuity of A and B we have, $\lim_{n\to+\infty} BA(\hat{s}_n) = B\tilde{u}$, $\lim_{n\to+\infty} AB(\hat{s}_n) = A\tilde{u}$. Furthermore,

$$(22)\ M(A\tilde{u}, B\tilde{u}) \le (M(A\tilde{u}, AB\tilde{u}) - m_{A\tilde{u},AB\tilde{u}}) + (M(AB\tilde{u}, B\tilde{u}) - m_{AB\tilde{u},A\tilde{u}}) + m_{A\tilde{u},B\tilde{u}}.$$

Thus, $M(A\tilde{u}, B\tilde{u}) = 0$ as $n \to +0$ in (22) hence result. □

Following result without continuity property B and L in Theorem 2.7.

Theorem 2.8. If $\hat{W}$ property in Theorem 2.7 that $\{B\hat{s}_n\} \subset \hat{W}$ non−decreasing such that $\lim_{n\to+\infty} B\hat{s}_n = B\hat{s} \in B\hat{W}$, and $B\hat{W} \subseteq \hat{W}$ is closed $B\hat{s}_n \preceq B\hat{s}$, $B\hat{s} \preceq B(B\hat{s})$ for n and $B\hat{s}_0 \preceq A\hat{s}_0$ for some $\hat{s}_0 \in \hat{W}$, then weakly compatible mappings A, B have coincidence point. When A and B commute there coincidence points, then A, B have a common fixed point in $\hat{W}$.

Proof . From Theorem 2.7, $\{\hat{s}_n\} = \{A\hat{s}_n\} = \{B\hat{s}_{n+1}\}$ is a Cauchy sequence. Since Bx closed, we have

$$\lim_{n\to+\infty} A\hat{s}_n = \lim_{n\to+\infty} B\hat{s}_{n+1} = B\hat{s} \text{ for } \hat{s} \in \hat{W}.$$

Thus, $B\hat{s}_n \preceq B\hat{s}$ for all n. Next show A, B have coincidence points. From (11),

$$(23)\ \phi(M(A\hat{s}_n, A\hat{s})) \le \phi(\text{Ç}_B(\hat{s}_n, \hat{s})) - \psi(\text{D}_B(\hat{s}_n, \hat{s})),$$

where

$$\text{Ç}_B(\hat{s}_n, \tilde{u}) = \max\left\{ \begin{array}{c} \frac{M(B\tilde{u},A\tilde{u})[1+M(B\hat{s}_n,A\hat{s}_n)]}{1+M(B\hat{s}_n,B\tilde{u})}, \frac{M(B\tilde{u},A\tilde{u})M(B\hat{s}_n,A\hat{s}_n)}{1+M(B\hat{s}_n,B\tilde{u})} \\ , M(B\tilde{u}, A\tilde{u}), M(B\hat{s}_n, A\hat{s}_n), M(B\hat{s}_n, B\tilde{u}) \end{array} \right\},$$

$\to \max\{M(B\tilde{u}, A\tilde{u}), 0, M(B\tilde{u}, A\tilde{u}), 0, 0\} = M(B\tilde{u}, A\tilde{u})$ as $n \to +\infty$

$$\text{D}_B(\hat{s}_n, \tilde{u}) = \max\left\{ \frac{M(B\tilde{u}, B\tilde{u})[1 + M(B\hat{s}_n, A\hat{s}_n)]}{1 + M(B\hat{s}_n, B\tilde{u})}, M(B\hat{s}_n, B\tilde{u}) \right\},$$

$\to \max\{M(B\tilde{u}, B\tilde{u}), 0\} = M(B\tilde{u}, B\tilde{u})$ as $n \to +\infty$. Thus Eq.(23) becomes

$$(24)\ \phi(\lim_{n\to+\infty} M(A\hat{s}_n, A\hat{s})) \le \phi(M(B\tilde{u}, A\tilde{u})) - \psi(M(B\tilde{u}, A\tilde{u})) < \phi(M(B\tilde{u}, A\tilde{u})).$$

As a result,

$$(25)\ \lim_{n\to+\infty} M(A\hat{s}_n, A\hat{s}) < M(B\tilde{u}, A\tilde{u}).$$

Furthermore, the triangular inequality of M,

$$(26)\ M(B\tilde{u}, A\tilde{u}) \le (M(B\tilde{u}, AB\tilde{u}) - m_{B\tilde{u},AB\tilde{u}}) + (M(AB\tilde{u}, A\tilde{u}) - m_{AB\tilde{u},A\tilde{u}}) + m_{B\tilde{u},A\tilde{u}}.$$

Eqs. (25) and (26) contradiction, if $B\tilde{u} \ne A\tilde{u}$. Hence, $B\tilde{u} = A\tilde{u}$. Let $B\tilde{u} \ne A\tilde{u} = p$, then $Ap = A(B\tilde{u}) = B(A\tilde{u}) = Bp$. Since $B\tilde{u} = B(B\tilde{u}) = Bp$, then Eq. (23) with $B\tilde{u} = A\tilde{u}$ & $Bp = Ap$, we get

$$(27)\ \phi(M(A\tilde{u}, Ap)) \le \phi(\text{Ç}_B(\tilde{u}, p)) - \psi(\text{D}_B(\tilde{u}, p)) < \phi(\text{Ç}_B(\tilde{u}, p)),$$

or equivalently,

$$M(A\tilde{u}, Ap) \le M(A\tilde{u}, Ap),$$

which contradiction, if $L\tilde{u} \ne Lp$. Thus, $A\tilde{u} = Lp = p \Rightarrow A\tilde{u} = Bp = p$. □

Definition 2.9. Consider a partially ordered m-metric space $(\hat{W}, M, \preceq)$. Mapping $A : \hat{W} \times \hat{W} \to \hat{W}$ is generalized $(\phi, \psi)-$contractive mapping with respect to $B : \hat{W} \to \hat{W}$,

$$(28)\ \phi(M(A(\hat{s}, \hat{c}), A(\tilde{u}, \hat{o})) \le \phi(\text{Ç}_B(\hat{s}, \hat{c}, \tilde{u}, \hat{o})) - \psi(\text{D}_B(\hat{s}, \hat{c}, \tilde{u}, \hat{o}))$$

if for all $\hat{s},\hat{c},\tilde{u},\hat{o}\in\hat{W}$ with $B\hat{s}\preceq B\tilde{u}$ and $B\hat{c}\succeq B\hat{o}$ $\phi\in\Phi$, $\psi\in\Psi$ and where

$$Ç_B(\hat{s},\hat{c},\tilde{u},\hat{o})=\max\left\{\begin{array}{c}\frac{M(B\tilde{u},A(\tilde{u},\hat{o}))[1+M(B\hat{s},A(\hat{s},\hat{c})]}{1+M(B\hat{s},B\tilde{u})},\frac{M(B\hat{s},A(\hat{s},\hat{c}))M(B\tilde{u},A(\tilde{u},\hat{o}))}{1+M(B\hat{s},B\tilde{u})}\\ ,M(B\hat{s},A(\hat{s},\hat{c})),M(B\tilde{u},A(\tilde{u},\hat{o})),M(B\hat{s},B\tilde{u})\end{array}\right\},$$

and

$$\mathrm{D}_B(\hat{s},\hat{c},\tilde{u},\hat{o})=\max\left\{\frac{M(B\tilde{u},A(\tilde{u},\hat{o}))[1+M(B\hat{s},A(\hat{s},\hat{c})]}{1+M(B\hat{s},B\tilde{u})},M(B\hat{s},B\tilde{u})\right\}.$$

Theorem 2.10. Let $(\hat{W},M,\preceq)$ be complete partially ordered M-metric space. Assume mapping $A:\hat{W}\times\hat{W}\to\hat{W}$ satisfies condition (28), A, B continuous, A mixed $B-$monotone property and commutes with B. Consider, if for $(\hat{s}_0,\hat{c}_0)\in\hat{W}\times\hat{W}$ such that $B\hat{s}_0\preceq A(\hat{s}_0,\hat{c}_0)$, $B\hat{c}_0\succeq A(\hat{c}_0,\hat{s}_0)$ and $A(\hat{W}\times\hat{W})\subseteq B(\hat{W})$, then A & B coupled coincidence point $\hat{W}$.

Proof . From Theorem 2.3, there are two $\{\hat{s}_n\}\{\hat{c}_n\}\subset\hat{W}$ such that in particular, the $\{B\hat{s}_n\}$ and $\{\hat{c}_n\}$ non-decreasing and non-increasing in $\hat{W}$. Put $\hat{s}=\hat{s}_n,\hat{c}=\hat{c}_n$, $\tilde{u}=\hat{s}_{n+1}$, and $\hat{o}=\hat{c}_{n+1}$ in (28),

$$\begin{aligned}\phi(M(A(\hat{s}_{n+1},\hat{s}_{n+1}))&=\phi(M(A(\hat{s}_n,\hat{c}_n),A(\hat{s}_{n+1},\hat{c}_{n+1}))\\&\le\phi(Ç_B(\hat{s}_n,\hat{c}_n,\hat{s}_{n+1},\hat{c}_{n+1}))-\psi(\mathrm{D}_B(\hat{s}_n,\hat{c}_n,\hat{s}_{n+1},\hat{c}_{n+1})),\end{aligned}\tag{2.1}$$

where

$$(30)\ Ç_B(\hat{s}_n,\hat{c}_n,\hat{s}_{n+1},\hat{c}_{n+1})\le\max\{M(B\hat{s}_n,B\hat{s}_{n+1}),M(B\hat{s}_{n+1},B\hat{s}_{n+2})\},$$

and

$$(31)\ \mathrm{D}\ _B(\hat{s}_n,\hat{c}_n,\hat{s}_{n+1},\hat{c}_{n+1})=\max\{M(B\hat{s}_n,B\hat{s}_{n+1}),M(B\hat{s}_{n+1},B\hat{s}_{n+2})\}.$$

Therefore from (29), we have

$$\begin{aligned}(32)\ \phi(M(B\hat{s}_{n+1},B\hat{s}_{n+2}))\ &\le\ \phi(\max\{M(B\hat{s}_n,B\hat{s}_{n+1}),M(B\hat{s}_{n+1},B\hat{s}_{n+2})\})\\&\quad-\psi(\max\{M(B\hat{s}_n,B\hat{s}_{n+1}),M(B\hat{s}_{n+1},B\hat{s}_{n+2})\}).\end{aligned}$$

Similarly by taking $\hat{s}=\hat{c}_{n+1}$, $\hat{c}=\hat{s}_{n+1}\tilde{u}=\hat{s}_n$, $\hat{o}=\hat{s}_n$ in (28), we get

$$(33)\ \phi(M(A(\hat{c}_{n+1},\hat{c}_{n+1}))\le\phi(\max\{M(B\hat{c}_n,B\hat{c}_{n+1}),M(B\hat{c}_{n+1},B\hat{c}_{n+2})\})-\psi(\max\{M(B\hat{c}_n,B\hat{c}_{n+1}),M(B\hat{c}_{n+1},B\hat{c}_{n+2})\}).$$

We know that

$$\max\{\phi(\theta_1),\phi(\theta_2)\}=\max\{\theta_1,\theta_2\}for\theta_1,\theta_2\in\{0,+\infty).$$

Then by adding Eqs.(32) and (33) together,

$$\begin{aligned}(34)\ \phi(\delta_n)\ &\le\ \phi(\max\{M(B\hat{s}_n,B\hat{s}_{n+1}),M(B\hat{s}_{n+1},B\hat{s}_{n+2}),M(B\hat{c}_n,B\hat{c}_{n+1}),M(B\hat{c}_{n+1},B\hat{c}_{n+2})\})\\&\quad-\psi(\max\{M(B\hat{s}_n,B\hat{s}_{n+1}),M(B\hat{s}_{n+1},B\hat{s}_{n+2}),M(B\hat{c}_n,B\hat{c}_{n+1}),M(B\hat{c}_{n+1},B\hat{c}_{n+2})\}),\end{aligned}$$

where

$$(35)\ \delta_n=\max\{M(B\hat{s}_{n+1},B\hat{s}_{n+2}),M(B\hat{c}_{n+1},B\hat{c}_{n+2})\}.$$

Let us denote that,

$$\nabla_n=\max\{M(B\hat{s}_n,B\hat{s}_{n+1}),M(B\hat{s}_{n+1},B\hat{s}_{n+2}),M(B\hat{c}_n,B\hat{c}_{n+1}),M(B\hat{c}_{n+1},B\hat{c}_{n+2})\}.$$

Hence from Eqs. (32)–(35), we obtain that

$$(37)\ \delta_n\le\nabla_n.$$

Now to claim that

$$(38)\ \delta_n\le\kappa\delta_{n-1},$$

for $n\ge1$ and $\kappa\in[0,1)$. Suppose that if $\nabla_n=\delta_n$ from (37), we give $\delta_n\le\delta_n$ this leads to $\delta_n=0$ since $\kappa>1$ and thus (38) holds. Consider $\nabla_n=\max\{M(B\hat{s}_n,B\hat{s}_{n+1}),M(B\hat{c}_n,B\hat{c}_{n+1})\}$, that is, $\nabla_n=\delta_{n-1}$ then (37) and (38). From (37) that $\delta_n\le\kappa^n\delta_0$ therefore, $M(B\hat{s}_{n+1},B\hat{s}_{n+2})\le\kappa^n\delta_0$ and $M(B\hat{c}_{n+1},B\hat{c}_{n+2})\le\kappa^n\delta_0$. Shows that $\{B\hat{s}_n\}$ and $\{B\hat{c}_n\}$ in $\hat{W}$ Cauchy sequences from Lemma(3.1) of [15]. Therefore, Result(2.2) that A and B in $\hat{W}$ coincidence point. □

Corollary 2.11. Suppose that $(\hat{W}, M, \preceq)$ is a complete partially ordered M-metric space. Suppose that continuous mapping $A : \hat{W} \times \hat{W} \to \hat{W}$ has a mixed monotone property and satisfies contraction conditions for any $\hat{s}, \hat{c}, \tilde{u}, \hat{o} \in \hat{W}$ such that $\hat{s} \preceq \tilde{u}$ and $\hat{c} \succeq \hat{o}$ and $\kappa > 2$, $\phi \in \Phi$ and $\psi \in \Psi$,

i. $\phi(\kappa M(A(\hat{s}, \hat{c}), A(\tilde{u}, \hat{o})) \leq \phi(\text{Ç}_B(\hat{s}, \hat{c}, \tilde{u}, \hat{o})) - \psi(\text{D}_B(\hat{s}, \hat{c}, \tilde{u}, \hat{o}))$,

ii. $\phi(M(A(\hat{s}, \hat{c}), A(\tilde{u}, \hat{o})) \leq \frac{1}{\kappa}\phi(\text{Ç}_B(\hat{s}, \hat{c}, \tilde{u}, \hat{o})) - \frac{1}{k}\psi(\text{D}_B(\hat{s}, \hat{c}, \tilde{u}, \hat{o}))$.

where

$$\text{Ç}_B(\hat{s}, \hat{c}, \tilde{u}, \hat{o}) = \max \left\{ \begin{array}{c} \frac{M(B\tilde{u}, A(\tilde{u}, \hat{o}))[1+M(B\hat{s}, A(\hat{s}, \hat{c})]}{1+M(B\hat{s}, B\tilde{u})}, \frac{M(B\hat{s}, A(\hat{s}, \hat{c}))M(B\tilde{u}, A(\tilde{u}, \hat{o}))}{1+M(B\hat{s}, B\tilde{u})} \\ , M(B\hat{s}, A(\hat{s}, \hat{c})), M(B\tilde{u}, A(\tilde{u}, \hat{o})), M(B\hat{s}, B\tilde{u}) \end{array} \right\},$$

and

$$\text{D}_B(\hat{s}, \hat{c}, \tilde{u}, \hat{o}) = \max \left\{ \frac{M(B\tilde{u}, A(\tilde{u}, \hat{o}))[1 + M(B\hat{s}, A(\hat{s}, \hat{c})]}{1 + M(B\hat{s}, B\tilde{u})}, M(B\hat{s}, B) \right\}.$$

If there exist, $(\hat{s}_0, \hat{c}_0) \in \hat{W} \times \hat{W}$ such that $\hat{s}_0 \preceq A(\hat{s}_0, \hat{c}_0)$ and $\hat{c}_0 \preceq A(\hat{c}_0, \hat{s}_0)$,then A has a coupled fixed point in $\hat{W}$.

Theorem 2.12. A unique coupled common fixed point as A and B exist in Result(2.9), if for $(\hat{s}, \hat{c}), (\kappa, \varrho) \in \hat{W} \times \hat{W}$ there some $(\lambda, \mu) \in \hat{W} \times \hat{W}$ such that $(A(\lambda, \mu), A(\mu, \lambda))$ comparable $(A(\hat{s}, \hat{c}), A(\hat{c}, \hat{s}))$ and $(A(\kappa, \varrho), A(\varrho, \kappa))$.

Proof . By Theorem 2.10, the mappings A & B coupled coincidence point $\hat{W}$. Let $(\hat{s}, \hat{c}), (\kappa, \varrho) \in \hat{W} \times \hat{W}$ two coupled coincidence points of A and B . Now, claim that $B\hat{s} = B\kappa$ and $B\hat{c} = B\varrho$. By hypotheses $(A(\lambda, \mu), A(\mu, \lambda))$ comparable to $(A(\hat{s}, \hat{c}), A(\hat{c}, \hat{s}))$ for some $(\lambda, \mu) \in \hat{W} \times \hat{W}$.

$$\begin{aligned} (A(\hat{s}, \hat{c}), A(\hat{c}, \hat{s})) &\leq (A(\lambda, \mu), A(\mu, \lambda)) \\ (A(\kappa, \varrho), A(\varrho, \kappa)) &\leq (A(\lambda, \mu), A(\mu, \lambda)). \end{aligned}$$

Suppose $\lambda_0 = \lambda$ and $\mu_0 = \mu$ there is point $(\lambda_1, \mu_1) \in \hat{W} \times \hat{W}$ such that

$$B(\lambda_1) = A(\lambda_0,\mu_0),\ B(\mu_1) = A(\mu_0,\lambda_0)\ (n \geq 1),$$

We have sequences $\{B\lambda_n\}$ and $\{B\mu_n\}$ in $\hat{W}$ repeated application above argument,

$$B(\lambda_{n+1}) = A(\lambda_n,\mu_n),\ B(\mu_{n+1}) = A(\mu_n,\lambda_n).$$

Similarly, defoe the sequences $\{B\hat{s}_n\}$, $\{B\hat{c}_n\}$ and $\{B\kappa_n\}$, $\{B\varrho_n\}$ in $\hat{W}$ by setting $\hat{s}_0 = \hat{s}$ and $\hat{c}_0 = \hat{c}$ and $\kappa_0 = \kappa$, $\varrho_0 = \varrho$. Furthermore, we have $B\hat{s}_n \to A(\hat{s}, \hat{c})$, $B\hat{c}_n \to A(\hat{c}, \hat{s})$, $B\kappa_n \to A(\kappa, \varrho)$, $B\varrho_n \to A(\varrho, \kappa)$, $(n \geq 1)$. Therefore by induction,

$$(B\hat{s}_n, B\hat{c}_n) \leq (B(\lambda_n),\ B(\mu_n))\,,\ n \geq 0.$$

Now from Eq. (28),

$$\begin{aligned} (39)\ \phi(M(B\hat{s}, B\lambda_{n+1})) &= \phi(M(A(\hat{s}, \hat{c}), A(\lambda_n, \mu_n)) \\ &\leq \phi(\text{Ç}_B(\hat{s}, \hat{c}, \lambda_n, \mu_n)) - \psi(\text{D}_B(\hat{s}, \hat{c}, \lambda_n, \mu_n)), \end{aligned}$$

where

$$\text{Ç}_B(\hat{s}, \hat{c}, \lambda_n, \mu_n) = \max \left\{ \begin{array}{c} \frac{M(B\lambda_n, A(\lambda_n, \mu_n))[1+M(B\hat{s}, A(\hat{s}, \hat{c})]}{1+M(B\hat{s}, B\lambda_n)}, \frac{M(B\hat{s}, A(\hat{s}, \hat{c}))M(B\lambda_n, A(\lambda_n, \mu_n))}{1+M(B\hat{s}, B\lambda_n)} \\ , M(B\hat{s}, A(\hat{s}, \hat{c})), M(B\lambda_n, A(\lambda_n, \mu_n)), M(B\hat{s}, B\lambda_n) \end{array} \right\},$$

and

$$\text{D}_B(\hat{s}, \hat{c}, \lambda_n, \mu_n) = \max\{\frac{M(B\lambda_n, A(\lambda_n, \mu_n))[1 + M(B\hat{s}, A(\hat{s}, \hat{c})]}{1 + M(B\hat{s}, B\lambda_n)}, M(B\hat{s}, B\lambda_n)\}.$$

As a result of Eq. (39),

$$(40)\ \phi(M(B\hat{s}, B\lambda_{n+1})) \leq \phi(M(B\hat{s}, B\lambda_n)) - \psi(M(B\hat{s}, B\lambda_n)).$$

As consequence of similar argument,

$$(41)\ \phi(M(B\hat{c}, B\mu_{n+1})) \leq \phi(M(B\hat{c}, B\mu_n)) - \psi(M(B\hat{c}, B\mu_n)).$$

Therefore from (40) and (41),

$$(42)\ \phi(\max\{M(B\hat{s},B\lambda_{n+1}),M(B\hat{c},B\mu_{n+1})\} \leq \phi(\max\{M(B\hat{s},B\lambda_n),M(B\hat{c},B\mu_n)\}) - \psi(\max\{M(B\hat{s},B\lambda_n),M(B\hat{c},B\mu_n)\}) < \phi(\max\{M(B\hat{s},B\lambda_n),M(B\hat{c},B\mu_n)\}).$$

The property ϕ implies that,

$$\max\{M(B\hat{s},B\lambda_{n+1}),M(B\hat{c},B\mu_{n+1})\} < \max\{M(B\hat{s},B\lambda_n),M(B\hat{c},B\mu_n)\}.$$

Hence, $\max\{M(B\hat{s},B\lambda_n),M(B\hat{c},B\mu_n)\}$ bounded below decreasing sequence of R^+ and by result,

$$\lim_{n\to+\infty}\max\{M(B\hat{s},B\lambda_n),M(B\hat{c},B\mu_n)\} = \Gamma,\ \Gamma \geq 0.$$

Therefore as $n \to +\infty$ in Eq.(42),

$$\phi(\Gamma) \leq \phi(\Gamma) - \psi(\Gamma),$$

derived $\psi(\Gamma) = 0$. Hence, $\Gamma = 0$.

$$\lim_{n\to+\infty}\max\{M(B\hat{s},B\lambda_n),M(B\hat{c},B\mu_n)\} = 0.$$

Thus,

$$(43)\ \lim_{n\to+\infty} M(B\hat{s},B\lambda_n) = 0 \text{ and } \lim_{n\to+\infty} M(B\hat{c},B\mu_n) = 0.$$

Also from above same argument,

$$(44)\ \lim_{n\to+\infty} M(B\kappa,B\lambda_n) = 0 \text{ and } \lim_{n\to+\infty} M(B\varrho,B\mu_n) = 0.$$

Therefore from (43) and (44), $B\hat{s} = B\kappa$ and $B\hat{c} = B\varrho$. Since $B\hat{s} = A(\hat{s},\hat{c})$ and $B\hat{c} = A(\hat{c},\hat{s})$ commutativity property of A and B,

$$(45)\ B(B\hat{s}) = B(A(\hat{s},\hat{c})) = A(B\hat{s},B\hat{c}) \text{ and } B(B\hat{c}) = B(A(\hat{c},\hat{s})) = A(B\hat{c},A\hat{s}).$$

If $B\hat{s} = \lambda$ and $B\hat{c} = \mu$ from (45), we get

$$(46)\ B(\lambda) = A(\lambda,\mu) \text{ and } B(\mu) = A(\mu,\lambda),$$

this shows that (λ,μ) coupled coincidence points of A and B . Hence, $B(\lambda) = B\kappa$ and $B(\mu) = B\varrho$ which in turn gives that $B(\lambda) = \lambda$ and $B(\mu) = \mu$. Therefore, we conclude from (46) that (λ,μ) is coupled common faxed point of A and B. Consider (λ,μ) another coupled common fixed point to A and B. Thus $\lambda = B(\lambda) = A(\lambda,\mu)$ and $\mu = B(\mu) = A(\mu,\lambda)$. But (μ,λ) is a coupled common fixed point of A and B then $B(\lambda) = B\hat{s} = \lambda$ and $B(\mu) = B\hat{c} = \mu$. Therefore, $\lambda = B\lambda = B\lambda = \lambda\ \lambda = B\mu = B\mu = \mu$. Hence the uniqueness. □

Theorem 2.13. If $B\hat{s}_0 \preceq B\hat{c}_0$ or $B\hat{s}_0 \succeq B\hat{c}_0$ in Result 2.11, then A and B have a unique common fixed point in $\hat{W}$.

Proof . Consider $(\hat{s},\hat{c}) \in \hat{W}$ is unique coupled common fixed point of A and B. Next show that $\hat{s} = \hat{c}$. Suppose that $B\hat{s}_0 \preceq B\hat{c}_0$ by induction, $\hat{s}_n \preceq B\hat{c}_n$, for all $n \geq 0$. Lemma 2 of [16], we have

$$\begin{aligned}
\phi(M(\hat{s},\hat{c})) &\leq \lim_{n\to+\infty}\sup\phi(M(\hat{s}_{n+1},\hat{c}_{n+1}))\\
&= \lim_{n\to+\infty}\sup\phi(M(A(\hat{s}_n,\hat{c}_n),A(\hat{c}_n,\hat{s}_n)))\\
&\leq \lim_{n\to+\infty}\sup\phi(\text{Ç}_B(\hat{s}_n,\hat{c}_n,\hat{c}_n,\hat{s}_n)) - \lim_{n\to+\infty}\sup\psi(\mathrm{D}_B(\hat{s}_n,\hat{c}_n,\hat{c}_n,\hat{s}_n))\\
&\leq \phi(M(\hat{s},\hat{c})) - \lim_{n\to+\infty}\sup\psi(\mathrm{D}_B(\hat{s}_n,\hat{c}_n,\hat{c}_n,\hat{s}_n))\\
&< \phi(M(\hat{s},\hat{c})),
\end{aligned}$$

which is a contradiction. Hence, $\hat{s} = \hat{c}$. □

3 Conclusion

In this manuscript, we derived several fixed point results in the context of partially ordered m-metric space. Our results are more generalized in the existing literature. This work can be extend in the context of many generalized spaces including fuzzy m-metric spaces, partially ordered fuzzy m-metric spaces, partially ordered metric-like spaces.

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