Int. J. Nonlinear Anal. Appl. 14 (2023) 5, 79–93
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.29877.4283

Analysis of an age structured SIR epidemic model with fractional Caputo derivative

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(Communicated by Mugur Alexandru Acu)

Abstract

In this paper, we consider a mathematical model with fractional derivatives in the sense of Caputo with respect to time. It describes the spread of an infectious disease that is directly transmitted in an age-structured population and whose transmission coefficient varies with age. We formulate the basic model as an abstract fractional Cauchy problem on a Banach space to prove the existence, and uniqueness of a local mild solution and ensure the global existence of a solution. Moreover, the results for the existence and uniqueness of non-trivial steady states are also demonstrated under the appropriate conditions.

Keywords: Age structure, Fractional epidemic model, Cauchy problem, SIR epidemic model
2020 MSC: Primary 92D30; Secondary 92D25, 35Q92

1 Introduction

Since the work of Mckendrick [25], It is well known that the dynamics of disease transmission are significantly influenced by the age composition of a population. Age-related differences in an individual's capacity for reproduction and survival are possible. For various age groups, diseases may have varying rates of infection and mortality.

Age differences may also affect an individual's behavior, and behavioral modifications are essential to the management and prevention of many infectious diseases. Young people typically participate more actively in interactions with or between populations and in the spread of disease. As a result, several age-structured epidemic models have been explored by various authors. As a result, a number of studies have been published that describe the stability of steady state solutions and analyze the overall behavior of these age-structured epidemic models, and find the threshold conditions for the disease to become endemic. Age structure should ideally be included in both births and deaths because, over time, the population's age distribution could alter, which could have an impact on the dynamics of the disease.[1, 3, 4, 9, 10, 24, 28, 29, 30, 32].

In this paper, we take a look at a mathematical model for an epidemic spreading in a population with an age structure, where the transmission coefficient varies with age. The model was developed for a SIR disease in a population, meaning that a susceptible person who contracts the disease will become contagious but will eventually recover with long-lasting immunity. We have one way to express the threshold phenomenon $\mathcal{R}_0$ is by referring to the spectral

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Received: February 2023 *Accepted:* March 2023

radius of a specific integral operator and an endemic steady state is possible if and only if $\mathcal{R}_0 > 1$ and if this state exists, it is unique. While the equilibrium with no disease present always exists.

To represent the dynamics of a SIR epidemic model with age structure, mathematical modeling using fractional differential equations (FDEs) is an appropriate method. Moreover, fractional differentiation is a generalization of classical differentiation and integration to arbitrary order. Since it naturally includes both memory and non-local effects, this is quite relevant to model the spread of epidemics.

The organization of the remainder of this article is as follows: In section 2, we give some known preliminary results to be used later. In sect 3, we consider the fractional order $(0 < \alpha < 1)$ SIR epidemic model with age structure with Caputo derivative. In section 4, in order to demonstrate the existence and uniqueness of its local mild solutions, we describe the fractional SIR epidemic model with age structure as an abstract fractional Cauchy problem on a Banach space. Next, we'll make sure the mild solution is present everywhere. Under the right circumstances, we will demonstrate the model's existence and uniqueness for non-trivial steady states in section 5. In section 6, we take into account the process of disease invasion to determine the disease's threshold.

2 Preliminaries

In this section we introduce notations, definitions and preliminary facts which are used throughout this paper. We denoted by $E^2 = L^1(0,\omega) \times L^1(0,\omega)$ the Banach space equipped with the norm $|\phi| = \sum_{i=1}^{2} |\phi_i|_1$ for $\phi(a) = (\phi_1(a), \phi_2(a))^T \in E$, where $|\cdot|_1$ is the ordinary norm of $E = L^1(0,\omega)$. Let $C((0,T], E^2)$ be the Banach space of continuous function from $[0,T]$ into E^2 with the norm $\|u\| = \sup_{t\in[0,T]} |u|$ where $u \in C((0,T], E^2)$. We need some basic definitions and properties of the fractional calculus theory. For more details, see [21, 31, 33].

Definition 2.1. ([21]) The fractional integral of the function $h \in L^1([a,b])$ of order $\alpha \in \mathbb{R}^+$is defined by

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where Γ is the gamma function.

Definition 2.2. ([21]) For a function h given on the interval $[a,b]$, the Caputo fractional order derivative of h, is given by

$${}^cD^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denote the integer part of α.

Suppose $M = \sup_{t\geq 0} \|T(t)\|$ and define

$$\mathcal{S}_\alpha(t) = \int_0^\infty h_\alpha(\theta) T\left(t^\alpha \theta\right) d\theta, \quad \mathcal{P}_\alpha(t) = \alpha \int_0^\infty \theta h_\alpha(\theta) T\left(t^\alpha \theta\right) d\theta, \quad t \geq 0,$$
$$h_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-1/\alpha} \psi_\alpha\left(\theta^{-1/\alpha}\right) \geq 0,$$
$$\psi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0,\infty),$$

where h_α is a probability density function defined on $(0,\infty)$, that is

$$h_\alpha(\theta) \geq 0, \quad \theta \in (0,\infty), \quad \int_0^\infty h_\alpha(\theta) d\theta = 1.$$

$$\text{For } \gamma \in [0,1], \int_0^\infty \theta^\gamma h_\alpha(\theta) \mathrm{d}\theta = \frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)}.$$

Lemma 2.3. ([33])

(i) For any fixed $t \geq 0$ and any

$$x \in E, \quad \|\mathcal{S}_\alpha(t)x\| \leq M\|x\| \text{ and } \|\mathcal{P}_\alpha(t)x\| \leq M\|x\|/\Gamma(\alpha).$$

(ii) $\{\mathcal{S}_\alpha(t) : t \geq 0\}$ and $\{\mathcal{P}_\alpha(t) : t \geq 0\}$ are strongly continuous.

(iii) For each $t > 0, \mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are compact operators if $T(t)$ is compact.

We recall a generalization of Gronwall's lemma that we will use in the sequel.

Lemma 2.4. (Generalized Gronwall inequality [31]) Let $v : [0, b] \longrightarrow [0, +\infty)$ be a real function and $\omega(\cdot)$ be a nonnegative, locally integrable function on $[0, b]$. Suppose that there exist $a > 0$ and $0 < \alpha < 1$ such that

$$v(t) \leq \omega(t) + a \int_0^t (t-s)^{-\alpha} v(s)ds.$$

Then there exists a constant $m = m(\alpha)$ such that

$$v(t) \leq \omega(t) + ma \int_0^t (t-s)^{-\alpha} \omega(s)ds, \quad \text{for} \quad t \in [0, b].$$

3 The basic model

First, we take into account a closed, one-sex, age-structured population within the context of demographic stability. Consider $P(t, a)$ to be the age distribution of the host population at time t, $\mu(a)$ to represent the age-specific natural death rate, and $f(a)$ to represent the age-specific fertility rate. The McKendrick equation would thus be used to represent the host population dynamics as follows:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) P(t, a) = -\mu(a)P(t, a), \tag{3.1}$$

$$P(t, 0) = \int_0^\omega f(a)P(t, a)da, \tag{3.2}$$

$$P(0, a) = P_0(a), \tag{3.3}$$

where $\omega < \infty$ is the maximum age and $P_0(a)$ contains the beginning data. The stable population model in demography is known as the system (3.1)-(3.3).

According to the stable population theory [14, 17], the system represented by (3.1) and (3.2) has a distinct persistent age profile that is as follows:

$$\psi(a) := \frac{e^{-r_0 a}\ell(a)}{\int_0^\omega e^{-r_0 a}\ell(a)da},$$

where $\ell(a)$ is the survival rate denoted by the expression

$$\ell(a) := \exp\left(-\int_0^a \mu(\sigma)d\sigma\right),$$

and the dominant real root of the Euler-Lotka characteristic equation gives the value of r_0, often known as the intrinsic rate of natural increase:

$$\int_0^\omega e^{-ra} f(a)\ell(a)da = 1. \tag{3.4}$$

Since ω is the oldest age that can be achieved, that is, $\ell(\omega) = 0$, we assume that $\mu \in L^1_{+,loc}(0, \omega)$ and $\int_0^\omega \mu(\sigma)d\sigma = \infty$.

Let's next divide the host population into three different groups based on their age-density functions: the susceptible class, the infected class, and the recovered class. These groups are represented by the symbols $S(t, a)$, $I(t, a)$, and $R(t, a)$, respectively. Let $\gamma(a)$ be the recovery rate at age a and $\beta(a, \sigma)$ be the transmission rate between susceptible individuals aged a and infected individuals aged σ. Moreover, we can suppose that the steady-state host population is a demographic stationary population provided by

$$P(t, a) = P_0(a) := B\ell(a) \quad \forall t > 0,$$

where B is the birth rate (number of new borns per unit time). Hence, $\psi(a) = b_0\ell(a)$, where $b_0 = \dfrac{1}{\int_0^\omega \ell(a)da}$ denotes the crude birth rate in the stationary population. Then we obtain the following system of equations that describe the dynamics of the model:

$$\begin{cases} \left(\dfrac{\partial^\alpha}{\partial t^\alpha} + \dfrac{\partial}{\partial a}\right) S(t,a) = -\lambda(t,a)S(t,a) - \mu(a)S(t,a), \\ \left(\dfrac{\partial^\alpha}{\partial t^\alpha} + \dfrac{\partial}{\partial a}\right) I(t,a) = \lambda(t,a)S(t,a) - \gamma(a)I(t,a) - \mu(a)I(t,a), \\ \left(\dfrac{\partial^\alpha}{\partial t^\alpha} + \dfrac{\partial}{\partial a}\right) R(t,a) = \gamma(a)I(t,a) - \mu(a)R(t,a), \\ S(t,0) = \displaystyle\int_0^\omega f(a)S(t,a)da, \\ I(t,0) = 0, \\ R(t,0) = 0, \end{cases} \tag{3.5}$$

where $\dfrac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative of order $0 < \alpha < 1$ and the force of infection $\lambda(t,a)$ is given by

$$\lambda(t,a) = \frac{1}{P(t)} \int_0^\omega \beta(a,\sigma)I(t,\sigma)d\sigma,$$

where

$$P(t) := \int_0^\omega P(t,a)da,$$

is the total size of the population and

$$P(t,a) = S(t,a) + I(t,a) + R(t,a).$$

We can introduce the ratio age distributions for each epidemiological class as follows since we assume that the epidemic has no effect on the demographic factors:

$$s(t,a) := \frac{S(t,a)}{P(t,a)}, \quad i(t,a) := \frac{I(t,a)}{P(t,a)}, \quad r(t,a) := \frac{R(t,a)}{P(t,a)}.$$

Then the new system is given by:

$$\begin{cases} \left(\dfrac{\partial^\alpha}{\partial t^\alpha} + \dfrac{\partial}{\partial a}\right) s(t,a) = -\lambda(t,a)s(t,a), \\ \left(\dfrac{\partial^\alpha}{\partial t^\alpha} + \dfrac{\partial}{\partial a}\right) i(t,a) = \lambda(t,a)s(t,a) - \gamma(a)i(t,a), \\ \left(\dfrac{\partial^\alpha}{\partial t^\alpha} + \dfrac{\partial}{\partial a}\right) r(t,a) = \gamma(a)i(t,a), \\ s(t,0) = 1, \\ i(t,0) = 0, \\ r(t,0) = 0. \end{cases} \tag{3.6}$$

Additionally, it follows from the definition that

$$s(t,a) + i(t,a) + r(t,a) = 1. \tag{3.7}$$

In the parts that follow, we mainly concentrate on the normalized system (3.6) under the condition (3.7) and the technical supposition that:

Assumption 3.1. $\beta \in L^\infty_+((0,\omega)\times(0,\omega))$ and $\gamma, f \in L^\infty_+(0,\omega)$.

Instead of the whole system (3.6), we can take into consideration the SI system from the normalized condition (3.7):

$$\begin{cases} \left(\dfrac{\partial^\alpha}{\partial t^\alpha} + \dfrac{\partial}{\partial a}\right) s(t,a) = -\lambda(a \mid i) s(t,a), \\ \left(\dfrac{\partial^\alpha}{\partial t^\alpha} + \dfrac{\partial}{\partial a}\right) i(t,a) = \lambda(a \mid i) s(t,a) - \gamma(a) i(t,a), \\ s(t,0) = 1, \\ i(t,0) = 0, \end{cases} \tag{3.8}$$

where $\lambda[a|i]$ is given by the integral operator defined by

$$\lambda[a|\phi] := \int_0^\omega \beta(a,\sigma)\psi(\sigma)\phi(\sigma)d\sigma, \qquad \phi \in L^1(0,\omega).$$

The state space of the system (3.8) is

$$\Omega := \left\{(s,i) \in E_+^2, 0 \leq s+i \leq 1\right\},$$

where E_+^2 is the positive cone of $E^2 = L^1(0,\omega) \times L^1(0,\omega)$. Let us define operators A and F on E^2 as follows:

$$(A\phi)(a) = \begin{pmatrix} -\frac{d}{da} & 0 \\ 0 & -\frac{d}{da} \end{pmatrix} \begin{pmatrix} \phi_1(a) \\ \phi_2(a) \end{pmatrix},$$

$$F(\phi)(a) = \begin{pmatrix} -\lambda\left[a \mid \phi_2\right] \phi_1(a) \\ \lambda\left[a \mid \phi_2\right] \phi_1(a) - \gamma(a)\phi_2(a) \end{pmatrix},$$

where $\phi = (\phi_1, \phi_2)^T \in E^2$, and the domain of the differential operator A is defined by

$$D(A) = \left\{\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in E^2 : \phi_j \in AC(0,\omega), \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\},$$

where $AC(0,\omega)$ represents the set of absolutely continuous functions on $(0,\omega)$. Let us define a E^2-valued function $u(t) = (s(t,\cdot), i(t,\cdot))^T$. As a result, the Cauchy problem in E^2 can be written as follows:

$$\frac{d^\alpha u(t)}{dt^\alpha} = Au(t) + F(u(t)), \quad u(0) = u_0. \tag{3.9}$$

The operator A generates a C_0 semigroup $(T(t))_{t\geq 0}$ such that $M = \sup_{t\geq 0} \|T(t)\|$, and $F : \Omega \longrightarrow E^2$ is lipschitz continuous;

$$\|F(u) - F(v)\| \leq L\|u - v\|,$$

and

$$N = \sup_{u \in E^2} \|F(u)\|.$$

4 Main results

It is suitable to rewrite the Cauchy problem (3.9) in the equivalent integral equation.

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(u(s))ds, \tag{4.1}$$

for $t \in [0,T]$. Note that the Laplace transform of an abstract function $f \in L^1(\mathbb{R}^+, X)$ is defined by $\widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt \quad (\lambda > 0)$. Applying the Laplace transform to (4.1) we get

$$\widehat{u}(\lambda) = \frac{u_0}{\lambda} - \frac{1}{\lambda^\alpha} A\widehat{u}(\lambda) + \frac{\widehat{F}(u(\lambda))}{\lambda^\alpha}$$

that is,

$$\widehat{u}(\lambda) = \lambda^{\alpha-1}\left(\lambda^{\alpha}+A\right)^{-1}u_0 + \left(\lambda^{\alpha}+A\right)^{-1}\widehat{F}(u(\lambda)).$$

$$\widehat{u}(\lambda) = \lambda^{\alpha-1}\int_0^{\infty} e^{-\lambda^{\alpha}}T(t)u_0dt + \int_0^{\infty} e^{-\lambda^{\alpha}t}T(t)\widehat{F}(\lambda)dt. \tag{4.2}$$

Consider the one-sides stable probability density

$$\psi_{\alpha}(\theta) == \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha).$$

Whose Laplace transform is given by

$$\int_0^{\infty} e^{-\lambda\theta}\psi_{\alpha}(\theta)d\theta = e^{-\lambda^{\alpha}} \qquad \alpha\in(0,1). \tag{4.3}$$

Using (4.2) and (4.3), we get

$$\widehat{u}(\lambda) = \int_0^{\infty} e^{-\lambda t}\Big[\int_0^{\infty}\psi_{\alpha}(\theta)T\left(\frac{t^{\alpha}}{\theta^{\alpha}}\right)d\theta + \alpha\int_0^t\int_0^{\infty}\psi_{\alpha}(\theta)T\left(\frac{(t-s)^{\alpha}}{\theta^{\alpha}}\right)(t-s)^{\alpha-1}\frac{(t-s)^{\alpha-1}}{\theta^{\alpha}}F(u(s))d\theta ds\Big]dt.$$

Now we can invert the last Laplace transform to get

$$u(t) = \mathcal{S}_{\alpha}(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_{\alpha}(t-s)F(u(s))ds, \quad t\in(0,T].$$

We give the following definition of the mild solution of (3.9).

Definition 4.1. By a mild solution of problem (3.9), we mean a function $u\in C((0,T];E^2)$ satisfying

$$u(t) = \mathcal{S}_{\alpha}(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_{\alpha}(t-s)F(u(s))ds, \quad t\in(0,T]. \tag{4.4}$$

Theorem 4.2. Let $u_0\in\Omega$ and assume that hypothesis $\frac{MLT^{\alpha}}{\Gamma(\alpha+1)}<1$ and $M\|u_0\|+\frac{MNT^{\alpha}}{\Gamma(\alpha+1)}<1$ hold, then the fractional problem (3.9) has a unique mild solution defined on $[0,T]$.

Proof . Consider the mapping H given by

$$(Hu)(t) = \mathcal{S}_{\alpha}(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_{\alpha}(t-s)F(u(s))ds, \quad t\in(0,T].$$

We see that $(Hu)(t)\in C((0,T];E^2)$. First, we show that $H(\Omega)\subset\Omega$. Let $t\in[0,T]$ and $u\in\Omega$

$$\begin{aligned}|(Hu)(t)| &\le |\mathcal{S}_{\alpha}(t)u_0| + \int_0^t (t-s)^{\alpha-1}|\mathcal{P}_{\alpha}(t-s)F(u(s))|ds\\ &\le M\mid u_0\mid + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mid F(u(s))\mid ds.\end{aligned}$$

$$\|H(u)\| \le M\|u_0\| + \frac{MNT^{\alpha}}{\Gamma(\alpha+1)} < 1.$$

Then H maps Ω into itself. Next, we shall show that H is a strict contraction on Ω which will ensure the existence of a unique mild solution. Let u and v two elements in Ω, by the assumption on F and $\frac{MLT^{\alpha}}{\Gamma(\alpha+1)}<1$, we have:

$$\begin{aligned}|(Hu)(t)-(Hv)(t)| &\le \int_0^t (t-s)^{\alpha-1}|\mathcal{P}_{\alpha}(t-s)(F(u(s))-F(v(s)))|ds\\ &\le \frac{ML}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u(s)-v(s)|ds\\ \|(Hu)-(Hv)\| &\le \frac{MLT^{\alpha}}{\Gamma(\alpha+1)}\|u-v\|.\end{aligned}$$

This yields that H is a contraction on Ω. So H has a unique fixed point $u \in \Omega$ by the Banach Fixed point Theorem, which is a mild solution to problem (3.9) on $[0, T]$. $\square$

Remark 4.3. If $u_0 \in D(A)$, the mild solution becomes a classical solution.

For concrete application, the global existence of the solution of the fractional differential equation always becomes a main concern.

Theorem 4.4. Let $u_0 \in \Omega$, the mapping $F : \Omega \longrightarrow E^2$ is Lipschitz continuous and there exist a positive constants c_1 et c_2 such that

$$\|F(u)\| \leq c_1 + c_2\|u\|. \tag{4.5}$$

Then the problem (3.9) has a global mild solution.

Proof . We have

$$F(\phi)(a) = \begin{pmatrix} -\lambda\,[a \mid \phi_2]\,\phi_1(a) \\ \lambda\,[a \mid \phi_2]\,\phi_1(a) - \gamma(a)\phi_2(a) \end{pmatrix}.$$

Let

$$F_1(\phi)(a) = -\lambda\,[a \mid \phi_2]\,\phi_1(a), \quad F_2(\phi)(a) = \lambda\,[a \mid \phi_2]\,\phi_1(a) - \gamma(a)\phi_2(a).$$

If we define $\lambda^+ = \sup \lambda$ and $\gamma^+ = \sup \gamma$, we obtain

$$\begin{aligned} |F(\phi)| &= \sum_{i=1}^{2} |F_i(\phi)|_1 \\ &= \lambda\,[a \mid \phi_2]\,|\phi_1|_1 + |\lambda\,[a \mid \phi_2]\,\phi_1 - \gamma(a)\phi_2|_1 \\ &\leq 2\lambda^+|\phi_1|_1 + \gamma^+|\phi_2|_1 \\ &\leq 2\lambda^+|\phi| + \gamma^+|\phi| \\ &\leq c_1 + c_2|\phi|, \end{aligned}$$

with $c_1 = 0$, $c_2 = 2\lambda^+ + \gamma^+$ and we go to the sup, we know that (4.5) holds. Next, we assume that the mild solution u admits a maximal existence interval $(0, T_{\max})$ ($T_{\max}$ is the maximum time of existence). Suppose there is a sequence $t_n \longrightarrow T_{\max}$ such that $|u(t_n)| \longrightarrow \infty$. Then for $0 < t < T_{max}$, substitution of (4.5) in the equation (4.4) leads to the following estimation.

$$|u(t)| \leq |\mathcal{S}_\alpha(t)u_0| + \int_0^t (t-s)^{\alpha-1}|\mathcal{P}_\alpha(t-s)|(c|u(s)|)ds.$$

Therefore,

$$|u(t)| \leq M|u_0| + \frac{Mc}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|u(s)|ds.$$

If we take

$$\omega(t) = M|u_0|,$$

which is bounded, and

$$a = \frac{Mc}{\Gamma(\alpha)},$$

it follows, in accordance with Lemma 2.4, that $v(t) = |u(t)|$ is bounded. Which contradicts the fact that $\lim_{t\to T_{max}} |u(t)| = \infty$. So $T_{\max} = \infty$. $\square$

5 Existence of steady states

Let $(s^*(a), i^*(a))^{\mathrm{T}}$ the endemic steady state solution satisfying the following ODE system:

$$\begin{aligned} &\frac{d}{da}s^*(a) = -\lambda^*(a)s^*(a), \\ &\frac{d}{da}i^*(a) = \lambda^*(a)s^*(a) - \gamma(a)i^*(a), \\ &s^*(0) = 1, \\ &i^*(0) = 0, \end{aligned} \tag{5.1}$$

where

$$\lambda^*(a) := \int_0^\omega \beta(a,\sigma)\psi(\sigma)i^*(\sigma)d\sigma, \tag{5.2}$$

$$\delta(a) := \exp\left(-\int_0^a \gamma(\sigma)d\sigma\right).$$

We have the following expressions after formally solving the preceding ODEs:

$$s^*(a) = e^{-\int_0^a \lambda^*(\sigma)d\sigma}, \tag{5.3}$$

$$i^*(a) = \int_0^a \frac{\delta(a)}{\delta(\sigma)}\lambda^*(\sigma)e^{-\int_0^\sigma \lambda^*(s)ds}d\sigma. \tag{5.4}$$

Substituting (5.4) into (5.2) and changing the order of integration, we obtain an equation for $\lambda^*(a)$.

$$\lambda^*(a) = \int_0^\omega \pi(a,\eta)\lambda^*(\eta)e^{-\int_0^\eta \lambda^*(s)ds}d\eta, \tag{5.5}$$

where

$$\pi(a,\eta) = \int_\eta^\omega \beta(a,\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma. \tag{5.6}$$

From (5.2), it follows that $|\lambda^*(a)| \le b_0\|\beta\|_\infty |i^*|_1$ where $\|\cdot\|_\infty$, $|\cdot|_1$ denote an $L^\infty-$norm and L^1-norm respectively. Then it follows from $i^* \in L^1(0,\omega)$ that $\lambda^* \in L_+^\infty(0,\omega)$. It is clear that one solution of (5.5) is $\lambda^*(a) \equiv 0$, which is equivalent to the disease-free equilibrium state. We define a nonlinear operator $\Phi(x)$ to study a nontrivial solution to (5.5) in the Banach space $E = L^1(0,\omega)$ with the positive cone $E_+ = \{\phi \in E, \phi > 0, a.e.\}$ by

$$(\Phi x)(a) = \int_0^\omega \pi(a,\eta)x(\eta)e^{-\int_0^\eta x(s)ds}d\eta, \quad x \in E. \tag{5.7}$$

As the range of Φ is included in $L^\infty(0,\omega)$, the solutions of (5.5) correspond to fixed points of Φ. Note that the operator Φ has a positive linear majorant K defined by

$$(Kx)(a) = \int_0^\omega \pi(a,\eta)x(\eta)d\eta, \quad x \in E. \tag{5.8}$$

Here, we review some fundamental nonsupporting operator notions. Let $B(Y)$ be the set of bounded linear operators from Y to Y, where Y is a Banach space with a positive cone Y_+. $L \in B(Y)$ is said to be positive if $L(Y_+) \subset Y_+$. $L \in B(Y)$ is said to be strongly positive if $\langle f, L\psi\rangle > 0$ for every pair $\psi \in Y_+\backslash\{0\}$, $f \in Y_+^*\backslash\{0\}$, where Y_+^* is the space of positive linear functionals on Y. For $L, V \in B(Y)$, we say $L \ge V$ if $(L-V)(Y_+) \subset Y_+$. A positive operator $L \in B(Y)$ is said to be nonsupporting if for every pair $\psi \in Y_+\backslash\{0\}$ and $f \in Y_+^*\backslash\{0\}$, there exists a positive integer $p = p(\psi, f)$ such that $\langle f, L^n\psi\rangle > 0$ for all $n \ge p$. $r(L)$ denotes the spectral radius of $L \in B(Y)$.

If a nonsupporting compact operator T has positive radius, the Perron-Frobenius type theorem holds (Sawashima 1964, Marek 1970).

Proposition 5.1. Let the cone E_+ be total, K is compact, nonsupporting with respect to E_+ and $r(K) > 0$. Then the following holds:

$r(K)$ is a point spectrum and is simple pole of the resolvent.

2. The eigenspace corresponding to $r(K)$ is one-dimensional and its eigenvector $v_0 \in E_+$ is a quasi-interior point. Any eigenvector in $E_+$is proportional to v_0.
3. The eigenspace of the dual operator K^* corresponding to $r(K)$ is also one-dimensional and is spanned by a strictly positive functional.
4. Let $S, T \in B(E)$ be compact and nonsupporting. Then $S \leq T$, $S \neq T$ and $r(T) \neq 0$ implies $r(S) < r(T)$.

After making the aforementioned preparations, we initially think about the nature of the majorant operator K described by (5.8). We'll assume the following in what follows:

Assumption 5.2. 1. There exist numbers $\delta_0 \in (0, \omega)$ and $\underline{\beta} > 0$ such that

$$\beta(a, \eta) \geq \underline{\beta} \quad \text{for almost all } (a, \eta) \in (0, \omega) \times (\omega - \delta_0, \omega)\,. \tag{5.9}$$

2. $\beta \in L^\infty_+((0, \omega) \times (0, \omega))$ is extended into $L^\infty_+\left(\mathbb{R}^2\right)$ by $\beta(a, \sigma) = 0$ for $(a, \sigma) \notin (0, \omega) \times (0, \omega)$ and satisfies

$$\lim_{h \to 0} \int_0^\omega |\beta(a + h, \eta) - \beta(a, \eta)| da = 0 \quad \text{uniformly for} \quad \eta \in \mathbb{R}. \tag{5.10}$$

Lemma 5.3. The operator $K : E \longrightarrow E$ is nonsupporting and compact under Assumption (5.2).

Proof . Let's define the positive linear functional $f_0 \in E^*_+$ by

$$\langle f_0, \phi \rangle := \int_0^\omega \int_\eta^\omega \beta_0(\sigma) \psi(\sigma) \frac{\delta(\sigma)}{\delta(\eta)} d\sigma \phi(\eta) d\eta,$$

where

$$\beta_0(\sigma) = \begin{cases} \underline{\beta} & \text{for } \sigma \in (\omega - \delta_0, \omega)\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{5.11}$$

Then $K\phi \geq \langle f_0, \phi \rangle e$, for all $\phi \in E_+$, where $e = 1 \in E_+$, which implies

$$K^{n+1} \phi \geq \langle f_0, \phi \rangle \langle f_0, e \rangle^n e, \quad \forall n \in \mathbb{N}.$$

Thus for arbitrary $F \in E^*_+ \backslash \{0\}, \phi \in E_+ \backslash \{0\}$ and $n \geq 1$,

$$\langle F, K^n \phi \rangle \geq \langle f_0, \phi \rangle \langle f_0, e \rangle^{n-1} \langle F, e \rangle > 0.$$

This shows K is nonsupporting. Next observe that

$$\int_0^\omega |\pi(a + h, \sigma) - \pi(a, \sigma)| da \leq b_0 \int_0^\omega \int_0^\omega |\beta(a + h, \sigma) - \beta(a, \sigma)| d\sigma da. \tag{5.12}$$

In order to prove the compactness of K, we identify the Banach space E with the subspace of $L^1(\mathbb{R})$ such that $E = \{\phi \in L^1(\mathbb{R}), \phi(a) = 0 \quad \text{for} \quad a \in (-\infty, 0) \cup (\omega, \infty)\}$. Then we can interpret K as an operator on $L^1(\mathbb{R})$ such that E is its invariant subspace, so it is sufficient to show that the operator K is compact in $L^1(\mathbb{R})$. Let T a bounded subset of $L^1(\mathbb{R})$. Then it follows immediately that $K(T)$ is also a bounded subset. Observe that

$$\begin{aligned} \int_{\mathbb{R}} |(K\phi)(a + h) - (K\phi)(a)| da \quad &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\pi(a + h, \sigma) - \pi(a, \sigma)||\phi(\sigma)| d\sigma da \\ &\leq \|\phi\| \sup_{0 \leq \sigma \leq \omega} \int_{\mathbb{R}} |\pi(a + h, \sigma) - \pi(a, \sigma)| da \end{aligned}$$

Together with conditions (5.10) and (5.12), It follows that the family $K(T)$ in the L^1-norm is an equicontinuous family. Moreover, it follows from $K(T) \subset E$ that

$$\int_{|\sigma|\geq\omega} |(K\phi)(\sigma)|d\sigma = 0, \quad \phi \in T.$$

Thus, we can use the Fréchet-Kolmogorov compactness criterion [8, 27], that is, $K(T)$ is relatively compact in $L^1(\mathbb{R})$. Thus, K is a compact operator. This completes the proof. □

It follows from the proposition that The unique positive eigenvalue with a positive eigenvector and an eigenvalue of the dual operator K^* with a strictly positive eigenfunctional is the spectral radius $r(K)$ of the operator K. Now we can show the main theorem in this section:

Theorem 5.4.
1. If $\mathcal{R}_0 \leq 1$, the equation $\Phi(x) = x$ has no solution except the trivial solution $x \equiv 0$.
2. if $\mathcal{R}_0 > 1$, the equation $\Phi(x) = x$ has at least one non-zero positive solution.

Proof . Supposing that $\mathcal{R}_0 = r(K) \leq 1$, it is easily checked that $K\phi - \Phi(\phi) \in E_+\backslash\{0\}$ for $\phi \in E_+\backslash\{0\}$. If there exists an $\phi_0 \in E_+\backslash\{0\}$ being a solution of $\phi = \Phi(\phi)$, then $\phi_0 = \Phi(\phi_0) \leq K(\phi_0)$. Let $F_0^* \in E_+^*\backslash\{0\}$ be the adjoint eigenvector for K that corresponds to $r(K)$. Taking duality pairing, we find

$$\langle F_0^*, K(\phi_0) - \phi_0\rangle = \langle (K^* - I^*) F_0^*, \phi_0\rangle = (r(K) - 1)\langle F_0^*, \phi_0\rangle > 0,$$

due to the fact that F_0^* is strictly positive and $K(\phi_0) - \phi_0 \in E_+\backslash\{0\}$. Then, $r(K) > 1$, a contradiction, results. Next, we suppose that $\mathcal{R}_0 = r(K) > 1$. The operator Φ is a completely continuous operator in the Banach space E, under Assumption (5.2), in the same way as the proof of Lemma (5.3). In addition, if the number M_0 is defined by

$$M_0 = \sup_{0\leq\sigma\leq\omega} \int_0^\omega \pi(a,\sigma)da,$$

the set $\Omega_0 := \{\phi \in E_+ : |\phi| \leq M_0\}$ is invariant (in fact $\Phi(E_+) \subset \Omega_0$) under the operator Φ. We define an operator Φ_r by:

$$\Phi_r(x) = \begin{cases} \Phi(x), & \text{if } |x|_1 \geq r, x \in E_+, \\ \Phi(x) + (r - |x|_1)x_0, & \text{if } |x|_1 \leq r, x \in E_+, \end{cases}$$

where x_0 is the positive eigenvector of K corresponding to $r(K) > 1$. Let

$$\Omega_r = \{x \in E_+, |x| \leq M_0 + r|x_0|\}.$$

Then Φ_r is completely continuous and transforms the set Ω_r into itself. Since Ω_r is bounded, convex and closed in E, Φ_r has a fixed point $x_r \in \Omega_r$ (Schauder's principle). Notice that the operator K does not have in $E+$ eigenvectors that correspond to the eigenvalue one of the function $\Phi(x)$ at $x = 0$. If r is sufficiently small, it can be demonstrated that the norms of these fixed points are greater than r by using the method of Krasnoselskii [22, Theorem 4.11]. That is, Φ has a positive fixed point. This completes the proof. □

We then introduce the idea of a concave operator [22], in order to study the uniqueness issue for nontrivial positive fixed points of the operator Φ.

Definition 5.5. Let E_+ be a cone in a real Banach space E and $<$ be the partial ordering defined by E_+. A positive operator $A : E_+ \longrightarrow E_+$ is called a concave operator if there exists a $u_0 \in E_+ \setminus \{0\}$ which satisfies the following

1. For any $x \in E_+ \setminus \{0\}$, there exist $\alpha = \alpha(x) > 0$ and $\beta = \beta(x) > 0$ such that $\alpha u_0 \leq Ax \leq \beta u_0$, that is, Ax is comparable with u_0.
2. $A(tx) > tAx$ for $0 < t < 1$ and for every $x \in E_+$ such that

$$\alpha(x)u_0 \leq x \leq \beta(x)u_0, \quad (\alpha(x) > 0, \beta(x) > 0).$$

Lemma 5.6. Suppose that the operator $A : E_+ \longrightarrow E_+$ is monotone and concave. If for any $x \in E_+$ satisfying $\alpha_1 u_0 \leq x \leq \beta_1 u_0 \quad (\alpha_1 = \alpha_1(x) > 0, \quad \beta_1 = \beta_1(x) > 0)$, and $0 < t < 1$, there exists $\eta = \eta(x,t) > 0$ such that

$$A(tx) \geq tAx + \eta u_0, \tag{5.13}$$

then A has at most one positive fixed point.

Another assumption is made in this case.

Assumption 5.7. For all $(a, \eta) \in [0, \omega) \times [0, \omega)$, the inequality

$$\beta(a, \eta)\psi(\eta) - \gamma(\eta)\pi(a, \eta) \geq 0 \tag{5.14}$$

holds.

Then we can prove the following.

Theorem 5.8. Suppose that Assumption (5.7) holds. If $r(K) > 1$, then Φ has only one positive fixed point.

Proof . From Lemma (5.6) and Theorem (5.4), it is sufficient to show that under Assumption (5.7), the operator Φ is a monotone concave operator satisfying condition (5.13). From (5.7) and (5.6), it follows that

$$\begin{aligned}
\Phi(x)(a) &= \int_0^\omega \pi(a, \eta)x(\eta)e^{-\int_0^\eta x(s)ds}d\eta, \\
&= \int_0^\omega \pi(a, \eta)\left(-\frac{d}{d\eta}e^{-\int_0^\eta x(s)ds}\right)d\eta, \\
&= \Big[-\pi(a, \eta)e^{-\int_0^\eta x(s)ds}\Big]_{\eta=0}^{\eta=\omega} + \int_0^\omega e^{-\int_0^\eta x(s)ds}\frac{d}{d\eta}\pi(a, \eta)d\eta, \\
&= \pi(a, 0) + \int_0^\omega e^{-\int_0^\eta x(s)ds} \times \Big[-\beta(a, \eta)\psi(\eta)\frac{\delta(\eta)}{\delta(\eta)} + \int_\eta^\omega \beta(a, s)\psi(s)\frac{d}{d\eta}\frac{\delta(s)}{\delta(\eta)}ds\Big]d\eta, \\
&= \pi(a, 0) + \int_0^\omega e^{-\int_0^\eta x(s)ds} \times \Big[-\beta(a, \eta)\psi(\eta) + \int_\eta^\omega \beta(a, s)\psi(s)\frac{\delta(s)}{\delta(\eta)}\gamma(\eta)ds\Big]d\eta, \\
&= \pi(a, 0) - \int_0^\omega e^{-\int_0^\eta x(s)ds}\Big[\beta(a, \eta)\psi(\eta) - \gamma(\eta)\pi(a, \eta)\Big]d\eta,
\end{aligned}$$

from which, together with Assumption (5.7), we know that Φ is a monotonic operator. Next from (5.7) and (5.6), we observe that

$$\alpha(x)u_0 \leq \Phi(x)(a) \leq \beta(x)u_0,$$

where $u_0 \equiv 1$ and

$$\begin{aligned}
\alpha(x) &= \int_0^\omega g(\eta)x(\eta)e^{-\int_0^\eta x(s)ds}d\eta, \\
\beta(x) &= \int_0^\omega h(\eta)x(\eta)e^{-\int_0^\eta x(s)ds}d\eta,
\end{aligned}$$

where $g(\eta)$ and $h(\eta)$ are defined by

$$\begin{aligned}
g(\eta) &:= \int_\eta^\omega \beta_0(\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma, \\
h(\eta) &:= \sup|\beta| \int_\eta^\omega \psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma.
\end{aligned}$$

It follows that $\alpha(x) > 0$ and $\beta(x) > 0$ for $x \in E_+ \setminus \{0\}$. Moreover, we obtain

$$\begin{aligned}
\Phi(tx)(a) - t\Phi(x)(a) &= t\int_0^\omega \pi(a, \eta)x(\eta)e^{-\int_0^\eta x(s)ds}\Big[e^{(1-t)\int_0^\eta x(s)ds} - 1\Big]d\eta, \\
&\geq t\int_0^\omega g(\eta)x(\eta)e^{-\int_0^\eta x(s)ds}\Big[e^{(1-t)\int_0^\eta x(s)ds} - 1\Big]d\eta,
\end{aligned}$$

from which we deduce that Φ is a concave operator and condition (5.13) holds by letting $u_0 \equiv 1$ and

$$\eta(x, t) = t\int_0^\omega g(\eta)x(\eta)e^{-\int_0^\eta x(s)ds}\Big[e^{(1-t)\int_0^\eta x(s)ds} - 1\Big]d\eta.$$

This completes the proof. □

Note that Assumption (5.7) holds if

$$\ell(a) \geq k\left(1 - e^{-(\omega-a)\|\gamma\|_\infty}\right), \tag{5.15}$$

where k is defined by

$$k := \frac{\sup \beta(a,b)}{\inf \beta(a,b)}, \tag{5.16}$$

is finite. In fact, we have

$$\begin{aligned}
\beta(a,\eta)\psi(\eta) - \gamma(\eta)\int_\eta^\omega \beta(a,\sigma)\psi(\sigma)\frac{\delta(\sigma}{\delta(\eta)}d\sigma =& \beta(a,\eta)b_0\ell(\eta) - \gamma(\eta)\int_\eta^\omega \beta(a,\sigma)b_0\ell(\sigma)e^{-\int_\eta^\sigma \gamma(s)ds}d\sigma,\\
&\geq \inf \beta(a,\eta)b_0\ell(\eta) - \|\gamma\|_\infty b_0 \sup \beta(a,\sigma)\int_\eta^\omega e^{-(\sigma-\eta)\|\gamma\|_\infty}d\sigma,\\
&\geq \inf \beta(a,\eta)b_0\left[\ell(\eta) - k\Big(1 - e^{-(\omega-\eta)\|\gamma\|_\infty}\Big)\right].
\end{aligned}$$

We adhere to the separable mixing assumption. This implies that there is no relationship between the age of the infected both those of the susceptible persons and those of the individuals.

Assumption 5.9. There exist $\beta_1, \beta_2 \in L_+^\infty(0,\omega)$ such that

$$\beta(a,\sigma) = \beta_1(a)\beta_2(\sigma).$$

with $\beta_1(a)$ representing susceptibility and $\beta_2(\sigma)$ denoting the infectiousness. In particular, we have the proportionate mixing assumption if $\beta_1 = \beta_2$. Without making a distinction between separable and proportional, these presumptions are traditionally referred to as the proportionate mixing assumption [7, 11]. Under the separable mixing assumption, (5.5) can be written as

$$\lambda^*(a) = \beta_1(a)\int_0^\omega \int_\eta^\omega \beta_2(\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma\lambda^*(\eta)e^{-\int_0^\eta \lambda^*(s)ds}d\eta. \tag{5.17}$$

As a result, (5.17) is reduced to a one-dimensional equation, and its solution is expressed as $\lambda^*(a) = c\beta_1(a)$, where c is a constant. We get at an equation for the unknown integer c by putting this expression into (5.17).

$$1 = \int_0^\omega \int_\eta^\omega \beta_2(\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma\beta_1(\eta)e^{-c\int_0^\eta \beta_1(s)ds}d\eta. \tag{5.18}$$

The right-hand side of equation (5.18) is a strictly decreasing function of c that goes to zero as $c \longrightarrow \infty$. As a result, if the condition

$$\int_0^\omega \beta_1(\eta)\int_\eta^\omega \beta_2(\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma d\eta > 1, \tag{5.19}$$

holds, the characteristic equation (5.18) has a unique positive solution.
The basic reproduction number $\mathcal{R}_0$ of the system (3.6) is provided on the left-hand side of the threshold condition (5.19). So, the following is how we can sum up the argument:

Proposition 5.10. If and only if the fundamental reproduction number is bigger than unity, there exists a unique endemic steady state for the normalized epidemic system (3.6) with the separable mixing assumption.

$\mathcal{R}_0$ for the separable mixing case is given by

$$\mathcal{R}_0 = \int_0^\omega \int_\eta^\omega \beta_2(\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma\beta_1(\eta)d\eta. \tag{5.20}$$

6 The invasion process

First let us consider the situation that very small infectious population invade into totally susceptible population. In this initial phase of epidemic, the growth of infected is described by the following linearized equation

$$\begin{cases} \left(\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\partial}{\partial a}\right) i(t,a) = P_0(a) \int_0^\omega \beta(a,\sigma) i(t,\sigma) d\sigma - \gamma(a) i(t,a), \\ i(t,0) = 0, \quad i(0,a) = i_0(a), \end{cases} \tag{6.1}$$

since we may ignore the fact that the infection process causes a drop in the density of susceptibles. Let $\hat{i}(a,\tau)$, $\tau \in \mathbb{C}$ be the Laplace transform of $i(t,a)$

$$\hat{i}(a,\tau) := \int_0^\infty e^{-\tau t} i(t,a) dt. \tag{6.2}$$

It is easily seen that using a priori estimate for $i(t,a)$, the integral (6.2) exists for τ with large real part. From (6.1), we have

$$\frac{\partial}{\partial a}\hat{i}(a,\tau) = \tau^{\alpha-1} i_0(a) - \left(\gamma(a) + \tau^\alpha\right)\hat{i}(a,\tau) + P_0(a) \int_0^\omega \beta(a,\sigma)\hat{i}(\sigma,\tau) d\sigma. \tag{6.3}$$

By the variation of constants formula, we have

$$\hat{i}(a,\tau) = \int_0^a e^{-\int_\sigma^a (\gamma(s)+\tau^\alpha) ds} \left[\tau^{\alpha-1} i_0(\sigma) + P_0(\sigma) \int_0^\omega \beta(a,\eta)\hat{i}(\eta,\tau) d\eta\right] d\sigma. \tag{6.4}$$

From (6.4), it follows that

$$\int_0^\omega \beta(a,\sigma)\hat{i}(\sigma,\tau) d\sigma = (I - T_\tau)^{-1} \int_0^\omega \beta(\cdot,\sigma) \int_0^\sigma e^{-\int_\eta^\sigma (\gamma(s)+\tau^\alpha) ds} \tau^{\alpha-1} i_0(\eta) d\eta d\sigma, \tag{6.5}$$

where the linear operator T_τ is defined by

$$(T_\tau x)(a) := \int_0^\omega \varphi_\tau(a,\eta) x(\eta) d\eta, \tag{6.6}$$

$$\varphi_\tau(a,\eta) \int_\eta^\omega \beta(a,\sigma) P_0(\sigma) e^{-\int_\eta^\sigma (\gamma(s)+\tau^\alpha) ds} d\sigma. \tag{6.7}$$

Since $I - T_\tau$ is invertible for τ with large real part, there exists a number σ such that the inverse Laplace transform exist.

$$i(t,a) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\tau t}\hat{i}(a,\tau) d\tau. \tag{6.8}$$

Therefore, we know that the behavior of $i(t,a)$ can be determined by the method of residues. Let $\Sigma := \{\tau \in \mathbb{C} : I - T_\tau \ \text{ is not invertible}\}$. Under our condition (5.10) that T_τ is a compact operator from $L^1(0,\omega)$ into $L^1(0,\omega)$ for all $\tau \in \mathbb{C}$. Then it follows that $\Sigma := \{\tau \in \mathbb{C} : 1 \in P_\sigma(T_\tau)\}$ and the function $\tau \longrightarrow (I - T_\tau)^{-1}$ is meromorphic in the complex domain. Hence Σ is a discrete set whose elements are pole of $(I - T_\tau)^{-1}$ of finite order. On real axis, T_τ is a positive operator and its spectral radius $r(T_\tau)$ is non-increasing with respect to τ. Since T_τ is a compact operator, it follows from Krein-Rutman theorem that nonzero spectral radius gives a positive eigenvalue. Hence, characteristic root of the equation $r(T_\tau) = 1$ become element of Σ.

Moreover, it follows from the condition (5.9) that for real τ, there exists a strictly positive functional F_τ and a quasi-interior point e with respect to natural cone L^1_+ such that $T_\tau x \geq \langle F_\tau, x\rangle e, \quad \lim_{\tau \longrightarrow -\infty}\langle F_\tau, e\rangle = +\infty$. Hence T_τ is a nonsupporting operator in the sense of Sawashima (1964) and from the comparison theorem of positive operator the spectral radius $r(T_\tau)$ is monotonically decreasing from $+\infty$ to zero with respect to $\tau \in \mathbb{R}$ and the characteristic equation has a unique root $\tau_0 \in \Sigma$. Therefore, From monotonicity of $r(T_\tau)$, we obtain that $\tau_0 > 0$ if $r(T_0) > 1$, $\tau_0 = 0$ if $r(T_0) = 1$, $\tau_0 < 0$ if $r(T_0) < 1$. Using the similar argument as Theorem 6.13 of Heijmans (1986)[33, Theorem 6.13], we can prove that τ_0 is the dominant singular point which determines the local stability of the disease-free steady state of the population, since τ_0 is the growth rate of the principal part of $i(t,a)$.

Lemma 6.1. if $\tau \in \Sigma$, $\tau \neq \tau_0$, then $\mathfrak{Re}(\tau) < \tau_0$.

Proof . For any $\tau \in \Sigma$, there exists a corresponding eigenfunction ψ_τ such that $T_\tau \psi_\tau = \psi_\tau$. Then we have

$$|\psi_\tau| = |T_\tau \psi_\tau| \leq T_{\mathfrak{Re}\tau} |\psi_\tau|.$$

Let $f_{\mathfrak{Re}(\tau)}$ be an adjoint positive eigenfunctional of $T_{\mathfrak{Re}(\tau)}$ corresponding to the eigenvalue unity. It follows that

$$\langle f_{\mathfrak{Re}(\tau)}, T_{\mathfrak{Re}(\tau)} |\psi_\tau| \rangle = r(T_{\mathfrak{Re}(\tau)} \langle f_{\mathfrak{Re}(\tau)}, |\psi_\tau| \rangle \geq \langle f_{\mathfrak{Re}(\tau)}, |\psi_\tau| \rangle.$$

Therefore, we have $r(T_{\mathfrak{Re}(\tau)} \geq 1$ and $\mathfrak{R}e(\tau) \leq \tau_0$ because $r(T_x)$ is monotone decreasing with respect to $x \in \mathbb{R}$ and $r(T_{\tau_0}) = 1$. If $\mathfrak{Re}(\tau) = \tau_0$, we obtain $T_{\tau_0}|\psi_\tau| = |\psi_\tau|$. In fact, if $T_{\tau_0}|\psi_{\tau_0}| > |\psi_{\tau_0}|$, applying the eigenfunctional f_{τ_0} to $r(T_{\tau_0}) = 1$, we have $\langle f_{\tau_0}, T_{\tau_0}|\psi_\tau| \rangle = \langle f_{\tau_0}, |\psi_\tau| \rangle > \langle f_{\tau_0}, |\psi_\tau| \rangle$, which is a contradiction. Therefore, we can write $|\psi_\tau| = c\psi_0$ using the eigenfunction ψ_0 corresponding to the eigenvalue $r(T_{\tau_0}) = 1$. Without loss of generality, we can assume that $c = 1$ and that there exists a real function $\alpha(a)$ such that $\psi_\tau(a) = \psi_0(a)e^{i\alpha(a)}$. Substituting this relation into

$$T_{\tau_0}\psi_0 = \psi_0 = \psi_\tau = |T_\tau \psi_\tau|,$$

we obtain the following

$$\int_0^\omega \int_\eta^\omega \beta(a,\sigma)P_0(\sigma)e^{-\int_\eta^\sigma(\gamma(s)+\tau^\alpha)ds}d\sigma\psi_0(\eta)d\eta = \Big|\int_0^\omega \int_\eta^\omega \beta(a,\sigma)P_0(\sigma)e^{-\int_\eta^\sigma[\gamma(s)+(\tau_0+\mathfrak{Im}(\tau))^\alpha]ds}d\sigma\psi_0(\eta)e^{i\alpha(\eta)}d\eta\Big|.$$

From [33, Lemma 6.12], it follows that, there exists a constant θ such that $-\mathfrak{Im}\tau(\sigma-\eta)+\alpha(\eta) = \theta$. It follows from $T_\tau\psi_\tau = \psi_\tau$ that $e^{i\theta}T_{\tau_0}\psi_0 = \psi_0 e^{i\theta}$. We then have $\theta = \alpha(a)$, which implies that $\mathfrak{Im}(\tau) = 0$. Thus, there is no element $\tau \in \Sigma \setminus \{\tau_0\}$ such that $\mathfrak{Re}(\tau) = \tau_0$ which complete the proof. □

Therefore, we have the threshold criterion: the disease can invade if $r(T_0) > 1$, whereas it can not if $r(T_0) < 1$. Then $r(T_0)$ can be interpreted as the basic production number $\mathcal{R}_0 = r(K)$.

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