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Existence of three solutions for fourth-order Kirchhoff type elliptic problems with Hardy potential

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Abstract

In this work, we establish existence results for the following fourth-order Kirchhoff-type elliptic problem with Hardy potential

$$M\Big(\int_\Omega |\Delta u|^p dx\Big)\Delta_p^2 u - \frac{a}{|x|^p}|u|^{p-2}u = \lambda f(x,u), \quad \text{in } \Omega,$$
$$u = \Delta u = 0, \quad \text{on } \partial\Omega.$$

Precisely, by using the classical Hardy inequality and critical point theory, we prove the existence of multiple weak solutions for the fourth-order Kirchhoff-type elliptic problem with Hardy potential.

Keywords: Kirchhoff-type, Multiple solutions, Critical points theory, Hardy potential, p-biharmonic type operator
2020 MSC: Primary 35A01; Secondary 35B38, 35D30

1 Introduction

Consider the following fourth-order Kirchhoff type elliptic problems with Hardy potential

$$\begin{aligned} &M\Big(\int_\Omega |\Delta u|^p dx\Big)\Delta_p^2 u - \frac{a}{|x|^p}|u|^{p-2}u = \lambda f(x,u), \quad \text{in } \Omega, \\ &u = \Delta u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ containing the origin and with smooth boundary $\partial\Omega, 1 < p < N, \Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is an operator of fourth order, so-called p-biharmonic operator, λ is a positive parameter, $M : [0, +\infty[\to \mathbb{R}$ is a continuous function, and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is an L^2-Carathéodory function.

Kirchhoff [30] first introduced a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

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which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. After that, many authors studied the following nonlocal elliptic boundary value problem

$$-M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u(x) = f(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on} \partial\Omega. \tag{1.3}$$

Problems like this are called the Kirchhoff type problems. In recent years, Kirchhoff type boundary value problems have been investigated in many papers, we refer to [1, 11, 14, 18, 20, 21, 23, 33, 37, 39, 45], in which the authors have used different methods to discuss the existence of solutions for nonlocal problems.

On the other hand, fourth-order boundary value problems which describe the deformations of an elastic beam in an equilibrium state whose both ends are simply supported have been extensively studied in the literature. The studying of existence and multiplicity of solutions for fourth-order problems which arise in the study of static equilibrium of an elastic body, has drawn the attention of many authors, see [6, 7, 19, 22, 24, 26, 32, 34, 35, 36, 40]. For example, Candito and Livrea in [7] by using critical point theory, established the existence of infinitely many weak solutions for a class of elliptic Navier boundary value problems depending on two parameters and involving the p-biharmonic operator. Liu et al. in [36] employing variational methods, studied the existence and multiplicity of nontrivial solutions for fourth-order elliptic equations. In [19, 26] based on variational methods and critical point theory, the existence of multiple solutions for $(p_1, \dots, p_n)$-biharmonic systems was discussed. Molica Bisci and Repovš in [40] exploiting variational methods, investigated the existence of multiple weak solutions for a class of elliptic Navier boundary problems involving the p-biharmonic operator, and presented a concrete example of an application. In [32], by using variational methods the existence and multiplicity of solutions for the following p-biharmonic equation

$$\Delta(|\Delta u|^{p-2}\Delta u) - div(|\Delta u|^{p-2}\nabla u) = \lambda f(x,u) + \mu g(x,u), \quad x \in \Omega, \\ u = \Delta u = 0, \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a non-empty bounded open set with a sufficiently smooth boundary $\partial\Omega, \lambda > 0, \mu > 0$ and $f, g : \Omega \times \mathbb{R}^N \to \mathbb{R}$ are two L^1-Carathéodory functions, were established.

The following fourth-order elliptic equations of Kirchhoff type

$$\Delta^2 u - \Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u(x) = \lambda f(x,u), \quad x \in \Omega, \\ u = \Delta u = 0, \quad \text{on } \partial\Omega,$$

which is related to the following stationary analogue of the equation of Kirchhoff type

$$u_{tt} + \Delta^2 u - \Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u(x) = \lambda f(x,u), \quad x \in \ \Omega,$$

has been studied by some researchers recently. Nonlocal fourth-order equations models the bending equilibrium of simply supported extensible beams on nonlinear foundations. Recently, many researchers have paid their attention to fourth-order Kirchhoff-type problems, we refer the reader to [12, 25, 38, 46] and the references therein. In [46], using the mountain pass theorem, Wang and An established the existence and multiplicity of solutions for the following fourth-order nonlocal elliptic problem

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x,u), & \text{in } \ \Omega, \\ u = \Delta u = 0, & \text{on } \ \partial\Omega. \end{cases}$$

In particular, in [12] using variational methods and critical point theory, multiplicity results of nontrivial and nonnegative solutions for a perturbed fourth-order Kirchhoff type elliptic problem were established.

Stationary problems involving singular nonlinearities, also the associated evolution equations, describe naturally several physical phenomena and applied economical models, see [16, 17, 44] and the references therein. For instance, nonlinear singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids and boundary layer phenomena for viscous fluids. Moreover, nonlinear singular elliptic equations are also encountered in glacial advance, intransport of coal slurries down conveyor belt sandin several other geophysical and industrial contents; see Callegari and Nachman [5]. Singular elliptic problems have been intensively studied in the last decades. Among others, we mention the works [2, 10, 13, 27, 28, 29, 31, 41, 42, 43, 47, 48, 49]. Xie and Wang in [48] proved that the problem

$$\Delta_p^2 u = \frac{|u|^{p-2}u}{|x|^{2p}} + g(\lambda, x, u), \quad \text{in } \Omega,$$
$$u = \Delta u = 0, \quad \text{on } \partial\Omega,$$

has infinitely many solutions with positive energy levels. Ferrara and Molica Bisic in [13] studied the existence of solutions for the elliptic problem with Hardy potential

$$\begin{aligned} -\Delta_p u &= \mu \frac{|u|^{p-2}u}{|x|^p} + \lambda f(x,u), \quad \text{in } \Omega, \\ u &= \Delta u = 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

Huang and Liu in [27] studied the sign-changing solutions for p-biharmonic equations with Hardy potential

$$\begin{aligned} \Delta_p^2 u - \frac{a}{|x|^{2p}}|u|^{p-2}u &= f(x,u), \quad \text{in } \Omega, \\ u &= \Delta u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

by using the method of invariant sets of descending flow. For instance, in [41] using variational methods and critical point theory the existence of at least three solutions for the following p-biharmonic equation with Hardy potential of Kirchhoff-type

$$M\Big(\int_\Omega |\Delta u|^p dx\Big)\Delta_p^2 u - \frac{a}{|x|^{2p}}|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u), \quad \text{in } \Omega,$$
$$u = \Delta u = 0, \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ containing the origin and with smooth boundary $\partial\Omega$, $1 < p < \frac{N}{2}$ was discussed. In [49] the authors, by using critical point theory, have investigated the existence of infinitely many weak solutions for a fourth-order Kirchhoff type elliptic problems with Hardy potential.

Motivated by the above facts, in the present paper, using two kinds of multi critical points theorems obtained in [3, 4] which we recall in the next section (Theorems 2.2, 2.1), we establish the existence of at least three and two weak solutions for the problem (1.1), see Theorems 3.1 - 3.2 Some recent results are extended and improved. Some examples are presented to demonstrate the applications of our main results.

2 Preliminaries

Let X be the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \Big(\int_\Omega |\Delta u|^p dx\Big)^{1/p}.$$

Since $1 < p < N$, we recall classical Hardy's inequality, which says that

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} \leq \frac{1}{H}\int_\Omega |\nabla u|^p dx \quad (\forall u \in X) \tag{2.1}$$

where $H := ((N-p)/p)^p$; see, for instance, the paper [15]. Set $p^* := pN/(N-p)$ then By the Sobolev embedding theorem there exists a positive constant C such that

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|, \qquad (\forall u \in X), \tag{2.2}$$

$$C := \frac{1}{N\sqrt{\pi}}\Big(\frac{N!\Gamma(\frac{N}{2})}{2\Gamma(\frac{N}{p})\Gamma(N+1+\frac{N}{p})}\Big)^{\frac{1}{N}}\Big(\frac{N(p-1)}{N-p}\Big)^{1-\frac{1}{p}}. \tag{2.3}$$

Fixing $q \in [1, p^*[$, again from the Sobolev embedding theorem, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \leq c_q\|u\|, \qquad (\forall u \in X), \tag{2.4}$$

and, in particular, the embedding $X \hookrightarrow L^q(\Omega)$ is compact. Due to (2.3), as simple consequence of Holder's inequality, it follows that

$$c_q \leq \frac{|\Omega|^{\frac{p^*-q}{p^*q}}}{N\sqrt{\pi}}\Big(\frac{N!\Gamma(\frac{N}{2})}{2\Gamma(\frac{N}{p})\Gamma(N+1+\frac{N}{p})}\Big)^{\frac{1}{N}}\Big(\frac{N(p-1)}{N-p}\Big)^{1-\frac{1}{p}},$$

where Γ denotes the Gamma function and $|\Omega|$ is the Lebesgue measure of Ω. Define the functionals $\Phi, \Psi : X \to \mathbb{R}$ by

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{a}{p}\int_\Omega \frac{|u(x)|^p}{|x|^p}dx,\\ \Psi(u) &= -\int_\Omega F(x,u(x))dx,\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}\widehat{M}(t) &= \int_0^t M(s)ds, \quad t \geq 0,\\ F(x,t) &= \int_0^t f(x,\xi)d\xi, \quad \Omega \times \mathbb{R}.\end{aligned}$$

It is easy to show that the functionals Φ and Ψ are well defined and continuously Gâteaux differentiable and whose derivative are

$$\begin{aligned}\Phi'(u)(v) = M\Big(\int_\Omega |\Delta u(x)|^p dx\Big)\int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)dx - \\ a\int_\Omega \frac{|u(x)|^{p-2}}{|x|^p}u(x)v(x)dx\end{aligned} \tag{2.6}$$

and

$$\Psi'(u)(v) = -\int_\Omega f(x,u(x))v(x)dx \tag{2.7}$$

for every $u, v \in X$. In this article, we assume that the following conditions hold,

(H1) $M : [0,+\infty[\to \mathbb{R}$ is a continuous function such that there are two positive constants m_0 and m_1 such that

$$m_0 \leq M(t) \leq m_1, \quad \forall t \geq 0. \tag{2.8}$$

(F) There exist positive constant $\gamma < p$ and a positive real function $\alpha(x) \in L^\infty(\Omega)$ such that

$$F(x,t) \leq \alpha(x)(1+|t|^\gamma) \qquad \text{for a.e. } x \in \Omega, \quad \forall t \in \mathbb{R}. \tag{2.9}$$

Define the functional $I : X \to \mathbb{R}$ given by $I = \Phi + \lambda\Psi$. By the conditions (H1) and (F), it is easy to see that $I \in C^1(X,\mathbb{R})$ and a critical point of I corresponds to a weak solution of the problem (1.1). Our main tools are two multiple critical points theorem without the Palais-Smale condition, the first one due to Bonanno in [11] and the second one an equivalent formulation [4, Theorem 2.3] of Ricceri's three critical points theorem [3, Theorem 1], which are recalled below.

Theorem 2.1. (see [4, Theorem 2.1]). Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous functions. Assume that Φ is (strongly) continuous and satisfies $\lim_{\|u\|\longrightarrow\infty}\Phi(u) = \infty$. Assume also that there exist two constants r_1 and r_2 such that

(i) $\inf_X \Phi < r_1 < r_2$,

(ii) $\varphi_1(r_1) < \varphi_2(r_1, r_2)$,

(iii) $\varphi_1(r_2) < \varphi_2(r_1, r_2)$,

where

$$\varphi_1(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\Psi(u) - \inf_{u\in\overline{\Phi^{-1}(-\infty,r)}^w}\Psi(u)}{r - \Phi(u)},$$

$$\varphi_2(r_1, r_2) = \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{u_1\in\Phi^{-1}[r_1,r_2[} \frac{\Psi(u)-\Psi(u_1)}{\Phi(u_1)-\Phi(u)},$$

Then, for each

$$\lambda \in \left]\frac{1}{\varphi_2(r_1,r_2)}, \min\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_1(r_2)}\}\right[.$$

the functional $\Phi+\lambda\Psi$ has two local minima which lie in $\Phi^{-1}(-\infty, r_1)$ and $\Phi^{-1}(r_1, r_2)$, respectively.

Theorem 2.2. (see [3, Theorem 2.3]). Let X be a separable and reflexive real Banach space. $\Phi : X \to \mathbb{R}$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Suppose that

(j) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$,

(jj) There are a real number r, and $u_0, u_1 \in X$ such that $\Phi(u_0) < r < \Phi(u_1)$,

(jjj) $\inf_{u\in\Phi^{-1}(-\infty,r)} \Psi(u) > \dfrac{(\Phi(u_1) - r)\Psi(u_0) + (r - \Phi(u_0))}{\Phi(u_1) - \Phi(u_0)}$.

Then there exists an open interval $\Lambda \subseteq [0,\infty]$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0,$$

has at least three weak solutions whose norms in X are less than ρ.

We refer to [9] in which Theorems 2.1 and 2.2 have been successfully employed to the existence of two solutions and three solutions of the a nonlocal elliptic system.

3 Main results

Pick $s > 0$ such that $B(0, s) \subset \Omega$, where $B(0, s)$ denotes the ball with center at 0 and radius of s. Let

$$L = \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_{\frac{s}{2}}^{s} \left|\frac{12(N+1)}{s^3}r - \frac{24N}{s^2} + \frac{9(N-1)}{s}\frac{1}{r}\right|^p r^{N-1}dr. \tag{3.1}$$

Define the function v by

$$v(x) = \begin{cases} 0, & x \in \overline{\Omega}\setminus B(0,s), \\ \frac{1}{h}\left(\frac{4}{s^3}\rho^3 - \frac{12}{s^2}\rho^2 + \frac{9}{s}\rho - 1\right), & x \in B(0,s)\setminus B(0,\frac{s}{2}), \\ \frac{1}{h}, & x \in B(0,\frac{s}{2}) \end{cases} \tag{3.2}$$

with $\rho = \mathrm{dist}(x, 0) = \sqrt{\sum_{i=1}^N x_i^2}$ and h is positive constant. Clearly $v \in X$. Let

$$B(\eta) = \int_\Omega F(x, v(x))dx - \int_\Omega \sup_{\|t\|_{L^q(\Omega)}\le\eta} F(x,t)dx.$$

Now we are ready to state our main results for the problem (1.1).

Theorem 3.1. Assume that (H1) holds and $0 < a < m_0H$ (with H is as in (2.1)). Suppose that there are three positive constants h, η_1, η_2 with

$$\frac{\eta_1^p}{c_q^p} < \frac{L}{h^p} < \frac{(m_0H - a)}{m_1Hc_q^p}\eta_2^p \tag{3.3}$$

such that

$$\frac{m_1 L}{h^p}\int_\Omega \sup_{\|t\|_{L^q(\Omega)}\le \eta_1} F(x,t)dx < \frac{(m_0H-a)\eta_1^p}{c_q^p H}B(\eta_1), \tag{3.4}$$

$$\frac{m_1 L}{h^p}\int_\Omega \sup_{\|t\|_{L^q(\Omega)}\le \eta_2} F(x,t)dx < \frac{(m_0H-a)\eta_2^p}{c_q^p H}B(\eta_1). \tag{3.5}$$

Then, for each

$$\lambda \in \left]\frac{m_1 L}{ph^pB(\eta_1)}, \frac{(m_0H-a)}{pHc_q^p}\min\{\frac{\eta_1^p}{\int_\Omega \sup_{\|t\|_{L^q(\Omega)}\le\eta_1} F(x,t)dx}, \frac{\eta_2^p}{\int_\Omega \sup_{\|t\|_{L^q(\Omega)}\le\eta_2} F(x,t)dx}\}\right[$$

there exists a positive real number ρ such that the problem (1.1) has at least two weak solutions $u_i \in X, i=1,2$ whose norms in $C^0(\Omega)$ are less than some positive constant ρ.

Proof . Our aim is to apply Theorem 2.1. Let Φ,Ψ be the functionals defined in (2.5). From the above, we know that the Gâteaux derivative of Φ and Ψ are given by (2.6) and (2.7), respectively. Note that $\Phi(0)=\Psi(0)=0$. By (H1), it follows that

$$\frac{(m_0H-a)}{pH}\|u\|^p \le \Phi(u) \le \frac{m_1}{p}\|u\|^p. \tag{3.6}$$

Therefore, (3.6) implies that

$$\lim_{\|u\|\longrightarrow\infty}\Phi(u) = +\infty,$$

it means Φ is coercive. Moreover, from the weakly lower semicontinuity of the norm, and the monotonicity and continuity of $\widehat{M}$, we known that Φ is sequentially weakly lower semicontinuous. The functional Ψ has compact derivative, hence it is sequentially weakly upper semicontinuous. Put $r_1 = \frac{(m_0H-a)\eta_1^p}{c_q^p pH}$ and $r_2 = \frac{(m_0H-a)\eta_2^p}{c_q^p pH}$. Let the function v be defined by (3.2). Direct calculations show

$$\frac{\partial v(x)}{\partial x_i} = \begin{cases} 0, & x \in (\overline{\Omega}\setminus B(0,s)) \cup B(0,\frac{s}{2}), \\ \frac{1}{h}\left(\frac{12\rho x_i}{s^3} - \frac{24x_i}{s^2} + \frac{9x_i}{s\rho}\right), & x\in B(0,s)\setminus B(0,\frac{s}{2}) \end{cases}$$

and

$$\frac{\partial^2 v(x)}{\partial x_i^2} = \begin{cases} 0, & x \in (\overline{\Omega}\setminus B(0,s)) \cup B(0,\frac{s}{2}), \\ \frac{1}{h}\left(\frac{12(x_i^2+\rho^2)}{s^3\rho} - \frac{24}{s^2} + \frac{9(\rho^2-x_i^2)}{s\rho^3}\right), & x\in B(0,s)\setminus B(0,\frac{s}{2}). \end{cases} \tag{3.7}$$

By (3.7) and (3.1) we have

$$\sum_{i=1}^N \frac{\partial^2 v(x)}{\partial x_i^2} = \begin{cases} 0, & x \in (\overline{\Omega}\setminus B(0,s)) \cup B(0,\frac{s}{2}), \\ \frac{1}{h}\left(\frac{12\rho(N+1)}{s^3} - \frac{24N}{s^2} + \frac{9(N-1)}{s\rho}\right), & x\in B(0,s)\setminus B(0,\frac{s}{2}), \end{cases}$$

and

$$\int_\Omega |\Delta v(x)|^p dx = \left(\frac{1}{h}\right)^p \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)}\int_{\frac{s}{2}}^s \left|\frac{12(N+1)}{s^3}r - \frac{24N}{s^2} + \frac{9(N-1)}{s}\frac{1}{r}\right|^p r^{N-1}dr = \frac{L}{h^p}. \tag{3.8}$$

Thus, we have by (2.8) and (3.8) that

$$\frac{(m_0H-a)L}{pHh^p} \le \Phi(v(x)) \le \frac{m_1L}{ph^p}. \tag{3.9}$$

Consequently, in view of (3.3) we get

$$r_1 < \Phi(v(x)) < r_2. \tag{3.10}$$

Furthermore, by (3.10) we have

$$\varphi_2(r_1,r_2) = \inf_{u\in\Phi^{-1}(-\infty,r_1)}\sup_{u_1\in\Phi^{-1}[r_1,r_2[}\frac{\Psi(u)-\Psi(u_1)}{\Phi(u_1)-\Phi(u)} \ge \inf_{u\in\Phi^{-1}(-\infty,r_1)}\frac{\Psi(u)-\Psi(v)}{\Phi(v)-\Phi(u)}. \tag{3.11}$$

On the other hand, from (3.3) and (3.4), one has

$$\int_\Omega F(x,v(x))dx > B(\eta_1) > \frac{\frac{m_1}{p}}{\frac{(m_0H-a)}{pH}} \int_\Omega \sup_{\|t\|_{L^q(\Omega)}\le\eta_1} F(x,t)dx > \int_\Omega \sup_{\|t\|_{L^q(\Omega)}\le\eta_1} F(x,t)dx. \tag{3.12}$$

By (2.4) and (3.6), we obtain

$$\begin{aligned}\Phi^{-1}(]-\infty,r_1[) &= \{u\in X : \Phi(u)<r_1\}\\ &\subset \Big\{u\in X : \frac{(m_0H-a)}{pH}\|u\|^p<r_1\Big\}\\ &\subset \Big\{u\in X : \|u\|_{L^q(\Omega)}<c_q\big(\frac{pHr_1}{m_0H-a}\big)^{1/p}=\eta_1\Big\}.\end{aligned} \tag{3.13}$$

Therefore, the combination of (3.12) and (3.13) implies

$$\begin{aligned}\frac{\Psi(u)-\Psi(v)}{\Phi(v)-\Phi(u)} &= \frac{\int_\Omega F(x,v)dx-\int_\Omega F(x,u)dx}{\Phi(v)-\Phi(u)}\\ &\ge \frac{\int_\Omega F(x,u)dx-\int_\Omega \sup_{\|u\|_{L^q(\Omega)}\le\eta_1} F(x,u)dx}{\Phi(v)-\Phi(u)}\\ &\ge \frac{\int_\Omega F(x,u)dx-\int_\Omega \sup_{\|u\|_{L^q(\Omega)}\le\eta_1} F(x,u)dx}{\Phi(v)}\\ &\ge \frac{\int_\Omega F(x,u)dx-\int_\Omega \sup_{\|u\|_{L^q(\Omega)}\le\eta_1} F(x,u)dx}{\frac{m_1}{p}\|v\|_p^p}\\ &= \frac{ph^p}{m_1L}B(\eta_1).\end{aligned} \tag{3.14}$$

By (3.11) and (3.4), we have

$$\varphi_2(r_1,r_2) \ge \frac{ph^p}{m_1L}B(\eta_1). \tag{3.15}$$

Similarly, for every $u\in X$ such that $\Phi(u)\le r$, where r is a positive real number, one has

$$\|u\|_{L^q(\Omega)} \le c_q\|u\|^p \le \frac{c_q r}{\frac{(m_0H-a)}{pH}}. \tag{3.16}$$

By virtue of Φ being sequentially weakly lower semicontinuous, then $\overline{\Phi^{-1}(\infty,r)^w}=\Phi^{-1}(\infty,r)$. Consequently,

$$\begin{aligned}\varphi_1(r) &= \inf_{u\in\Phi^{-1}(\infty,r)} \frac{\Psi(u)-\inf_{\overline{\Phi^{-1}(\infty,r)^w}}\Psi(u)}{r-\Phi(u)}\\ &\le \frac{\Psi(0)-\inf_{\overline{\Phi^{-1}(\infty,r)^w}}\Psi(u)}{r-\Phi(0)}\\ &\le \frac{-\inf_{\overline{\Phi^{-1}(\infty,r)^w}}\Psi(u)}{r}\\ &\le \frac{\int_\Omega \sup_{\|u\|_{L^q(\Omega)}\le \frac{c_q r}{\frac{(m_0H-a)}{pH}}} F(x,u)dx}{r}.\end{aligned} \tag{3.17}$$

It implies that

$$\varphi_1(r_1) \le \frac{\int_\Omega \sup_{\|t\|_{L^q(\Omega)}\le\eta_1} F(x,t)dx}{r_1} = \frac{pHc_q^p}{(m_0H-a)\eta_1^p}\int_\Omega \sup_{\|t\|_{L^q(\Omega)}\le\eta_1} F(x,t)dx < \frac{ph^p}{m_1L}B(\eta_1), \tag{3.18}$$

$$\varphi_1(r_2) \le \frac{\int_\Omega \sup_{\|t\|_{L^q(\Omega)}\le\eta_2} F(x,t)dx}{r_2} = \frac{pHc_q^p}{(m_0H-a)\eta_2^p}\int_\Omega \sup_{\|t\|_{L^q(\Omega)}\le\eta_2} F(x,t)dx < \frac{ph^p}{m_1L}B(\eta_1). \tag{3.19}$$

By (3.15) -(3.19), we conclude

$$\varphi_1(r_1) \leq \varphi_2(r_1, r_2), \qquad \varphi_1(r_2) \leq \varphi_2(r_1, r_2), \tag{3.20}$$

Therefore, the conditions (i), (ii), and (iii) in Theorem 2.1 are satisfied. Consequently, by above facts, the functional $\Phi + \lambda\Psi$ has two local minima $u_1, u_2 \in X$, which lie in $\Phi^{-1}(\infty, r_1)$ and $\Phi^{-1}[r_1, r_2)$, respectively. Since $I = \Phi + \lambda\Psi \in C^1, u_1, u_2 \in X$ are the solutions of the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0.$$

Then $u_1, u_2 \in X$ are the weak solutions of problem (1.1). Since $\Phi(u_i) < r_2, i = 1, 2$, by (2.3) and (3.6)

$$\|u_i\|_{L^q(\Omega)} < c_q\Big(\frac{pHr_2}{m_0H - a}\Big)^{1/p} = \eta_2, i = 1, 2;$$

which implies there exists a positive real number ρ such that the norms of $u_i, \in X, i = 1, 2, \in C^0(\Omega)$ are less than some positive constant ρ. This completes the proof. □

Theorem 3.2. Assume that (F) and $(H1)$ hold and $0 < 2a < m_0H$ (with H is as in (2.1)). Suppose that there are two positive constants h, η with

$$\frac{\eta^p}{c_q^p} < \frac{L}{h^p} \tag{3.21}$$

such that

(k) $F(x, t) \geq 0 \quad \forall x \in \Omega\backslash B(0, s/2)$ and for all $t \in [0, \frac{1}{h}]$,

(kk) $\frac{m_1L}{h^p}|\Omega| \sup_{(x,t)\in\Omega\times\{t\in\mathbb{R}:\|t\|_{L^q(\Omega)}\leq c_q\left(\frac{pHr}{m_0H-a}\right)^{1/p}\}} F(x, t) < \frac{(m_0H - a)\eta^p}{Hc_q^p} \int_{B(0,s/2)} F(x, \frac{1}{h})dx.$

Then, there exists an open interval $\Lambda \subseteq [0, \infty]$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, the problem (1.1) has at least three weak solutions $u_i \in X, i = 1, 2, 3$ whose norms are less than ρ.

Proof . By (k) and (3.6), we have

$$\begin{aligned}\Phi(u) + \lambda\Psi(u) =& \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{a}{p}\int_\Omega \frac{|u(x)|^p}{|x|^p}dx - \lambda\int_\Omega F(x, u(x))dx \\ \geq& \frac{(m_0H - a)}{pH}\|u\|^p - \frac{a}{pH}\|u\|^p - \lambda\frac{a}{p}\int_\Omega \alpha(x)(1 + |u(x)|^\gamma) \\ \geq& \frac{(m_0H - 2a)}{pH}\|u\|^p - \lambda\|\alpha\|_\infty(|\Omega| + k_1\|u\|^\gamma),\end{aligned} \tag{3.22}$$

where k_1, are positive constant. Since $\gamma < p$, (3.22) implies that

$$\lim_{\|u\|\longrightarrow\infty} \Phi(u) + \lambda\Psi(u) = \infty, \tag{3.23}$$

The same as in (3.2), defining a function $v(x)$. Choosing $r = \frac{(m_0H - a)\eta^p}{pHc_q^p}$, by (3.21) we conclude

$$\Phi(v) \geq \frac{(m_0H - a)}{pH}\|v\|_p^p = \frac{(m_0H - a)}{pH}\frac{L}{h^p} > r.$$

By (kk) and the definitions of v, one has

$$\begin{aligned}|\Omega| \sup_{(x,t)\in\Omega\times\{t\in\mathbb{R}:\|t\|_{L^q(\Omega)}\leq c_q\left(\frac{pHr}{m_0H-a}\right)^{1/p}\}} F(x,t) &< \frac{(m_0H-a)h^p\eta^p}{m_1LHc_q^p}\int_{B(0,s/2)} F(x,\frac{1}{h})dx \\ &= \frac{(m_0H-a)\eta^p}{pHc_q^p}\frac{\int_{B(0,s/2)}F(x,\frac{1}{h})dx}{\frac{m_1L}{ph^p}} \\ &\leq \frac{(m_0H-a)\eta^p}{pHc_q^p}\frac{\int_{\Omega\setminus B(0,s/2)}F(x,v(x))dx+\int_{B(0,s/2)}F(x,\frac{1}{h})dx}{\frac{m_1L}{ph^p}} \\ &\leq \frac{(m_0H-a)\eta^p}{pHc_q^p}\frac{\int_\Omega F(x,v(x))dx}{\Phi(v(x))}.\end{aligned} \tag{3.24}$$

For every $u\in X$ such that $\Phi(u)\leq r$, and $x\in\Omega$, one has By (2.4) and (3.6), we obtain

$$\begin{aligned}\Phi^{-1}(]-\infty,r[) &= \{u\in X:\Phi(u)<r\} \\ &\subset \Big\{u\in X:\frac{(m_0H-a)}{pH}\|u\|^p<r\Big\} \\ &\subset \Big\{u\in X:\|u\|_{L^q(\Omega)}<c_q\big(\frac{pHr}{m_0H-a}\big)^{1/p}=\eta\Big\}.\end{aligned} \tag{3.25}$$

So

$$\begin{aligned}\sup_{u\in\Phi^{-1}(-\infty,r)}(-\Psi(u)) &\leq \sup_{u\in\Phi^{-1}(-\infty,r)}\int_\Omega F(x,u)dx \\ &\leq \sup_{\|u\|_{L^q(\Omega)}\leq\eta}\int_\Omega F(x,u)dx \\ &\leq \int_\Omega \sup_{\|u\|_{L^q(\Omega)}\leq\eta} F(x,u)dx \\ &\leq |\Omega| \sup_{(x,u)\in\Omega\times\{u\in X:\|u\|_{L^q(\Omega)}\leq\eta\}} F(x,u)dx \\ &\leq \frac{(m_0H-a)\eta^p}{c_q^p pH}\frac{\int_\Omega F(x,v(x))}{\Phi(v(x))} \\ &= r\frac{-\Psi(v)}{\Phi(v)},\end{aligned} \tag{3.26}$$

Therefore,

$$\inf_{u\in\Phi^{-1}(-\infty,r)}\Psi(u) > r\frac{\Psi(v(x))}{\Phi(v(x))}.$$

Note that $\Phi(0)=\Psi(0)=0$, we conclude that

$$\inf_{u\in\Phi^{-1}(-\infty,r)}\Psi(u) > \frac{\Big(\Phi(v(x))-r\Big)\Psi(0)+\Big(r-\Phi(0)\Big)\Psi(v(x))}{\Phi(v(x))-\Phi(0)}.$$

Hence, above facts, Φ and Ψ satisfy all conditions of Theorem 2.2; then the conclusion directly follows from Theorem 2.2. □

We end this paper by giving the following examples to illustrate Theorems 3.1 and 3.2, respectively.

Example 3.3. Consider the problem

$$\begin{aligned}M\Big(\int_\Omega|\Delta u|^2dx\Big)\Delta_2 u-\frac{0.0001}{|x|^2}u &= \lambda f(x,u) \quad \text{in } \Omega \\ u=\Delta u &= 0 \quad \text{on } \partial\Omega,\end{aligned} \tag{3.27}$$

where $\Omega := \{x \in \mathbb{R}^3 : |x| < 1\}$ and $M(t) = 1 + \dfrac{sin(t)}{100}, \quad t \geq 0$, and define $f(x,t) = cos(2\sqrt{x_1^2 + x_2^2 + x_3^2})cos(t)$ for every $(x,t) \in \Omega \times \mathbb{R}$. By the expression of f we have $F(x,t) = cos(2\sqrt{x_1^2 + x_2^2 + x_3^2})sin(t)$ for every $(x,t) \in \Omega \times \mathbb{R}$. Taking $\eta_1 = 13$ and $\eta_2 = 20$, $h = 2, p = 2$ since in this case, $H = \left(\frac{3-2}{2}\right)^2 = 0.25$, by simple calculations we observe that all conditions in Theorem 3.1 are satisfied. Therefore, for each $\lambda \in]182.84, 447.3[$ the problem (3.27) has at least two weak solutions.

Example 3.4. Consider the problem

$$\begin{gathered} M\Big(\int_\Omega |\Delta u|^2 dx\Big)\Delta_2 u - \frac{0.01}{|x|^2}u = \lambda f(x,u) \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.28}$$

where $\Omega := \{x \in \mathbb{R}^3 : |x| < 1\}$ and $M(t) = 2 - \dfrac{t}{4t+1}, \quad t \geq 0$ and define $f(x,t) = \dfrac{cos(t)}{2 + (x_1^2 + x_2^2 + x_3^2)}$ for every $(x,t) \in \Omega \times [0,1]$. By the expression of f we have $F(x,t) = \dfrac{sin(t)}{2 + (x_1^2 + x_2^2 + x_3^2)}$ for every $(x,t) \in \Omega \times [0,1]$. Choose $\eta = 27$ and $h = 1, p = 2$. We get $H = \left(\frac{3-2}{2}\right)^2 = 0.25$. By simple calculations then all conditions in Theorem 3.2 are fulfilled. Then there exist an open interval $\Lambda \subseteq [0,\infty]$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, the problem (3.28) has at least three weak solutions.

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