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On partial fuzzy metric-preserving functions

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Abstract

The target of this paper is to present partial fuzzy metric-preserving functions and characterize the functions $f : [0,1] \to [0,1]$ with this aspect. We give a characterization for partial fuzzy metric-preserving functions considering the different t-norms. Also, we show that the topology induced by partial fuzzy metric does not preserve under these functions with an example. Then we give a characterization of those partial fuzzy metric-preserving functions which preserve completeness and contractivity under some conditions. Finally, we discussed the relation between fuzzy metric preserving and partial fuzzy preserving functions.

Keywords: fuzzy partial metric spaces, metric-preserving functions, contraction mapping, completeness
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1 Introduction

Zadeh [23] introduced the theory of fuzzy sets in 1965, then this theory has been very popular which has been used frequently in contemporary studies concerning the logical and set-theoretical foundations of mathematics. The idea of applying this theory to the classical notions of metric and metric spaces was presented by Kramosil and Michalek [14] in 1975 and this notion called the fuzzy metric spaces. Then George and Veeramani [8] made a small change of the definition of fuzzy metric space to gain significant topological properties of these structures. This structure was studied extensively with the topological perspective such as completeness, compactness, countability, separability and etc. by many authors [8, 11, 12].

In 1994, Matthews [16] introduced the notion of partial fuzzy metric spaces which is a generalization of metric spaces with the aim of developing an appropriate mathematical aspect for modeling different processes that have seen in especially computer sciences. As a considering the axioms of partial metric and fuzzy metric together, Sedghi et al. [21] introduced the concept of partial fuzzy metric spaces by giving some properties of these spaces and they obtained some fixed point results. Some more studies related to the generalizations of metric structures can be found in [2, 4, 6, 7, 9, 10, 13, 18, 22].

As different from the theoretical aspects, J. Borsik and J. Dobos [5] characterized the functions that transform a metric into a new one and named these functions as metric-preserving functions in 1981. The transformation made with this way preserves the main properties of the metric space to be transformed. One of the reasons for doing this is that clarify if a particular property is fulfilled by a subset or a mapping. Another one can be thought of as to reduce the running time of computing of an algorithm used to solve a problem or to express certain cost measures

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when modeling some mathematical or physical problems. Then, S. Massanet and O. Valero [15] applied the idea of the metric-preserving function to the partial metric spaces. They gave the notion of partial metric-preserving functions and also obtained some characterizations of these functions. After, Minana and Valero [17] proved that the topology induced by partial metric is coincident with the topology induced by the transformed partial metric under the condition of continuity. Also they give some characterizations of these functions which preserves completeness and contractivity. And the obtained some relationships between metric-preserving and partial metric-preserving functions.

In recent studies, aggregation functions play a crucial role in the necessity of merging information contained in a collection of pieces of information into a single one in the numerical calculations especially decision-making problems, image processing, clustering analysis and etc. Another motivation where aggregation function theory has been successfully applied and fuzzy binary relations have shown to be particularly very useful is provided by the so-called fuzzy databases where uncertain information can be managed. One technique to generate new fuzzy binary relations is based on merging a collection of them into a new one by means of the use of a function. With this base, Pedraza et al. [19] presented a novel study which functions allow us to merge a collection of fuzzy (quasi-) metrics (in the sense of Kramosil and Michalek) into a single one. They given a characterization of such functions in terms of $*$-triangular triplets, isotonicity and $*$-supmultiplicativity, where $*$ is a t-norm. They also proved that this characterization does not depend on the symmetry of the fuzzy quasi-metrics. This study is innovative and important as that has shown there are relationships between those functions aggregating fuzzy preorders and indistinguishability operators.

In this paper, we aim to characterize the functions $f : [0,1] \to [0,1]$ in the perspective of partial fuzzy metric-preserving with the consideration of the partial metric-preserving and fuzzy metric-preserving functions. Not only to obtain the relation between fuzzy metric-preserving functions and partial fuzzy metric-preserving functions but also to have a more general view, we first revise the definition of partial fuzzy metric spaces given by Sedghi et al. [21]. Then we give some characterizations for these functions and also, we investigate under which conditions partial fuzzy metric-preserving functions preserve completeness and contractivity. Finally, we discuss the relations between fuzzy metric-preserving and partial fuzzy metric-preserving functions.

2 Preliminaries

In this section, we recollect some fundamental notions such as t-norm, (partial) fuzzy metric, fuzzy metric-preserving functions which will play an important role throughout this study.

Definition 2.1 [20] A t-norm is a binary operation $*$ on the unit interval $[0,1]$ which is monotone, commutative, associative and has 1 as neutral element. If $*$ is also continuous then we will say that it is a continuous t-norm. The followings are examples of continuous t-norms:

$$a *_{\wedge} b := \min(a,b)$$
$$a *_{p} b := a.b$$
$$a *_{\mathfrak{L}} b := \max(x+y-1,0)$$
$$a *_{D} b := \begin{cases} \min(a,b), & a=1 \text{ or } b=1 \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.1. [19] Let $*$ be a t-norm. A triplet $(a,b,c) \in [0,1]^3$ is said to be

(i) asymmetric $*$-triangular if $a * b \leq c$.

(ii) $*$-triangular if $a * b \leq c$, $b * c \leq a$ and $a * c \leq b$.

If (a,b,c) is (asymmetric) $*$-triangular for every t-norm, then we say that (a,b,c) is a (asymmetric) triangular triplet.

Definition 2.2. [14] A fuzzy metric (in the sense of Kramosil and Michalek) on a non-empty set X is a pair $M,*)$ such that $*$ is a continuous t-norm and P is a fuzzy set on $X \times X \times [0,\infty)$ satisfying the following conditions for all $x,y,z \in X$ and $s,t > 0$:

(PFM1) $M(x,y,0) = 0$,

(PFM2) $M(x,y,t) = 1 \Leftrightarrow x = y$,

(PFM3) $M(x,y,t) = M(y,x,t)$,

(PFM4) $M(x, y, s+t) \geq M(x, z, s) * M(z, y, t)$,

(PFM5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left-continuous.

A fuzzy metric space is a triple $(X, M, *)$ such that X is a non-empty set and $(M, *)$ is a fuzzy metric on X.

Definition 2.3. [19] Let $f : [0, 1] \to [0, 1]$ be a function. Then

(1) f is said to be an aggregation function if the followings are hold:

(i) f is isotone. i.e., if $a \leq b$ then $f(a) \leq f(b)$,

(ii) $f(0) = 0$ and $f(1) = 1$.

(2) f is said to be $*$-supmultiplicative if the following inequality is satisfied for all $a, b \in [0, 1]$ where $*$ is a t-norm:

$$f(a) * f(b) \leq f(a * b).$$

(3) f is said to preserve (asymmetric) $*$-triangular triplets if $(f(a), f(b), f(c))$ is a (symmetric) $*$-triangular triplet whenever (a, b, c) is a (symmetric) $*$-triangular triplet. If f preserves (asymmetric) $*$-triangular for every t-norm, then we say that f preserves (asymmetric) triangular triplet.

(4) $*$-(stationary) fuzzy metric-preserving ($*$-(s)fmp) function if $(X, M, *)$ is a (stationary) fuzzy metric space then $(X, f \circ M = M_f, *)$ is a (stationary) fuzzy metric space.

Definition 2.4. [19] The core of a function $f : [0, 1] \to [0, 1]$ is the set $f^{-1}(1)$. We say that f has a trivial core if $f^{-1}(1) = \{1\}$.

Theorem 2.5. [19] Let $f : [0, 1] \to [0, 1]$ be a function and $*$ be a t-norm. The following statements are equivalent:

(i) f is a $*$-fmp function.

(ii) f is an aggregation function, $*$-supmultiplicative, left-continuous and f has a trivial core.

(iii) $f(0) = 0$, f is left-continuous with a trivial core and f preserves asymmetric $*$-triangular triplets.

Corollary 2.6. [19] Let f be a function. Then f is a $*_\wedge$-fmp function if and only if f is an aggregation function, left-continuous and f has a trivial core.

Theorem 2.7. [19] Let $f : [0, 1] \to [0, 1]$ be a function and $*$ be a t-norm. The following statements are equivalent:

(i) f is a $*$-sfmp function.

(ii) f is an aggregation function, $*$-supmultiplicative and f has a trivial core.

(iii) $f(0) = 0$, f has a trivial core and f preserves asymmetric $*$-triangular triplets.

Definition 2.8. [21] A partial fuzzy metric in the sense of Sedghi et al. (SE-partial fuzzy metric, for short) on a non-empty set X is a pair $(P, *)$ such that $*$ is a continuous t-norm and P is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:

(PFM1) $P(x, y, t) > 0$,

(PFM2) $x = y \Leftrightarrow P(x, x, t) = P(x, y, t) = P(y, y, t)$,

(PFM3) $P(x, y, t) = P(y, x, t)$,

(PFM4) $P(x, y, \max(s, t)) * P(z, z, \max(s, t)) \geq P(x, z, s) * P(z, y, t)$,

(PFM5) $P(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

A SE-partial fuzzy metric space is a triple $(X, P, *)$ such that X is a non-empty set and $(P, *)$ is a SE-partial fuzzy metric on X.

There are some difference between SE-partial fuzzy metric spaces and fuzzy metric spaces. One of them, in a fuzzy metric space $(X, M, *)$, $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is non-decreasing function for all $x, y \in X$, but in a SE-partial fuzzy metric space $(X, P, *)$, $P(x, y, \cdot) : (0, \infty) \to [0, 1]$ may not be non-decreasing function for all $x, y \in X$. An example showing this situation is provided in [1].

Proposition 2.9. [21] Let $(X, P, *)$ be a SE-partial fuzzy metric space. If $b \geq c$ whenever $a * b \geq a * c$ for all $a, b, c \in [0, 1]$, then $P(x, y, \cdot) : (0, \infty) \to [0, 1]$ is non-decreasing function for all $x, y \in X$.

Definition 2.10. [1] Let $(X, P, *)$ be a SE-partial fuzzy metric space and (x_n) be a sequence in X.

(i) (x_n) is said to converge to $x \in X$ if $P(x, x, t) = \lim_{n\to\infty} P(x_n, x, t)$ for all $t > 0$.

(ii) (x_n) is said to be a Cauchy sequence if $\lim_{n,m\to\infty} P(x_n, x_m, t)$ exists.
If $\lim_{n,m\to\infty} P(x_n, x_m, t) = 1$, then (x_n) is called a 1-Cauchy sequence.

(iii) A SE-partial fuzzy metric space $(X, P, *)$ is said to be complete (resp. 1-complete) if every Cauchy (resp. 1-Cauchy) sequence $(x_n) \subset X$ converges to $x \in X$ such that $\lim_{n,m\to\infty} P(x_n, x_m, t) = P(x, x, t)$.

Clearly, every 1-Cauchy sequence (x_n) in $(X, P, *)$ is also a Cauchy sequence and every complete SE-partial fuzzy metric space is a 1-complete space.

3 Partial fuzzy metric-preserving functions

In this section, we obtain some properties of partial fuzzy metric-preserving functions by giving some characterization for those functions and also investigate some relations between partial fuzzy metric-preserving functions and fuzzy metric-preserving functions. To obtain these relations, we first give the notion of partial fuzzy metric space in the meaning of Kramosil and Michalek which is a more general view of the concept given by Sedghi et al. [21]:

Definition 3.1. A partial fuzzy metric on a non-empty set X is a pair $(P, *)$ such that $*$ is a continuous t-norm and P is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:

(PFM1) $P(x, y, 0) = 0$,

(PFM2) $x = y \Leftrightarrow P(x, x, t) = P(x, y, t) = P(y, y, t)$,

(PFM3) $P(x, y, t) = P(y, x, t)$,

(PFM4) $P(x, y, \max(s, t)) * P(z, z, \max(s, t)) \geq P(x, z, s) * P(z, y, t)$,

(PFM5) $P(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left-continuous.

A partial fuzzy metric space is a triple $(X, P, *)$ such that X is a non-empty set and $(P, *)$ is a partial fuzzy metric on X.

Remark 3.2. If $(X, P, *)$ is a SE-partial fuzzy metric space, then $(X, \overline{P}, *)$ is a partial fuzzy metric space where

$$\overline{P}(x, y, t) = \begin{cases} 0, & t = 0 \\ P(x, y, t), & t > 0 \end{cases}$$

for all $x, y \in X$ and $t \geq 0$.

Example 3.3. Let (X, p) be a partial metric space. Define the mapping $P : X \times X \times [0, \infty)$ by

$$P(x, y, t) = \begin{cases} 0, & t = 0 \\ \frac{t}{t + p(x,y)}, & t > 0 \end{cases}$$

for all $x, y \in X$. Then $(X, P, *_p)$ is a partial fuzzy metric space.

Definition 3.4. A (partial) fuzzy metric space is $(X, P, *)$ is said to be stationary (or $(P, *)$ is a stationary (partial) fuzzy metric on X) if the function $P(x, y, \cdot) : (0, \infty) \to [0, 1]$ is constant for every $x, y \in X$.

Example 3.5. Let $X = (0, 1]$ and $P : X \times X \times [0, \infty) \to [0, 1]$ be defined by

$$P(x, y, t) = \begin{cases} 0, & t = 0 \\ 1 - \frac{1}{2}\max(x, y), & t > 0 \end{cases}$$

for all $x, y \in X$. Then $(X, P, *_p)$ is a stationary partial fuzzy metric space.

Here, we note that Proposition 2.9, Definition 2.10 and Definition 2.11 are valid for partial fuzzy metric spaces.

Definition 3.6. A function $f : [0, 1] \to [0, 1]$ is said to be a $*$-(stationary) partial fuzzy metric-preserving ($*$-(s)pfmp) function provided that, for each (stationary) partial fuzzy metric space $(X, P, *)$, the triple $(X, P_f, *)$ is a (stationary) partial fuzzy metric space where $P_f : X \times X \times [0, \infty) \to [0, 1]$ is a mapping defined by $P_f(x, y, t) = f(P(x, y, t))$ for all $x, y \in X$ and $t \geq 0$.

Proposition 3.7. If $f : [0, 1] \to [0, 1]$ is a $*$-pfmp function, then f is isotone and $f(0) = 0$.

Proof . Let $X = [0, 1]$ and $P : X \times X \times [0, \infty)$ be defined by

$$P(x, y, t) = \begin{cases} t. \min(x, y), & 0 \leq t < 1 \\ \min(x, y), & t \geq 1 \end{cases}$$

for all $x, y \in [0, 1]$. Then $(X, P, *)$ is a partial fuzzy metric space where $*$ is a continuous t-norm. Suppose that $f : [0, 1] \to [0, 1]$ is a $*$-pfmp function and $a \leq b$ for $a, b \in [0, 1]$. Then

$$P_f(a, b, 3) \leq P_f(b, b, 3)$$

since $(P_f, *)$ is a partial fuzzy metric on X. Hence, we obtain that $f(P(a, b, 3)) \leq f(P(b, b, 3))$ which means $f(a) \leq f(b)$.

To show $f(0) = 0$, let $(X, P, *)$ be any partial fuzzy metric space. Then, we have

$$f(0) = f(P(x, y, 0)) = P_f(x, y, 0) = 0$$

for all $x, y \in X$, since $(X, P_f, *)$ is a partial fuzzy metric space. □ The next example illustrates that the property $f(1) = 1$ satisfying for $*$-fmp function does not hold for $*$-pfmp function.

Example 3.8. Let $([0, 1], P, *)$ be a partial fuzzy metric space and $a \in (0, 1)$. Then $P_1 : X \times X \times [0, \infty) \to [0, 1]$ defined by $P_1(x, y, t) = P(x, y, t) \wedge a$, is a partial fuzzy metric on X. Therefore, $f(x) = \min(x, a)$ is a $*$-pfmp function. But $F(1) = \min(a, 1) = a \neq 1$.

Proposition 3.9. If $f : [0, 1] \to [0, 1]$ is a $*$-pfmp function, then f is $*$-supmultiplicative.

Proof . Let $f : [0, 1] \to [0, 1]$ is a $*$-pfmp function. With a similar idea to the proof of Theorem 4.15 in [19], let us take $X = \{x_1, x_2, x_3\}$ and define $P : X \times X \times [0, \infty) \to [0, 1]$ by

$$P(x, y, t) = P(y, x, t) = \begin{cases} 0, & t = 0 \\ a, & x = x_1, y \in \{x_1, x_2\}, t > 0 \\ b, & x \in \{x_2, x_3\}, y = x_3, t > 0 \\ a * b, & x = x_1, y = x_3, t > 0 \\ 1, & x = x_2, y = x_2, t > 0 \end{cases}$$

where $*$ is a continuous t-norm. It is simple to check that $(X, P, *)$ is a partial fuzzy metric space. Then, we have the followings:

$$P_f(x_1, x_2, 2) * P_f(x_2, x_3, 2) \leq P_f(x_1, x_3, 2) * P_f(x_2, x_2, 2) \Rightarrow f(a) * f(b) \leq f(a * b) * f(1) \leq f(a * b).$$

□

Corollary 3.10. If $f : [0, 1] \to [0, 1]$ is a $*$-pfmp function, then f is isotone, $*$-supmultiplicative and $f(0) = 0$.

Proposition 3.11. If $* = \wedge$ or $*$ satisfies the condition $b \geq c$ whenever $a * b \geq a * c$ for all $a, b, c \in [0, 1]$ and f is a $*$-pfmp function, then $P_f : X \times X \times [0, \infty) \to [0, 1]$ is non-decreasing for all $x, y \in X$.

Proof . If $*$ satisfies the condition $b \geq c$ whenever $a * b \geq a * c$ for all $a, b, c \in [0, 1]$ and f is a $*$-pfmp function, then it is clear from Proposition 2.9 that $P_f : X \times X \times [0, \infty) \to [0, 1]$ is non-decreasing for all $x, y \in X$. Now, let us take $* = \wedge$. Suppose that $P_f(x, y, t) < P_f(x, y, s)$ whenever $t \geq s$. Then, we have $P_f(x, y, t) \wedge P_f(y, y, t) \geq P_f(x, y, s) \wedge P_f(y, y, t)$

for all $x, y \in X$. This follows that $P_f(x,y,t) \geq P_f(x,y,s) \wedge P_f(y,y,t)$ since $P_f(x,y,t) \leq P_f(y,y,t)$ for all $x, y \in X$ and $t > 0$. Hence, we obtain $P_f(x,y,t) \geq P_f(x,y,s) \wedge P_f(y,y,t)$. If $P_f(x,y,s) \wedge P_f(y,y,t) = P_f(x,y,s)$, then we have $P_f(x,y,t) \geq P_f(x,y,s) > P_f(x,y,t)$ which is a contradiction. Otherwise, if $P_f(x,y,s) \wedge P_f(y,y,t) = P_f(y,y,t)$, then we obtain $P_f(y,y,t) = P_f(x,y,t)$ and $P_f(x,x,t) = P_f(x,y,t)$ with similar consideration and so we have $x = y$ which contradicts that we have this hypothesis for all $x, y \in X$. As a result, $P_f : X \times X \times [0,\infty) \to [0,1]$ is a non-decreasing function for all $x, y \in X$ when $* = \wedge$. □

Theorem 3.12. Let $f : [0,1] \to [0,1]$ be a function and let $*$ be a t-norm. If f is a $*$-pfmp function, then f holds the following properties for all $a, b, c, d \in [0,1]$:

(1) f is left-continuous,

(2) $f(a) * f(b) \leq f(c) * f(d)$ whenever $max(a,b) \leq d$ and $a * b \leq c * d$,

(3) If $a \leq \min(b,c)$ and $f(a) = f(b) = f(c)$, then $a = b = c$.

Proof . (1): Let (t_n) be a non-decreasing sequence in $[0,1]$ and (t_n) converges to $t \in [0,1]$. Consider the partial fuzzy metric space $(X, P, *_p)$ where

$$P(x,y,t) = \begin{cases} 0, & t = 0 \\ \frac{t}{t+\max(x,y)}, & t > 0 \end{cases}$$

for all $x, y \in X$. Now, we can find a non-decreasing sequence $(s_n) \subseteq [0,1]$ such that $\frac{s_n}{s_n+1} = t_n$. Therefore, $P(1,1,s_n) = t_n$ and we can denote $\frac{s_n}{s_n+1} \to \frac{s}{s+1} = t$. So, we have

$$P(1,1,s_n) = t_n \to t = \frac{s}{s+1} = P(1,1,s).$$

Since $P_f : X \times X \times [0,\infty) \to [0,1]$ is left-continuous for all $x, y \in X$, we obtain

$$P_f(1,1,s_n) \to P_f(1,1,s)$$

which means that $f(t_n) \to f(t)$ and so f is left-continuous.

(2): Let $max(a,b) \leq d$ and $a * b \leq c * d$. If $a \leq b \leq c \leq d$, $a \leq c \leq b \leq d$, $a \leq b \leq d \leq c$, $b \leq a \leq c \leq d$, $b \leq a \leq d \leq c$ or $b \leq c \leq a \leq d$, then we obtain $f(a) * f(b) \leq f(c) * f(d)$ directly. If $a * b \leq c * d$ whenever $c \leq a \leq b \leq d$ or $c \leq b \leq a \leq d$, then we have $f(a) * f(b) \leq f(c) * f(d)$ since f is isotone.

(3): Let $a \leq min(b,c)$, f be a $*$-pfmp function and $f(a) = f(b) = f(c)$. Consider the partial fuzzy metric space $(X, P, *_p)$ where

$$P(x,y,t) = \begin{cases} 0, & t = 0 \\ \frac{t}{t+\max(x,y)}, & t > 0 \end{cases}$$

for all $x, y \in X$. If $a, b, c \in (0,1]$, then there exist $x_1, x_2 \in X$ such that $a = P(x_3,x_3,t) \leq P(x_1,x_2,t)$, $b = P(x_1,x_1,t)$ and $c = P(x_2,x_2,t)$ for all $t > 0$. Then, we have $P_f(x_3,x_3,t) \leq P_f(x_1,x_2,t) \leq P_f(x_1,x_1,t)$. It follows that $P_f(x_1,x_2,t) = P_f(x_1,x_1,t) = P_f(x_2,x_2,t) = P_f(x_3,x_3,t)$ since $f(a) = f(b) = f(c)$. And we obtain that $x_1 = x_2$ since $(P_f, *)$ is a partial fuzzy metric on X. If $P(x_3,x_3,t) \leq P(x_1,x_1,t)$, then we have $\frac{t}{t+x_3} \leq \frac{t}{t+x_1}$ which follows that $P_f(x_1,x_3,t) = P_f(x_3,x_3,t)$. As a consequence, we prove that $x_1 = x_2 = x_3$ and so $a = b = c$. □

Theorem 3.13. Let $f : [0,1] \to [0,1]$ be a function and let $*$ be satisfied the condition $b \geq c$ whenever $a * b \geq a * c$ for all $a, b, c \in [0,1]$. If f satisfies the following properties:

(1) $f(0) = 0$,

(2) f is left-continuous,

(3) $f(a) * f(b) \leq f(c) * f(d)$ whenever $max(a,b) \leq d$ and $a * b \leq c * d$,

(4) If $a \leq \min(b,c)$ and $f(a) = f(b) = f(c)$, then $a = b = c$,

for all $a, b, c, d \in [0,1]$, then f is a $*$-pfmp function.

Proof . Let the conditions (1)-(4) hold and let $(X, P, *)$ be a partial fuzzy metric space. Then, for all $x, y, z \in X$ and $s, t > 0$, we have the followings:

(PFM1) $P_f(x,y,0) = f(P(x,y,0)) = f(0) = 0$ (from the condition (1))

(PFM2) Let $P_f(x,y,t) = P_f(x,x,t) = P_f(y,y,t)$. Since $P(x,y,t) \leq P(x,x,t)$ and $P(x,y,t) \leq P(y,y,t)$, we have, from (4), $P(x,y,t) = P(x,x,t) = P(y,y,t)$. It follows that $x = y$.

(PFM3) $P_f(x,y,t) = f(P(x,y,t)) = f(P(y,x,t)) = P_f(y,x,t)$.

(PFM4) Since $P(x,z,t) \leq P(z,z,\max(s,t))$, $P(z,y,s) \leq P(z,z,\max(s,t))$ and $P(x,y,max(s,t)) * P(z,z,\max(s,t)) \geq P(x,z,t) * P(z,y,s)$, from (2.3), the following hold:

$$P_f(x,y,\max(s,t)) * P_f(z,z,\max(s,t)) \geq P_f(x,z,t) * P_f(z,y,s)$$

(PFM5) Since $P(x,y,\cdot) : [0,\infty) \to [0,1]$ and $f : [0,1] \to [0,1]$ are left-continuous, $P_f(x,y,\cdot) = f(P(x,y,\cdot)) : [0,\infty) \to [0,1]$ is left-continuous.

Therefore, $(X, P_f, *)$ is a partial fuzzy metric space and this implies that f is a $*$-pfmp function. □

Theorem 3.14. Let $f : [0,1] \to [0,1]$ be a function. Then f is a $*_\wedge$-pfmp function if and only if f satisfies the following conditions for all $a,b,c,d \in [0,1]$:

(1) $f(0) = 0$,

(2) f is left-continuous,

(3) If $a \leq \min(b,c)$ and $f(a) = f(b) = f(c)$, then $a = b = c$,

Proof . The proof can be completed in a similar way to the proof of Theorem 3.13. □

Corollary 3.15. Let $f : [0,1] \to [0,1]$ be a function and let $* = *_\wedge$ or $*$ be satisfied the condition $b \geq c$ whenever $a * b \geq a * c$ for all $a,b,c \in [0,1]$. Then the followings are equivalent:

(1) f is a $*$-pfmp function.

(2) f satisfies the following properties for all $a,b,c,d \in [0,1]$:

(2.1) $f(0) = 0$,

(2.2) f is left-continuous,

(2.3) $f(a) * f(b) \leq f(c) * f(d)$ whenever $\max(a,b) \leq d$ and $a * b \leq c * d$,

(2.4) If $a \leq \min(b,c)$ and $f(a) = f(b) = f(c)$, then $a = b = c$,

Proof . The proof is obtained directly from Theorem 3.12 and Theorem 3.13. □

Corollary 3.16. Let $f : [0,1] \to [0,1]$ be a function and $*$ be a continuous t-norm. Then the followings are equivalent:

(1) f is a $*$-spfmp function.

(2) f holds the following properties for all $a,b,c,d \in [0,1]$:

(2.1) $f(0) = 0$,

(2.2) $f(a) * f(b) \leq f(c) * f(d)$ whenever $\max(a,b) \leq d$ and $a * b \leq c * d$,

(2.3) If $a \leq \min(b,c)$ and $f(a) = f(b) = f(c)$, then $a = b = c$.

Proof . We can prove easily this assertion by taking a stationary partial fuzzy metric in the proof of Theorem 3.13. □

Corollary 3.17. Let $f : [0,1] \to [0,1]$ be a function. Then the followings are equivalent:

(1) f is a $*_\wedge$-spfmp function.

(2) f holds the following properties for all $a,b,c,d \in [0,1]$:

(1) $f(0) = 0$,

(2) If $a \leq \min(b,c)$ and $f(a) = f(b) = f(c)$, then $a = b = c$.

Proof . We can prove easily this assertion by taking a stationary partial fuzzy metric in the proof of Theorem 3.14. □

Corollary 3.18. If $f:[0,1]\to[0,1]$ is a $*$-pfmp function which has a trivial core, then f is a $*$-fmp function.

Proof . If f has a trivial core, we have $f(1)=1$. Since, $f(0)=0$, $f(1)=1$ and f is isotone, then f is a aggregation function. Also, f is $*$-supmultiplicative and left-continuous. Therefore, from Theorem 2.5, it follows that f is a $*$-fmp function. □

4 Strong partial fuzzy metric-preserving functions

If $(X,P,*_\wedge)$ is a partial fuzzy metric space, then the family $\{B_P(x,r,t)|x\in X, r\in(0,1), t>0\}$ where $B_P(x,r,t)=\{y|P(x,y,t)>P(x,x,t)-r\}$ is a base for a topology on X (see [3]).

The following example shows that $*_\wedge$-spfmp function does not preserve topologies. i.e., The topology induced by the transformed partial fuzzy metric may not be coincident with the topology induced by the partial fuzzy metric to be transform through the $*_\wedge$-spfmp function.

Example 4.1. Let us consider the function $f:[0,1]\to[0,1]$ given by

$$f(x)=\begin{cases}0, & x=0\\ \frac{1+x}{2+x}, & x\in(0,1]\end{cases}.$$

From Corollary 3.17,f is a $*_\wedge$-spfmp function. Consider the partial fuzzy metric space $(X,P,*_\wedge)$ where $X=\mathbb{R}^+$ and

$$P(x,y,t)=\begin{cases}0, & t=0\\ \frac{1}{1+\max(x,y)}, & t>0\end{cases}$$

for all $x,y\in\mathbb{R}^+$. Now, we have

$$P_f(x,y,t)=\begin{cases}0, & t=0\\ \frac{2+\max(x,y)}{3+\max(x,y)}, & t>0\end{cases}$$

for all $x,y\in\mathbb{R}^+$. It is easy to see that $B_{P_f}(0,\frac{1}{4})=\{0\}$ and $B_P(0,\delta)=[0,\frac{\delta}{1-\delta}),\forall\delta\in(0,1)$. So, we can not find a $\delta\in(0,1)$ such that $[0,\frac{\delta}{1-\delta})\subseteq\{0\}$. Consequently, τ_{P_f} and τ_P are not same.

Definition 4.2. A $*$-spfmp function $f:[0,1]\to[0,1]$ is said to be strong if for each (stationary) partial fuzzy metric space $(X,P,*)$, the partial fuzzy metrics $(P,*)$ and $(P_f,*)$ are topologically equivalent.

Theorem 4.3. If $f:[0,1]\to[0,1]$ is a $*_\wedge$-spfmp ($*_p$-spfmp) function, then f is continuous at 1.

Proof . Let us consider the stationary partial fuzzy metric space $((0,1],P,*_\wedge)$ where

$$P(x,y,t)=\begin{cases}0, & t=0\\ \min(x,y), & t>0\end{cases}$$

for all $x,y\in(0,1]$. Since f is strong $*_\wedge$-spfmp function,we have that the partial fuzzy metrics $(P,*)$ and $(P_f,*)$ are topologically equivalent. Let $\varepsilon>0$. Then there is $\delta>0$ such that $B_P(1,\delta t)\subseteq B_{P_f}(1,\varepsilon,t)$. Hence, if $y\in B_P(1,\delta t)$ then $y\in B_{P_f}(1,\varepsilon,t)$. As a consequence, we have $f(y)>1-\varepsilon$ whenever $y>1-\delta$. This follows that f is continuous at 1.

Since $((0,1],P,*_p)$ is a stationary partial fuzzy metric where P is defined as above, the proof can be completed with the similar way to show that f is continuous at 1 when f is a strong $*_p$-spfmp function. □

Theorem 4.4. Let $f:[0,1]\to[0,1]$ be a strong $*_\wedge$-spfmp function such that $f(1)=1$. Then the followings are equivalent:

(i) $(X,P,*_\wedge)$ is a 1-complete stationary partial fuzzy metric space.

(ii) $(X,P_f,*_\wedge)$ is a 1-complete stationary partial fuzzy metric space.

Proof . $(i) \Rightarrow (ii)$: Let $(X, P, *_\wedge)$ be a 1-complete stationary partial fuzzy metric space and (x_n) be a 1-Cauchy sequence in $(X, P_f, *_\wedge)$. Then, we have $lim_{n,m\to\infty} P_f(x_n, x_m, t) = 1$ for all $t > 0$. Since f is strictly increasing function, we obtain $f^{-1} : [0, f(1)] \to [0, 1]$ is continuous and by using this continuity, $\lim_{n,m\to\infty} P(x_n, x_m, t) = 1$ holds for all $t > 0$. This follows that (x_n) is a 1-Cauchy sequence in $(X, P, *_\wedge)$. Since $(X, P, *_\wedge)$ is 1-complete, there exist a point $x \in X$ such that $\lim_{n,m\to\infty} P(x_n, x_m, t) = \lim_{n\to\infty} P(x_n, x, t) = P(x, x, t) = 1$ for all $t > 0$. And from the continuity of f at 1, the following is obtained:

$$\lim_{n,m\to\infty} f(P(x_n, x_m, t)) = \lim_{n\to\infty} f(P(x_n, x, t)) = f(P(x, x, t)) = f(1) = 1$$

for all $t > 0$, which means that $(X, P_f, *_\wedge)$ is 1-complete.

$(ii) \Rightarrow (i)$: Now, suppose that $(X, P_f, *_\wedge)$ is a 1-complete stationary partial fuzzy metric space and (x_n) be a 1-Cauchy sequence in $(X, P, *_\wedge)$. From this hypothesis, we have $lim_{n,m\to\infty} P(x_n, x_m, t) = 1$ for all $t > 0$. Since f is continuous at 1, we get $\lim_{n,m\to\infty} f(P(x_n, x_m, t)) = f(1) = 1$. This means that (x_n) is 1-Cauchy sequence in $(X, P_f, *_\wedge)$. Since $(X, P_f, *_\wedge)$ is 1-complete, there exists $x \in X$ such that $\lim_{n,m\to\infty} P_f(x_n, x_m, t) = \lim_{n\to\infty} P_f(x_n, x, t) = P_f(x, x, t) = 1 = f(1)$ for all $t > 0$. Hence, we have $P(x, x, t) = 1$ by using the strictly monotonicity of f. Now, we have to show that $\lim_{n,m\to\infty} P(x_n, x_m, t) = \lim_{n\to\infty} P(x_n, x, t) = P(x, x, t) = 1$ for all $t > 0$. Let $\varepsilon \in (0, 1)$. Since f is strong $*_\wedge$-spfmp function, there exists a $\delta \in (0, 1)$ such that $B_{P_f}(x, \delta, t) \subseteq B_P(x, \varepsilon, t)$ for all $x \in X$. This follows that $x_n \in B_P(x, \varepsilon, t)$ since $\lim_{n\to\infty} P_f(x_n, x, t) = P_f(x, x, t) = 1$. Therefore, we have $\lim_{n,m\to\infty} P(x_n, x_m, t) = \lim_{n\to\infty} P(x_n, x, t) = P(x, x, t) = 1$ which means that $(X, P, *_\wedge)$ is 1-complete. □

Theorem 4.5. Let $f : [0, 1] \to [0, 1]$ be a strictly increasing, surjective strong $*$-spfmp function. Then the followings are equivalent:

(i) $(X, P, *)$ is a complete stationary partial fuzzy metric space.

(ii) $(X, P_f, *)$ is a complete stationary partial fuzzy metric space.

Proof . $(i) \Rightarrow (ii)$: Let $(X, P, *)$ be a complete stationary partial fuzzy metric space and (x_n) be a Cauchy sequence in $(X, P_f, *)$. Then, there is $b_0 \in [0, 1]$ such that $\lim_{n,m\to\infty} P_f(x_n, x_m, t) = b_0$ for all $t > 0$. By the surjectivity of f, there exists $a_0 \in [0, 1]$ such that $f(a_0) = b_0$. Since f is strictly increasing function, then we have $f^{-1} : [0, f(1)] \to [0, 1]$ is continuous and by using this continuity $\lim_{n,m\to\infty} P(x_n, x_m, t) = a_0$ is obtained for all $t > 0$. This shows that (x_n) is a Cauchy sequence in $(X, P, *)$. Since $(X, P, *)$ is complete, there exists $x \in X$ such that $\lim_{n,m\to\infty} P(x_n, x_m, t) = \lim_{n\to\infty} P(x_n, x, t) = P(x, x, t) = a_0$. Also, f is continuous since f is strictly increasing and surjective. So, we have $\lim_{n,m\to\infty} P_f(x_n, x_m, t) = \lim_{n\to\infty} P_f(x_n, x, t) = P_f(x, x, t) = b_0$ which means that $(X, P_f, *)$ is complete.

$(ii) \Rightarrow (i)$: Now, suppose that $(X, P_f, *)$ is a complete stationary partial fuzzy metric space and (x_n) be a Cauchy sequence in $(X, P, *)$. From this hypothesis, there exists $a_0 \in [0, 1]$ such that $\lim_{n,m\to\infty} P(x_n, x_m, t) = a_0$ for all $t > 0$. From the continuity of f, we have $lim_{n,m\to\infty} P_f(x_n, x_m, t) = f(a_0)$ which means that (x_n) is Cauchy sequence in $(X, P_f, *)$. Since $(X, P_f, *)$ is complete, there exist a point $x \in X$ such that $\lim_{n,m\to\infty} P_f(x_n, x_m, t) = \lim_{n\to\infty} P_f(x_n, x, t) = P_f(x, x, t) = f(a_0)$ for all $t > 0$. Hence, we have $P(x, x, t) = a_0$ by using the strictly monotonicity of f. Now, we have to show that $\lim_{n,m\to\infty} P(x_n, x_m, t) = \lim_{n\to\infty} P(x_n, x, t) = P(x, x, t) = a_0$ for all $t > 0$. Let $\varepsilon \in (0, 1)$. Since f is strong $*$-spfmp function, there exists $\delta \in (0, 1)$ such that $B_{P_f}(x, \delta, t) \subseteq B_P(x, \varepsilon, t)$ for all $x \in X$. This follows that $x_n \in B_P(x, \varepsilon, t)$ since $\lim_{n\to\infty} P_f(x_n, x, t) = P_f(x, x, t)$. Therefore, we have $\lim_{n,m\to\infty} P(x_n, x_m, t) = \lim_{n\to\infty} P(x_n, x, t) = P(x, x, t) = a_0$ which means that $(X, P, *)$ is complete. □

5 Contraction-Preserving Functions

In this section, we discussed whether there are necessary conditions or not that the $*$-pfmp functions keep the contractivity of self-mappings.

We first recall the definitions of contraction mapping, given by [1], defined on partial fuzzy metric spaces.

Definition 5.1. [1] Let $(X, P, *)$ be a SE-partial fuzzy metric space. The mapping $T : X \to X$ is called

(i) a KM-contraction if there exists $k \in (0, 1)$ such that

$$P(T(x), T(y), kt) \geq P(x, y, t)$$

for all $x, y \in X$ and $t > 0$.

(ii) a GS-contraction if there exists $k \in (0, 1)$ such that

$$P(T(x), T(y), t) \geq \frac{P(x, y, t)}{k + (1 - k)P(x, y, t)}$$

for all $x, y \in X$ and $t > 0$.

Now, we are able to present contraction-preserving mapping by obtaining some characterizations of them.

Definition 5.2. Let $f : [0, 1] \to [0, 1]$ be a $*$-pfmp function. Then we say that f is (KM/GS)-contraction-preserving if for each partial fuzzy metric space $(X, P, *)$, every (KM/GS)-P-contraction is a (KM/GS)-P_f-contraction.

Remark 5.3. Each $*$-pfmp function is KM-contraction-preserving. But as seen in the following example, a $*$-pfmp function may not be a GS-contraction-preserving function.

Example 5.4. Let $X = [0, \infty)$ and define $P : X \times X \times [0, \infty) \to [0, 1]$ by

$$P(x, y, t) = \begin{cases} 0, & t \leq |x - y| \\ 1, & t > |x - y| \end{cases}$$

for all $x, y \in [0, \infty)$. Then $(X, P, *_\wedge)$ is a partial fuzzy metric space. Take a function $T : X \to X$ as $T(x) = x + 1$ for all $x \in X$. It obviously seen that T is a GS-contraction mapping in $(X, P, *_\wedge)$. If we take the $*_\wedge$-pfmp function $f : [0, 1] \to [0, 1]$ as $f(x) = \frac{x}{2}$, then we obtain

$$P_f(x, y, t) = \begin{cases} 0, & t \leq |x - y| \\ \frac{1}{2}, & t > |x - y| \end{cases}$$

for all $x, y \in [0, \infty)$. Then, we have

$$P_f(T(1), T(2), 3) = P_f(2, 3, 3) = \frac{1}{2} < \frac{1}{k + 1} = \frac{P_f(1, 2, 3)}{k + (1 - k)P_f(1, 2, 3)}$$

for all $k \in (0, 1)$, which means that T is not a GS-contraction mapping in $(X, P_f, *_\wedge)$.

Theorem 5.5. Let f be a $*$-pfmp function. Then the followings are equivalent:

(i) f is GS-contraction-preserving.

(ii) For each $k \in (0, 1)$ there exists $c \in (0, 1)$ such that $\frac{f(x)}{c+(1-c)f(x)} \leq f(\frac{x}{k+(1-k)x})$ for all $x \in [0, 1]$.

Proof . $(i) \Rightarrow (ii)$: Let f be a GS-contraction-preserving function. Suppose that there exists k_0 such that for each $c \in (0, 1)$, $x_c \in [0, 1]$ can be found such that $\frac{f(x_c)}{c+(1-c)f(x_c)} > f(\frac{x_c}{\frac{k_0}{2}+(1-\frac{k_0}{2})x_c})$. Now, we show that f is not GS-contraction-preserving. Let us consider the partial fuzzy metric space $((0, 1], P, *_p)$ where

$$P(x, y, t) = \begin{cases} 0, & t = 0 \\ 1 - \frac{1}{2}\max(x, y), & t > 0 \end{cases}$$

for all $x, y \in (0, 1]$. And define the mapping $T : [0, 1] \to [0, 1]$ by $T(x) = k_0 x$ for all $x \in [0, 1]$. Then we can find k (with choosing $k = k_0$) such that

$$\begin{aligned} P(T(x), T(y), t) &= P(k_0 x, k_0 y, t) = 1 - \frac{k_0}{2}\max(x, y) \geq \frac{2 - \max(x, y)}{2 - \max(x, y) + k\max(x, y)} \\ &= \frac{P(x, y, t)}{k + (1 - k)P(x, y, t)} \end{aligned}$$

for all $x, y \in X$ and $t > 0$. Hence, T is a GS-P-contraction mapping. However, for all $t > 0$, we obtain

$$\begin{aligned} P_f(T(x_c), T(1), t) &= f(P(T(x_c), T(1), t)) = f(k_0x_c), k_0, t)) = f(1 - \frac{k_0}{2}) \leq f(\frac{2}{2+k_0}) \\ &= f(\frac{1}{1+\frac{k_0}{2}}) = f(\frac{\frac{1}{2}}{\frac{1}{2}+\frac{k_0}{2}}) = f(\frac{P(x_c, 1, t)}{\frac{k_0}{2} + (1-\frac{k_0}{2})P(x_c, 1, t)}) \\ &< \frac{f(P(x_c, 1, t))}{c + (1-c)f(P(x_c, 1, t))} = \frac{P_f(x_c, 1, t)}{c + (1-c)P_f(x_c, 1, t)} \end{aligned}$$

which contradicts to the fact that f is GS-contraction-preserving.

$(ii) \Rightarrow (i)$: Let $(X, P, *)$ be a partial fuzzy metric space and $T : X \to X$ be a GS-P-contraction. Then there is $k_0 \in (0, 1)$ such that

$$P(T(x), T(y), t) \geq \frac{P(x, y, t)}{k_0 + (1-k_0)P(x, y, t)}$$

for all $x, y \in X$ and $t > 0$. For a given $k_0 \in (0, 1)$, there exists $c \in (0, 1)$ such that $\frac{f(x)}{c+(1-c)f(x)} \leq f(\frac{x}{k+(1-k)x})$ for all $x \in [0, 1]$. Since f is isotone, it follows that

$$\begin{aligned} P_f(T(x), T(y), t) &= f(P(T(x), T(y), t)) \geq f(\frac{P(x, y, t)}{k_0 + (1-k_0)P(x, y, t)}) \\ &\geq \frac{f(P(x, y, t))}{c + (1-c)f(P(x, y, t))} = \frac{P_f(x, y, t)}{c + (1-c)P_f(x, y, t)} \end{aligned}$$

for all $x, y \in X$ and $t > 0$. Therefore, T is a GS-P_f-contraction mapping and so f is a GS-contraction-preserving function. □

Corollary 5.6. If $f : [0, 1] \to [0, 1]$ is $*$-pfmp function which is GS-contraction-preserving, then f is a $*$-fmp function.

Proof . The proof is clear since it is completed by the consideration of Corollary 3.18 and Theorem 5.5. □

In Definition 5.2, if we take $*$-fmp functions and fuzzy metric spaces instead of $*$-pfmp functions and partial fuzzy metric spaces, respectively, then we obtain GS/KM-contraction-preserving for fuzzy metric spaces. This types of functions will be called fuzzy metric GS/KM-contraction-preserving. The following theorem gives us a relation between GS/KM-contraction-preserving and fuzzy metric GS/KM-contraction-preserving similar to the classical case.

Theorem 5.7. Let $f : [0, 1] \to [0, 1]$ be a $*$-pfmp function. Then the followings are equivalent:

(i) f is GS-contraction-preserving.

(ii) f is fuzzy metric GS-contraction-preserving.

Proof . $(i) \Rightarrow (ii)$: Let f be a GS-contraction-preserving function. Then by Corollary 5.6, we obtain that $(X, M_f = f \circ M, *)$ is a fuzzy metric space when $(X, M, *)$ is a fuzzy metric space. By Theorem 5.5, we have that for each $k \in (0, 1)$ there exists $c \in (0, 1)$ such that $\frac{f(x)}{c+(1-c)f(x)} \leq f(\frac{x}{k+(1-k)x})$ for all $x \in [0, 1]$. Take a fuzzy metric GS-contraction $T : X \to X$ with contractive constant k_0. Then there exists $c \in (0, 1)$ such that

$$\begin{aligned} M_f(T(x), T(y), t) &= f(M(T(x), T(y), t)) \geq f(\frac{M(x, y, t)}{c + (1-c)M(x, y, t)}) \\ &\geq \frac{f(M(x, y, t))}{k_0 + (1-k_0)f(M(x, y, t))} = \frac{M_f(x, y, t)}{k_0 + (1-k_0)M_f(x, y, t)} \end{aligned}$$

for all $x, y \in X$ and $t > 0$. Hence, we have that T is a GS-contraction mapping in $(X, M_f, *)$ and this follows that f is fuzzy metric GS-contraction-preserving.

$(ii) \Rightarrow (i)$: Let us suppose that f is not a GS-contraction-preserving function. By Theorem 5.5, there exists $k_0 \in (0, 1)$ such that for each $c \in (0, 1)$ we can find $x_c \in [0, 1]$ satisfying $\frac{f(x_c)}{c+(1-c)f(x_c)} > f(\frac{x_c}{k_0+(1-k_0)x_c})$. Consider the partial fuzzy metric space $([0, \infty), P, *_p)$ where

$$P(x, y, t) = \begin{cases} 0, & t = 0 \\ \frac{t}{t+\max(x,y)}, & t > 0 \end{cases}$$

for all $x, y \in [0, \infty)$ and define the mapping $T : [0, \infty) \to [0, \infty)$ by $T(x) = k_0 x$ for all $x \in [0, \infty)$. Then it is easily seen that T is a GS-contraction mapping. Define the mapping $M_P : [0, \infty) \times [0, \infty) \times [0, \infty) \to [0, 1]$ by

$$M_P(x, y, t) = \begin{cases} 1, & x = y \\ P(x, y, t), & x \neq y \end{cases}$$

for all $x, y \in [0, \infty)$. It is shown that $(X, M_P, *_p)$ is a fuzzy metric space in [3]. Also, it is easily seen that T is a GS-contraction mapping in $([0, \infty), M_P, *)$. Since f is a fuzzy metric GS-contraction-preserving function, we have that T is GS-contraction in $([0, \infty), (M_P)_f, *_p)$. Hence, there exists $k_1 \in (0, 1)$ such that

$$(M_P)_f(T(x), T(y), t) \geq \frac{(M_P)_f(x, y, t)}{k_1 + (1 - k_1)(M_P)_f(x, y, t)}$$

for all $x, y \in [0, \infty)$ and $t > 0$. This means that

$$P_f(T(x), T(y), t) \geq \frac{P_f(x, y, t)}{k_1 + (1 - k_1)P_f(x, y, t)}$$

for all $x, y \in [0, \infty)$ with $x \neq y$ and $t > 0$. From the above inequality, for $y = 1$ and $t = 0.5$ we have

$$P_f(T(x), T(1), 0.5) = P_f(k_0 x, k_0, 0.5) \geq \frac{P_f(x, 1, 0.5)}{k_1 + (1 - k_1)P_f(x, 1, 0.5)}$$

for all $x \in [0, \infty)$. It follows from the hypothesis that

$$\frac{f(\frac{0.5}{0.5+1})}{k_1 + (1 - k_1)f(\frac{0.5}{0.5+1})} > f(\frac{0.5}{0.5 + k_0}) \geq \frac{f(\frac{0.5}{0.5+1})}{k_1 + (1 - k_1)f(\frac{0.5}{0.5+1})}$$

which is a contradiction. Therefore, f is a GS-contraction-preserving function. □

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