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Bifurcation analysis and chaos in a discretized prey-predator system with Holling type III

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Abstract

In this paper, we investigate a discrete-time prey-predator model. The model is formulated by using the piecewise constant argument method for differential equations and taking into account Holling type III. The existence and local behavior of equilibria are studied. We established that the system experienced both Neimark-Sacker and period-doubling bifurcations analytically by using bifurcation theory and the center manifold theorem. In order to control chaos and bifurcations, the state feedback method is implemented. Numerical simulations are also provided for the theoretical discussion.

Keywords: Difference equations, Asymptotic stability, Bifurcation analysis, Holling's type III, Chaos control
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1 Introduction

Predator-prey models offer a broad variety of ecological and biological applications [14]. Despite the fact that numerous essential features of continuous-time nonlinear prey-predator models have been explored, discrete-time Predator-prey systems remain relatively poorly understood.

In 1970, May [10] realized that a simple-discrete model may exhibit complicated dynamics, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random fluctuations. Discrete-time models may offer more complex and rich dynamics (such as, bifurcation and chaos) than their continuous counterparts represented by differential equations [3, 7]. Discritization of continuous systems is an important way to attain discrete models. To achieve this purpose, many schemes are applied. The most popular methods to obtain the discrete-time counterparts of continuous-time models are the forward Euler scheme and the piecewise constant arguments for differential equations. Singh et al. [17] discussed period-doubling and Neimark-Sacker bifurcations of a discrete-time prey-predator model derived following the forward Euler's scheme and by choosing step size in the discretization method as the bifurcation parameter. It is observed that both types of bifurcations occur for larger values of step size used in Euler's scheme, and this fact violates the accuracy of the numerical method for discretization. In order to remove this deficiency, Din gives an alternate discretization exponential model, formulated by the method of piecewise constant arguments [6]. The analysis of local dynamics seems to be more challenging compared to the Euler scheme developed in [17]. Further, different chaos control methods are implemented to avoid chaotic orbits.

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In 1995, Hsu and Huang [8] studied the following prey-predator system of the Leslie type with sigmoidal functional response:

$$\frac{dx}{dt} = rx(1-\frac{x}{K}) - \frac{mx^2}{(A+x)(B+x)}y, \tag{1.1}$$

$$\frac{dy}{dt} = y\Big(s(1-\frac{hy}{x})\Big), \tag{1.2}$$

where

$$x(0) > 0,\ y(0) > 0.$$

In the system (1.1)-(1.2), x and y denote the numbers of prey and predator, respectively, and all parameters are nonnegative. It is assumed that the prey grows logistically with carrying capacity K and an intrinsic growth rate r in the absence of predation. The prey is consumed by the predator in accordance with the Holling type-III functional response $\frac{mx^2}{(A+x)(B+x)}y$, and the predator grows logistically with an intrinsic rate s and carrying capacity that are proportionate to the population size of the prey. The model's parameter m stands for the maximum specific rate of product formation. Some results on global stability are established for a non-dimensional form of the system (1.1)-(1.2).

Motivated by the above discussions, we propose the following discrete-time prey-predator model:

$$x_{n+1} = x_n \exp\left(r(1-\frac{x_n}{K}) - \frac{mx_n}{(A+x_n)(B+x_n)}y_n\right), \tag{1.3}$$

$$y_{n+1} = y_n \exp\left(s(1-\frac{hy_n}{x_n})\right). \tag{1.4}$$

It is our aim to analyze the asymptotic stability of the system's equilibria (1.3)-(1.4). We will use bifurcation theory and the center manifold theorem with numerical simulation to study the Neimark-Sacker bifurcation and period-doubling bifurcation in the system. In addition, we implement the state feedback control (SFC) method to achieve the stability of unstable orbits. Some recent works on the stability and chaos in a discrete-time dynamical system can be found, among many others, in [1]-[2].

The remainder of this manuscript is organized as follows: The existence and local dynamics of the equilibria are developed in Section 2. The bifurcation analysis is carried out in Sections 3 and 4. In Section 5, we employ the SFC method to stabilize chaotic orbits. Numerical simulations are developed in Section 6. Finally, Section 7 draws the conclusion to this paper.

2 Local dynamics

The results about the existence of the equilibria of the system (1.3)-(1.4) in $\mathbb{R}^2_+$, are summarized as follows:

Lemma 2.1. The system (1.3)-(1.4) has a boundary equilibrium and a unique interior equilibrium in $\mathbb{R}^2_+$.

1. For all positive parametric values, the system has one boundary equilibrium $B(K, 0)$.
2. If $s^2 + 4\frac{p^3}{27} > 0$ then, the system (1.3)-(1.4) has a unique interior equilibrium

$$C(x^*, y^*) = \left(\sqrt[3]{\frac{-s+\sqrt{s^2+4\frac{p^3}{27}}}{2}} + \sqrt[3]{\frac{-s-\sqrt{s^2+4\frac{p^3}{27}}}{2}}, \frac{x^*}{h}\right).$$

Proof . The equilibria of the system (1.3)-(1.4), satisfy the two isocline equations

$$r(1-\frac{x}{K}) - \frac{mx^2}{h(A+x)(B+x)} = 0, \tag{2.1}$$

$$x = hy. \tag{2.2}$$

Eq.(2.1) implies

$$x^3 + ax^2 + bx + c = 0, \tag{2.3}$$

where

$$a = \frac{h(r(A+B) - Kr)}{hr}, \quad b = \frac{h(rAB - rAK - rBK))}{hr}, \quad c = -KAB.$$

By using the transformation $x_1 = x + \frac{a}{3}$, the equation (2.3) reduces to

$$x_1^3 + px_1 + s = 0, \tag{2.4}$$

where

$$p = b - \frac{a^2}{3} \text{and } \; s = c - \frac{ab}{3} + \frac{2a^3}{27}.$$

Let $x_1 = v + w$, then (2.4) becomes

$$v^3 + w^3 + (v+w)(3vw + p) + s = 0, \tag{2.5}$$

which is equivalent to

$$v^3 + w^3 = -s, \quad 3vw = -p. \tag{2.6}$$

Let again $V = v^3$ and $W = w^3$, yields

$$V + W = -s, \quad VW = -\frac{p^3}{27}, \tag{2.7}$$

thus, V and W are the roots of the second polynomial degree below

$$Z^2 + sZ - \frac{p^3}{27} = 0. \tag{2.8}$$

The discriminant of (2.8) is $\Delta = s^2 + 4\frac{p^3}{27}$, Now if $s^2 + 4\frac{p^3}{27} > 0$, the the roots of (2.8) are

$$V = v^3 = \frac{-s + \sqrt{s^2 + 4\frac{p^3}{27}}}{2} \; and \; W = w^3 = \frac{-s - \sqrt{s^2 + 4\frac{p^3}{27}}}{2}. \tag{2.9}$$

Since $x = v + w$,

$$x = \sqrt[3]{\frac{-s + \sqrt{s^2 + 4\frac{p^3}{27}}}{2}} + \sqrt[3]{\frac{-s - \sqrt{s^2 + 4\frac{p^3}{27}}}{2}}, \tag{2.10}$$

Thus, the unique interior equilibrium is

$$C(x^*, y^*) = \left(\sqrt[3]{\frac{-s + \sqrt{s^2 + 4\frac{p^3}{27}}}{2}} + \sqrt[3]{\frac{-s - \sqrt{s^2 + 4\frac{p^3}{27}}}{2}}, \frac{x^*}{h} \right).$$

Now, for $y = 0$, the system (1.3)-(1.4) has a boundary equilibrium noted $D(K, 0)$. The Jacobian matrix of the system (1.3)-(1.4) at any equilibrium (x, y) is given by

$$J(x,y) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}, \tag{2.11}$$

where

$$\begin{aligned}
j_{11} &= \exp\left(r(1 - \frac{x}{K}) - \frac{mxy}{(A+x)(B+x)} \right) \left(1 - x\left(\frac{r}{K} + \frac{my(AB - x^2)}{(A+x)^2(B+x)^2} \right) \right), \\
j_{12} &= -\frac{x^2 m}{(A+x)(B+x)} \exp\left(r(1 - \frac{x}{K}) - \frac{mxy}{(A+x)(B+x)} \right), \\
j_{21} &= \frac{hsy^2}{x^2} \exp\left(s(1 - \frac{hy}{x}) \right), \\
j_{22} &= \exp\left(s(1 - h\frac{y}{x}) \right) \left(1 - \frac{ysh}{x} \right).
\end{aligned}$$

For the equilibrium $B(K,0)$, we have

$$J(B) = \begin{pmatrix} 1 & -\frac{K^2 m}{(A+K)(B+K)} \\ 0 & \exp(s) \end{pmatrix}. \quad (2.12)$$

The equilibrium $B(K,0)$ is non-hyperbolic since $\lambda_1 = 1$. For the equilibrium $C(x^*, y^*)$, we have

$$J(C) = \begin{pmatrix} 1 - x^*\left(\frac{r}{K} + \frac{my(AB-x^{*2})}{(A+x^*)^2(B+x^*)^2}\right) & -\frac{x^{*2}m}{(A+x^*)(B+x^*)} \\ \frac{s}{h} & 1-s \end{pmatrix}. \quad (2.13)$$

The characteristic equation associated to (2.13) is

$$\omega^2 - trJ(C)\omega + \det J(C) = 0. \quad (2.14)$$

where

$$T := trJ(C) = 2 - x^*\left(\frac{r}{K} + \frac{my(AB-x^{*2})}{(A+x^*)^2(B+x^*)^2}\right) - s, \quad (2.15)$$

and

$$D := \det J(C) = \left(1 - x^*\left(\frac{r}{K} + \frac{my(AB-x^{*2})}{(A+x^*)^2(B+x^*)^2}\right)\right)\left(1-s\right) + \frac{x^{*2}sm}{h(A+x^*)(B+x^*)}. \quad (2.16)$$

□

The following lemma describes the various conditions associated with the local stability analysis of feasible equilibria.

Lemma 2.2. [3] Let $\psi(\omega) = \omega^2 - T\omega + D$. Suppose that $\psi(1) > 0$, ω_1, ω_2 are two roots of $\psi(\omega) = 0$. Then

- $|\omega_1| < 1$ and $|\omega_2| < 1$ if and only if $\psi(-1) > 0$ and $D < 1$;
- ($|\omega_1| > 1$ and $|\omega_2| < 1$) or ($|\omega_1| < 1$ and $|\omega_2| > 1$) if and only if $\psi(-1) < 0$;
- $|\omega_1| > 1$ and $|\omega_2| > 1$ if and only if $\psi(-1) > 0$ and $D > 1$;
- $\omega_1 = -1$ and $|\omega_2| \neq 1$ if and only if $\psi(-1) = 0$ and $D \neq 1$;
- ρ_1 and ρ_2 are complex and $|\omega_1| = 1$ and $|\omega_2| = 1$ if and only if $T^2 - 4D < 0$ and $D = 1$.

Let ω_1 and ω_2 be two roots of (2.14), which called eigenvalues of the equilibrium (x,y). The following typological classifications are considered:

1. (x,y) is locally asymptotically stable if $|\omega_1| < 1$ and $|\omega_2| < 1$.
2. (x,y) is called a source if $|\omega_1| > 1$ and $|\omega_2| > 1$. A source is locally unstable.
3. (x,y) is called a saddle if $|\omega_1| < 1$ and $|\omega_2| > 1$ or ($|\omega_1| > 1$ and $|\omega_2| < 1$).
4. (x,y) is called non-hyperbolic if either $|\omega_1| = 1$ or $|\omega_2| = 1$.

We set

$$\mathcal{A} = \frac{2h\left(x^*\left(\frac{r}{K} + \frac{my(AB-x^{*2})}{(A+x^*)^2(B+x^*)^2}\right) - 2\right)}{h\left(x^*\left(c + \frac{my(AB-x^{*2})}{(A+x^*)^2(B+x^*)^2}\right) - 2\right) + \frac{x^{*2}m}{h(A+x^*)(B+x^*)}},$$

and

$$\mathcal{B} = \frac{hx^*\left(\frac{r}{K} + \frac{my(AB-x^{*2})}{(A+x^*)^2(B+x^*)^2}\right)}{h\left(x^*\left(\frac{r}{K} + \frac{my(AB-x^{*2})}{(A+x^*)^2(B+x^*)^2}\right) - 1\right) + \frac{x^{*2}m}{h(A+x^*)(B+x^*)}}.$$

The following proposition summarizes the local dynamics of the interior equilibrium C:

Proposition 2.3. Let $x^*\left(\frac{r}{K}+\frac{my(AB-x^{*2})}{(A+x^*)^2(B+x^*)^2}\right)>2$.

- The interior equilibrium $C(x^*,y^*)$ is locally asymptotically stable if
$$\mathcal{A}<s<\mathcal{B}. \tag{2.17}$$
- The interior equilibrium $C(x^*,y^*)$ is source if
$$s<\mathcal{A}. \tag{2.18}$$
- The interior equilibrium $C(x^*,y^*)$ is saddle if
$$s>\max\left\{\mathcal{A},\mathcal{B}\right\}. \tag{2.19}$$
- The interior equilibrium $C(x^*,y^*)$ is non-hyperbolic if
$$s=\mathcal{A}, \tag{2.20}$$
  or
$$s=\mathcal{B}. \tag{2.21}$$

If the non-hyperbolic condition (2.20) in (2.3) holds, then one of the eigenvalues of $C(x^*,y^*)$ is -1 and the other is neither 1 nor -1. Thus (2.20) can be written as

$$P_d=\left\{(r,K,m,A,B,h,s)>0,\ T>0,\ \ s=\mathcal{A}\right\}. \tag{2.22}$$

If the non-hyperbolic condition (2.21) in (2.3) holds, then the eigenvalues of $C(x^*,y^*)$ are a pair of complex conjugate numbers with modulus 1. Thus (2.21) can be written as

$$N_s=\left\{(r,K,m,A,B,h,s)>0,\ |\ T\ |<2,\ \ s=\mathcal{B}\right\}. \tag{2.23}$$

3 Period-doubling bifurcation

For the fixed point (x^*,y^*) associated to the system (1.3)-(1.4). The set (2.22) can be written as

$$P_d=\left\{s=\widehat{s},\quad T>0,\quad (r,\delta,m,K,s,A)>\ 0\right\}. \tag{3.1}$$

Thus, the system (1.3)-(1.4) undergoes a period-doubling bifurcation at the interior equilibrium (x^*,y^*) if s varies in the small neighborhood of $s=\widehat{s}$ and the other model's parameters are kept constant in P_d. Giving a perturbation s^* ($s^*\ll 1$) of the parameter s in the neighborhood of $s=\widehat{s}$ in the system (1.3)-(1.4), we obtain

$$x_{n+1}=x_n\exp\left(r(1-\frac{x_n}{K})-\frac{mx_n}{(A+x_n)(B+x_n)}y_n\right)=f(x_n,y_n,s^*), \tag{3.2a}$$
$$y_{n+1}=y_n\exp\left((s+s^*)(1-\frac{hy_n}{x_n})\right)=g(x_n,y_n,s^*). \tag{3.2b}$$

The system (3.2) can be linearized near the origin as follows:

$$\begin{aligned} v_{n+1} &= \alpha_1v_n+\alpha_2w_n+\alpha_{12}v_nw_n+\alpha_{11}v_n^2+\alpha_{22}w_n^2,\\ w_{n+1} &= \beta_1v_n+\beta_2w_n+\beta_{12}v_nw_n+\beta_{11}v_n^2+\beta_{22}w_n^2+\\ &\quad \beta_{13}s^*v_n+\beta_{23}s^*w_n+\beta_{123}s^*v_nw_n+\beta_{113}s^*v_n^2+\beta_{223}s^*w_n^2. \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
\alpha_1 =& f_x(x^*,y^*,0) = 1 - x^*\left(\frac{r}{K} + \frac{my(AB-x^{*2})}{(A+x^*)^2(B+x^*)^2}\right), \quad \alpha_2 = f_y(x^*,y^*,0) = -\frac{x^{*2}m}{(A+x^*)(B+x^*)},\\
\alpha_{12} =& f_{xy}(x^*,y^*,0) = -\frac{2m}{(A+x^*)(B+x^*)} + \frac{(A+B+2x^*)x^{*2}m}{(A+x^*)^2(B+x^*)^2} + \frac{m^2x^{*3}}{h(A+x^*)^2(B+x^*)^2}\\
&- \frac{m^2x^{*4}(A+B+2x^*)}{h(A+x^*)^3(B+x^*)^3} + \frac{rmx^{*2}}{K(A+x^*)(B+x^*)},\\
\alpha_{11} =& f_{xx}(x^*,y^*,0) = -\frac{mx^*y^*}{x^*(A+x^*)(B+x^*)} + \frac{(A+B+2x^*)mx^*y^*}{(A+x^*)^2(B+x^*)^2} - \frac{r}{K} + \frac{(A+B+2x^*)mx^{*2}y^*}{(A+x^*)^2(B+x^*)^2}\\
&+ \frac{mx^{*2}y^*}{(A+x^*)^2(B+x^*)^2} - \frac{(A+B+2x^*)^2x^{*2}y}{(A+x^*)^3(B+x^*)^3} + \frac{m^2x^*y^{*2}}{2(A+x^*)^2(B+x^*)^2} + \frac{(A+B+2x^*)m^2x^{*2}y^{*2}}{(A+x^*)^3(B+x^*)^3}\\
&+ \frac{m^2x^{*3}y^{*2}(A+B+2x^*)^2}{2(A+x^*)^4(B+x^*)^4} - \frac{rmx^*y^*}{K(A+x^*)(B+x^*)} - \frac{rmx^{*2}y^*(A+B+2x^*)}{K(A+x^*)^2(B+x^*)^2} + \frac{r^2x}{2K^2},\\
\alpha_{22} =& f_{yy}(x^*,y^*,0) = \frac{m^2x^{*3}}{2(A+x^*)^2(B+x^*)^2}, \beta_1 = g_x(x^*,y^*,0) = \frac{s}{h},\ \beta_2 = g_y(x^*,y^*,0) = 1-s,\\
\beta_{12} =& g_{xy}(x^*,y^*,0) = \frac{s}{x^*}(2-\frac{s}{x^*}), \beta_{13} = g_{xs^*}(x^*,y^*,0) = \frac{1}{h}, \beta_{11} = g_{xx}(x^*,y^*,0) = \frac{sx^*}{2h}(-1+\frac{s}{x^{*2}}),\\
\beta_{22} =& g_{yy}(x^*,y^*,0) = \frac{sh}{x^*}(-1+\frac{s}{2}), \beta_{23} = g_{ys^*}(x^*,y^*,0) = -1, \beta_{123} = g_{xys^*}(x^*,y^*,0) = \frac{2}{x^*}(1-\frac{s}{x^*}),\\
\beta_{113} =& g_{xxs^*}(x^*,y^*,0) = \frac{-x^*}{2h} + \frac{s}{hx^*}, \beta_{223} = g_{yys^*}(x^*,y^*,0) = \frac{h}{x^*}(-1+\frac{s}{2}) + \frac{sh}{2x^*}.
\end{aligned}$$

Let us define the invertible matrix $T_1 = \begin{pmatrix} \alpha_2 & \alpha_2 \\ -1-\alpha_1 & \omega_2-\alpha_1 \end{pmatrix}$, with the transformation $\begin{pmatrix} v_n \\ w_n \end{pmatrix} = T_1 \begin{pmatrix} X_n \\ Y_n \end{pmatrix}$. Writing $v_n = \alpha_2(X_n+Y_n)$, $w_n = -(1+\alpha_1)X_n + (\omega_2-\alpha)Y_n$. Thus, the system (3.3) becomes

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \omega_2 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} F_1(X_n,Y_n,s^*) \\ G_1(X_n,Y_n,s^*) \end{pmatrix}, \tag{3.4}$$

where

$$\begin{aligned}
F_1(X_n,Y_n,s^*) =& \frac{\omega_2-\alpha_1}{\alpha_2(1+\omega_2)}\Bigg(\Big(-\alpha_{12}\alpha_2(1+\alpha_1) + \alpha_{11}\alpha_2^2 + \alpha_{22}(1+\alpha_1)^2\Big)X_n^2\\
&+ \Big(\alpha_{12}\alpha_2(\omega_2-\alpha_1) - \alpha_{12}\alpha_2(1+\alpha_1) + 2\alpha_{11}\alpha_2^2 - 2\alpha_{22}(1+\alpha_1)(\omega_2-\alpha_1)\Big)X_nY_n\\
&+ \Big(\alpha_{12}\alpha_2(\omega_2-\alpha_1) + \alpha_{11}\alpha_2^2 + \alpha_{22}(\omega_2-\alpha_1)^2\Big)Y_n^2 - \frac{1}{\omega_2+1}\Bigg(\Big(-\beta_{12}\alpha_2(1+\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(1+\alpha_1)^2\Big)X_n^2\\
&+ \Big(\beta_{113}\alpha_2^2 + \beta_{223}(1+\alpha_1)^2 - \beta_{123}\alpha_2(1+\alpha_1)\Big)X_n^2s^*\\
&+ \Big(\beta_{12}\alpha_2(\omega_2-\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(\omega_2-\alpha_1)^2\Big)Y_t^2 + \Big(\beta_{223}\alpha_2^2 + \beta_{223}(\omega_2-\alpha_1)^2 + \beta_{123}\alpha_2(\omega_2-\alpha_1)\Big)Y_n^2s^*\\
&+ \Big(\beta_{12}\alpha_2(\omega_2-\alpha_1) - \beta_{12}\alpha_2(1+\alpha_1) + 2\beta_{11}\alpha_2^2 - 2\beta_{22}(1+\alpha_1)(\omega_2-\alpha_1)\Big)X_nY_n\\
&+ \Big(2\beta_{113}\alpha_2^2 + 2\beta_{223}(1+\alpha_1)(\omega_2-\alpha_1) + \beta_{123}\alpha_2(\omega_2-\alpha_1) - \beta_{123}\alpha_2(1+\alpha_1)\Big)X_nY_ns^*\\
&+ \Big(\beta_{13}\alpha_2 - \beta_{23}(1+\alpha_1)\Big)X_ns^* + \Big(\beta_{13}\alpha_2 + \beta_{23}(\omega_2-\alpha_1)\Big)Y_ns^*\Bigg)\Bigg).
\end{aligned}$$

and

$$
\begin{aligned}
G_1(X_n,Y_n,s^*) &= \frac{1+\alpha_1}{\alpha_2(1+\omega_2)}\Bigg(\Bigg(-\alpha_{12}\alpha_2(1+\alpha_1)+\alpha_{11}\alpha_2^2+\alpha_{22}(1+\alpha_1)^2\Bigg)X_n^2\\
&+\Bigg(\alpha_{12}\alpha_2(\omega_2-\alpha_1)-\alpha_{12}\alpha_2(1+\alpha_1)+2\alpha_{11}\alpha_2^2-2\alpha_{22}(1+\alpha_1)(\omega_2-\alpha_1)\Bigg)X_nY_n\\
&+\Bigg(\alpha_{12}\alpha_2(\omega_2-\alpha_1)+\alpha_{11}\alpha_2^2+\alpha_{22}(\omega_2-\alpha_1)^2\Bigg)Y_n^2+\frac{1}{\omega_2+1}\Bigg(\Bigg(-\beta_{12}\alpha_2(1+\alpha_1)+\beta_{11}\alpha_2^2+\beta_{22}(1+\alpha_1)^2\Bigg)X_n^2\\
&+\Bigg(\beta_{113}\alpha_2^2+\beta_{223}(1+\alpha_1)^2-\beta_{123}\alpha_2(1+\alpha_1)\Bigg)X_n^2s^*\\
&+\Bigg(\beta_{12}\alpha_2(\omega_2-\alpha_1)+\beta_{11}\alpha_2^2+\beta_{22}(\omega_2-\alpha_1)^2\Bigg)Y_t^2+\Bigg(\beta_{223}\alpha_2^2+\beta_{223}(\omega_2-\alpha_1)^2+\beta_{123}\alpha_2(\omega_2-\alpha_1)\Bigg)Y_n^2s^*\\
&+\Bigg(\beta_{12}\alpha_2(\omega_2-\alpha_1)-\beta_{12}\alpha_2(1+\alpha_1)+2\beta_{11}\alpha_2^2-2\beta_{22}(1+\alpha_1)(\omega_2-\alpha_1)\Bigg)X_nY_n\\
&+\Bigg(2\beta_{113}\alpha_2^2+2\beta_{223}(1+\alpha_1)(\omega_2-\alpha_1)+\beta_{123}\alpha_2(\omega_2-\alpha_1)-\beta_{123}\alpha_2(1+\alpha_1)\Bigg)X_nY_ns^*\\
&+\Bigg(\beta_{13}\alpha_2-\beta_{23}(1+\alpha_1)\Bigg)X_ns^*+\Bigg(\beta_{13}\alpha_2+\beta_{23}(\omega_2-\alpha_1)\Bigg)Y_ns^*\Bigg)\Bigg).
\end{aligned}
$$

Hereafter, we determine the center manifold $\mathcal{W}_c(0,0)$ of (3.4) about $(0,0)$ in a small neighborhood of s^*. By center manifold theorem [3], there exists a center manifold $\mathcal{W}_c(0,0)$ that can be represented as follows:

$$\mathcal{W}_c(0,0,0)\{(X_n,Y_n,s^*)\in\mathbb{R}_+^3 : Y_n=h(X_n,s^*)=a_1X_n^2+a_2X_ns^*+a_3s^{*2}+O((\mid X_n\mid+\mid s^*\mid)^2)\}, \tag{3.5}$$

where $O((\mid X_n\mid+\mid s^*\mid)^2)$ is a function with order at least three in their variables (X_n,s^*), Moreover, the center manifold must satisfy

$$h\Bigg(-X_n+F_1(X_n,h(X_n,s^*)),s^*),s^*\Bigg)-\omega_2h(X_n,s^*)-G_1(X_n,h(X_n,s^*),s^*)=0. \tag{3.6}$$

By equating (3.6), we obtain

$$
\begin{aligned}
a_1 &= \frac{1+\alpha_1}{\alpha_2(1-\omega_2^2)}\Bigg(-\alpha_{12}\alpha_2(1+\alpha_1)+\alpha_{11}\alpha_2^2+\alpha_{22}(1+\alpha_1)^2\Bigg)+\frac{1}{1-\omega_2^2}\Bigg(-\beta_{12}\alpha_2(1+\alpha_1)+\beta_{11}\alpha_2^2+\beta_{22}(1+\alpha_1)^2\Bigg)\Bigg),\\
a_2 &= \frac{-1}{(1+\omega_2)^2}\Bigg(\beta_{13}\alpha_2-\beta_{23}(1+\alpha_1)\Bigg),\\
a_3 &= 0.
\end{aligned}
$$

Therefore, we consider the map which is the map (3.4) restricted to the center manifold $\mathcal{W}_c(0,0)$

$$f=X_{n+1}=-X_n+c_1X_ns^*+c_2X_n^2+c_3X_n^2s^*+c_4X_n^3, \tag{3.7}$$

where

$$c_1=-\frac{1}{1+\omega_2}\Bigg(\beta_{13}\alpha_2-\beta_{23}(1+\alpha_1)\Bigg),$$

$$c_2=\frac{\omega_2-\alpha_1}{\alpha_2(1+\omega_2)}\Bigg(-\alpha_{12}\alpha_2(1+\alpha_1)+\alpha_{11}\alpha_2^2+\alpha_{22}(1+\alpha_1)^2\Bigg)-\frac{1}{\omega_2+1}\Bigg(-\beta_{12}\alpha_2(1+\alpha_1)+\beta_{11}\alpha_2^2+\beta_{22}(1+\alpha_1)^2\Bigg),$$

$$
\begin{aligned}
c_3 =&\frac{-1}{1+\omega_2}\Bigg(\beta_{13}\alpha_2-\beta_{23}(1+\alpha_1)\Bigg)\Bigg[\frac{\omega_2-\alpha_1}{\alpha_2(1+\omega_2)}\Bigg(\alpha_{12}\alpha_2(\omega_2-\alpha_1)-\alpha_{12}\alpha_2(1+\alpha_1)+2\alpha_{11}\alpha_2^2-2\alpha_{22}(1+\alpha_1)(\omega_2-\alpha_1)\Bigg)\\
&-\frac{1}{1+\omega_2)}\Bigg(\beta_{12}\alpha_2(\omega_2-\alpha_1)-\beta_{12}\alpha_2(1+\alpha_1)+2\beta_{11}\alpha_2^2-2\beta_{22}(1+\alpha_1)(\omega_2-\alpha_1)\Bigg)\Bigg]\\
&-\frac{1}{1+\omega_2}\Bigg(\beta_{113}\alpha_2^2+\beta_{223}(1+\alpha_1)^2-\beta_{123}\alpha_2(1+\alpha_1)\Bigg)-\Bigg(\frac{1+\alpha_1}{\alpha_2(1-\omega_2^2)(1+\omega_2)}\Bigg(-\alpha_{12}\alpha_2(1+\alpha_1)\\
&+\alpha_{11}\alpha_2^2+\alpha_{22}(1+\alpha_1)^2\Bigg)+\frac{1}{(1-\omega_2^2)(1+\omega_2)}\Bigg(-\beta_{12}\alpha_2(1+\alpha_1)+\beta_{11}\alpha_2^2+\beta_{22}(1+\alpha_1)^2\Bigg)\Bigg)\Bigg(\beta_{13}\alpha_2+\beta_{23}(\omega_2-\alpha_1)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
c_4 =&\Big(\frac{1+\alpha_1}{\alpha_2(1-\omega_2^2)}\Big(-\alpha_{12}\alpha_2(1+\alpha_1)+\alpha_{11}\alpha_2^2+\alpha_{22}(1+\alpha_1)^2\Big)+\frac{1}{(1-\omega_2^2)}\Big(-\beta_{12}\alpha_2(1+\alpha_1)+\beta_{11}\alpha_2^2+\beta_{22}(1+\alpha_1)^2\Big)\Big)\\
&\Big[\frac{\omega_2-\alpha_1}{\alpha_2(1+\omega_2)}\Big(\alpha_{12}\alpha_2(\omega_2-\alpha_1)-\alpha_{12}\alpha_2(1+\alpha_1)+2\alpha_{11}\alpha_2^2\Big)\\
&-\frac{1}{1+\omega_2}\Big(\beta_{12}\alpha_2(\omega_2-\alpha_1)-\beta_{12}\alpha_2(1+\alpha_1)+2\beta_{11}\alpha_2^2-2\beta_{22}(1+\alpha_1)(\omega_2-\alpha_1)\Big)\Big].
\end{aligned}
$$

In order for the map (3.7) to undergo a period-doubling bifurcation, we require that the following discriminatory quantities are non zero [3, 9]:

$$\sigma_1 = \left(\frac{\partial^2 f}{\partial X_n \partial s^*} + \frac{1}{2}\frac{\partial f}{\partial s^*}\frac{\partial^2 f}{\partial^2 X_n}\right)|_{(0,0)} \neq 0,$$

$$\sigma_2 = \left(\frac{1}{6}\frac{\partial^3 f}{\partial X_n^3} + (\frac{1}{2}\frac{\partial^2 f}{\partial X_n^2})^2\right)|_{(0,0)} \neq 0.$$

After calculating we get

$$\sigma_1 = c_1,$$

$$\sigma_2 = c_4 + c_2^2.$$

From the above analysis, we have the following theorem.

Theorem 3.1. If $\sigma_2 \neq 0$, and $\sigma_1 \neq 0$ the system (1.3)-(1.4) undergoes period-doubling bifurcation about the interior fixed point $C(x^*, y^*)$ when s^* varies in a small neighborhood of $O(0,0)$. Moreover, if $\sigma_2 > 0$ (resp $\sigma_2 < 0$), then the period 2 points that bifurcate from $C(x^*, y^*)$ are stable (unstable).

4 Neimark-Sacker bifurcation

Neimark-Sacker bifurcation occurs when the roots of (2.14) at $C(x^*, y^*)$ are pair of complex conjugate numbers ω_1, ω_2 given by

$$\omega_{1,2} = \frac{trJ(C) \pm i\sqrt{4\det J(C) - (trJ(C))^2}}{2}, \tag{4.1}$$

with $trJ(C)$ and $\det J(C)$ are given in (2.15)and(2.16) respectively. Let Neimark-Sacker bifurcation occurs for $s = \overline{s}$, we construct then the following set

$$NSB = \Big\{(r, K, m, A, B, h, s) > 0,\ s = \overline{s},\ |\ trJ(C)\ |< 2\Big\}.$$

If we vary s in the neighborhood of $s = \overline{s}$ keeping other parameters in NSB constant, then the interior equilibrium $C(x^*, y^*)$ undergoes Neimark-Sacker bifurcation. Taking a perturbation s^* ($s^* \ll 1$) of the parameter s in the neighborhood of $s = \overline{s}$ in the system (1.3)-(1.4), we have

$$x_{n+1} = x_n \exp\left(r(1-\frac{x_n}{K}) - \frac{m x_n}{(A+x_n)(B+x_n)}y_n\right) = f(x_n, y_n, s^*), \tag{4.2a}$$

$$y_{n+1} = y_n \exp\left((s+s^*)(1-\frac{h y_n}{x_n})\right) = g(x_n, y_n, s^*). \tag{4.2b}$$

Let $v_n = x_n - x^*$, $w_n = y_n - y^*$, Then from (4.2) we obtain

$$v_{n+1} = (v_n + x^*)\exp\left(r(1-\frac{(v_n+x^*)}{K}) - \frac{m(v_n+x^*)}{(A+(v_n+x^*))(B+(v_n+x^*))}(w_n+y^*)\right) - x^*, \tag{4.3a}$$

$$w_{n+1} = (w_n + y^*)\exp\left((s+s^*)(1-\frac{h(w_n+y^*)}{(v_n+x^*)})\right) - y^*. \tag{4.3b}$$

Expanding the above in Taylor series at $(v_n, w_n) = (0,0)$ considering the terms up to second order, we have

$$v_{n+1} = \alpha_1 v_n + \alpha_2 w_n + \alpha_{12} v_n w_n + \alpha_{11} v_n^2 + \alpha_{22} w_n^2 + O\Big((\mid v_n \mid + \mid w_n \mid)^2\Big), \tag{4.4a}$$

$$w_{n+1} = \beta_1 v_n + \beta_2 w_n + \beta_{12} v_n w_n + \beta_{11} v_n^2 + \beta_{22} w_n^2 + O\Big((\mid v_n \mid + \mid w_n \mid)^2\Big), \tag{4.4b}$$

where the expression of α_i, α_{ij}, β_i, β_{ij}, for $i,j = 1,2$ are given in (3.3). The roots of the characteristic equation associated with the linearized map (4.4) at $(v_t, w_t) = (0,0)$ are given by

$$\omega_{1,2}(s*) = \frac{trJ(s^*) \pm i\sqrt{4\det J(s^*) - (tr(s^*))^2}}{2}, \tag{4.5}$$

and

$$|\omega_{1,2}(s^*)| = \sqrt{\det J(s^*)}.$$

when $s^* = 0$, we have

$$\det(J(0)) = 1 \ \ , and \ \ \frac{d|\omega_{1,2}|}{ds^*} \ |_{s^*=0} \neq 0. \tag{4.6}$$

Additionally, we required that when $s^* = 0$, $\omega_{1,2}^m \neq 1$, $m = 1,2,3,4$. This is equivalent to $trJ(0) \neq -2,-1,1,2$. Let $\eta = Re(\omega_{1,2})$, and $\xi = Im(\omega_{1,2})$,. The model (4.4) is written as

$$\begin{pmatrix} v_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} v_n \\ w_n \end{pmatrix} + \begin{pmatrix} \alpha_{12} v_n w_n + \alpha_{11} v_n^2 + \alpha_{22} w_n^2 \\ \beta_{12} v_n w_n + \beta_{11} v_n^2 + \beta_{22} w_n^2 \end{pmatrix}. \tag{4.7}$$

Let consider the invertible matric P associated to the eigenvalue $\omega_{1,2} = \eta \pm i\xi$

$$P = \begin{pmatrix} \alpha_2 & 0 \\ \eta - \alpha_1 & -\xi \end{pmatrix}.$$

Using the following translation

$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = \begin{pmatrix} \alpha_2 & 0 \\ \eta - \alpha_1 & -\xi \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}.$$

The system (4.7) becomes

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \eta & -\xi \\ -\xi & \eta \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} F(X_n, Y_n) \\ G(X_n, Y_n) \end{pmatrix}, \tag{4.8}$$

with

$$F(X_n,Y_n) = \frac{1}{\alpha_2}\Big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2 + \alpha_{22}(\eta-\alpha_1)^2\Big)X_n^2 - \frac{1}{\alpha_2}\Big(\alpha_{12}\alpha_2\xi + 2(\eta-\alpha_1)\alpha_{22}\xi\Big)X_nY_n + \frac{1}{\alpha_2}\alpha_{22}\xi^2 Y_n^2,$$

and

$$G(X_n,Y_n) = \bigg(\frac{\eta-\alpha_1}{\xi\alpha_2}\Big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2 + \alpha_{22}(\eta-\alpha_1)^2\Big) - \frac{1}{\xi}\Big(\beta_{12}\alpha_2(\eta-\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(\eta-\alpha_1)^2\Big)\bigg)X_n^2$$
$$-\bigg(\frac{\eta-\alpha_1}{\xi\alpha_2}\Big(\alpha_{12}\alpha_2\xi + 2(\eta-\alpha_1)\alpha_{22}\xi\Big) - \frac{1}{\xi}\Big(\beta_{12}\alpha_2\xi + 2(\eta-\alpha_1)\beta_{22}\xi\Big)\bigg)X_nY_n + \bigg(\frac{(\eta-\alpha_1)\alpha_{22}\xi^2}{\xi\alpha_2} - \frac{\beta_{22}\xi^2}{\xi}\bigg)Y_n^2.$$

In order for (4.8) to undergo a Neimark Sacker bifurcation, it is mandatory that the following discriminatory quantity, (i.e, $M \neq 0$ [9]),

$$M = -\Re\bigg[\frac{(1-2\overline{\omega})\overline{\omega}^2}{1-\omega}\rho_{11}\rho_{20}\bigg] - \frac{1}{2} \mid \rho_{11} \mid^2 - \mid \rho_{02} \mid^2 + \Re(\overline{\omega}\rho_{21}), \tag{4.9}$$

where

$$
\begin{aligned}
\rho_{02} =& \frac{1}{8}[F_{X_nX_n} - F_{Y_nY_n} - 2G_{X_nY_n} + i(G_{X_nX_n} - G_{Y_nY_n} + 2F_{X_nY_n})]_{(0,0)} \\
=& \frac{1}{4}\Bigg[\Bigg(\frac{1}{\alpha_2}\Big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2 + \alpha_{22}(\eta-\alpha_1)^2\Big) - \frac{1}{\alpha_2}\alpha_{22}\xi^2 + \Bigg(\frac{\eta-\alpha_1}{\xi\alpha_2}\Big(\alpha_{12}\alpha_2\xi + 2(\eta-\alpha_1)\alpha_{22}\xi\Big) \\
&- \frac{1}{\xi}\Big(\beta_{12}\alpha_2\xi + 2(\eta-\alpha_1)\beta_{22}\xi\Big)\Bigg)\Bigg) + i\Bigg(\Bigg(\frac{\eta-\alpha_1}{\xi\alpha_2}\Big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2 + \alpha_{22}(\eta-\alpha_1)^2\Big) \\
&- \frac{1}{\xi}\Big(\beta_{12}\alpha_2(\eta-\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(\eta-\alpha_1)^2\Big)\Bigg) - \Bigg(\frac{(\eta-\alpha_1)\alpha_{22}\xi^2}{\xi\alpha_2} - \frac{\beta_{22}\xi^2}{\xi}\Bigg) - \frac{1}{\alpha_2}\Big(\alpha_{12}\alpha_2\xi + 2(\eta-\alpha_1)\alpha_{22}\xi\Big)\Bigg)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\rho_{11} =& \frac{1}{4}[F_{X_nX_n} + F_{Y_nY_n} + i(G_{X_nX_n} + G_{Y_nY_n})]_{(0,0)} \\
=& \frac{1}{2}\Bigg[\Bigg(\frac{1}{\alpha_2}\Big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2 + \alpha_{22}(\eta-\alpha_1)^2\Big) + \frac{1}{\alpha_2}\alpha_{22}\xi^2\Bigg) \\
&+ i\Bigg(\Bigg(\frac{\eta-\alpha_1}{\xi\alpha_2}\Big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2 + \alpha_{22}(\eta-\alpha_1)^2\Big) \\
&- \frac{1}{\xi}\Big(\beta_{12}\alpha_2(\eta-\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(\eta-\alpha_1)^2\Big)\Bigg) + \Bigg(\frac{(\eta-\alpha_1)\alpha_{22}\xi^2}{\xi\alpha_2} - \frac{\beta_{22}\xi^2}{\xi}\Bigg)\Bigg)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\rho_{20} =& \frac{1}{8}[F_{X_nX_n} - F_{Y_nY_n} + 2G_{X_nY_n} + i(G_{X_nX_n} - G_{Y_nY_n} - 2F_{X_nY_n})]_{(0,0)} \\
=& \frac{1}{4}\Bigg[\Bigg(\frac{1}{\alpha_2}\Big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2 + \alpha_{22}(\eta-\alpha_1)^2\Big) - \frac{1}{\alpha_2}\alpha_{22}\xi^2 - \Bigg(\frac{\eta-\alpha_1}{\xi\alpha_2}\Big(\alpha_{12}\alpha_2\xi + 2(\eta-\alpha_1)\alpha_{22}\xi\Big) \\
&- \frac{1}{\xi}\Big(\beta_{12}\alpha_2\xi + 2(\eta-\alpha_1)\beta_{22}\xi\Big)\Bigg)\Bigg) + i\Bigg(\Bigg(\frac{\eta-\alpha_1}{\xi\alpha_2}\Big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2 + \alpha_{22}(\eta-\alpha_1)^2\Big) \\
&- \frac{1}{\xi}\Big(\beta_{12}\alpha_2(\eta-\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(\eta-\alpha_1)^2\Big)\Bigg) + \Bigg(\frac{\eta-\alpha_1}{\xi\alpha_2}\Big(\alpha_{12}\alpha_2\xi + 2(\eta-\alpha_1)\alpha_{22}\xi\Big) \\
&- \frac{1}{\xi}\Big(\beta_{12}\alpha_2\xi + 2(\eta-\alpha_1)\beta_{22}\xi\Big)\Bigg) + \frac{1}{\alpha_2}\Big(\alpha_{12}\alpha_2\xi + 2(\eta-\alpha_1)\alpha_{22}\xi\Big)\Bigg)\Bigg],
\end{aligned}
$$

$$
\rho_{21} = \frac{1}{16}[F_{X_nX_nX_n} + F_{X_nY_nY_n} + G_{X_nX_nY_n} + G_{Y_nY_nY_n} + i(G_{X_nX_nX_n} + G_{X_nY_nY_n} - F_{X_nX_nY_n} - F_{Y_nY_nY_n})]_{(0,0)} = 0.
$$

Based on the above analysis, we state the following result on Neimark-Sacker bifurcation:

Theorem 4.1. If the condition (4.6) holds and M defined in (4.9) is nonzero, then the system (1.3)-(1.4) experiences Neimark-Sacker bifurcation at the interior fixed point $C(x^*, y^*)$ when s^* varies near the origin and $(r, K, m, A, B, h) \in NSB$. Moreover, if $M < 0$ ($M > 0$) then an attracting (respectively repelling) invariant closed curve bifurcates from the fixed point $C(x^*, y^*)$ for $s > \overline{s}$ (respectively, $s < \overline{s}$).

5 Chaos Control

Controlling chaos attempts to stabilize an unstable orbit in a given system. To do this, we apply small perturbations to the values of certain parameters known as bifurcation parameters. In this paper, we employ The state feedback method [6]. We consider the following controlled system associated to (1.3)-(1.4)

$$
x_{n+1} = x_n \exp\left(r(1 - \frac{x_n}{K}) - \frac{m x_n}{(A + x_n)(B + x_n)} y_n\right), \tag{5.1}
$$

$$
y_{n+1} = y_n \exp\left(\widehat{s}(1 - \frac{h y_n}{x_n})\right) - P_n, \tag{5.2}
$$

$$
P_n = u(x_n - x^*) + v(y_n - y^*), \tag{5.3}
$$

where P_n is the feedback control force with feedback gains u and v. Here, the value of s belongs to some chaotic regions and noted $\widehat{s}$. The Jacobian matrix of (5.1)-(5.2) evaluated at $C(x^*, y^*)$ is

$$J(x,y) = \begin{pmatrix} 1 - x\left(\frac{r}{K} + \frac{my(AB-x^2)}{(A+x)^2(B+x)^2}\right) & -\frac{x^2 m}{(A+x)(B+x)} \\ \frac{\widehat{s}}{h} - u & 1 - \widehat{s} - v \end{pmatrix}. \tag{5.4}$$

The corresponding characteristic equation of (5.4) is

$$\lambda^2 - \left(1 - x\left(\frac{r}{K} + \frac{my(AB-x^2)}{(A+x)^2(B+x)^2}\right) + 1 - \widehat{s} - v\right)\lambda + \tag{5.5}$$

$$\left(1 - x\left(\frac{r}{K} + \frac{my(AB-x^2)}{(A+x)^2(B+x)^2}\right)\right)\left(1 - \widehat{s} - v\right) + \frac{x^2 m}{(A+x)(B+x)}\left(\frac{\widehat{s}}{h} - u\right) = 0.$$

Let λ_1 , λ_2 are the eigenvalues of Eq.(5.4), then the sum and the product of their roots are given by

$$\lambda_1 + \lambda_2 = 1 - x\left(\frac{r}{K} + \frac{my(AB-x^2)}{(A+x)^2(B+x)^2}\right) + 1 - \widehat{s} - v, \tag{5.6}$$

$$\lambda_1\lambda_2 = \left(1 - x\left(\frac{r}{K} + \frac{my(AB-x^2)}{(A+x)^2(B+x)^2}\right)\right)\left(1 - \widehat{s} - v\right) + \frac{x^2 m}{(A+x)(B+x)}\left(\frac{\widehat{s}}{h} - u\right). \tag{5.7}$$

Lemma 5.1. The system (5.1)-(5.2) is locally asymptotically stable if all the eigenvalues of the characteristic Eq.(5.4) lie in an open unit disc.

Proof . The marginal stability lines can be obtained from the conditions $\lambda_1 = \pm 1$, $\lambda_1\lambda_2 = 1$. For the conditions $\lambda_1\lambda_2 = 1$, Eq.(5.7) gives

$$L_1 : \frac{x^2 m}{(A+x)(B+x)}u + \left(1 - x\left(\frac{r}{K} + \frac{my(AB-x^2)}{(A+x)^2(B+x)^2}\right)\right)v = -1 + \tag{5.8}$$

$$\left(1 - x\left(\frac{r}{K} + \frac{my(AB-x^2)}{(A+x)^2(B+x)^2}\right)\right)\left(1 - \widehat{s}\right) + \frac{x^2 m\widehat{s}}{h(A+x)(B+x)}.$$

The Eq. (5.8) expresses the first condition for marginal stability. For $\lambda_1 = 1$, the Eq. (5.6) yields

$$L_2 : \frac{x^2 m}{(A+x)(B+x)}u - x\left(\frac{r}{K} + \frac{my(AB-x^2)}{(A+x)^2(B+x)^2}\right)v = \widehat{s}x\left(\frac{r}{K} + \frac{my(AB-x^2)}{(A+x)^2(B+x)^2}\right) + \tag{5.9}$$

$$\frac{x^2 m\widehat{s}}{h(A+x)(B+x)},$$

similarly for $\lambda_1 = -1$, it gives

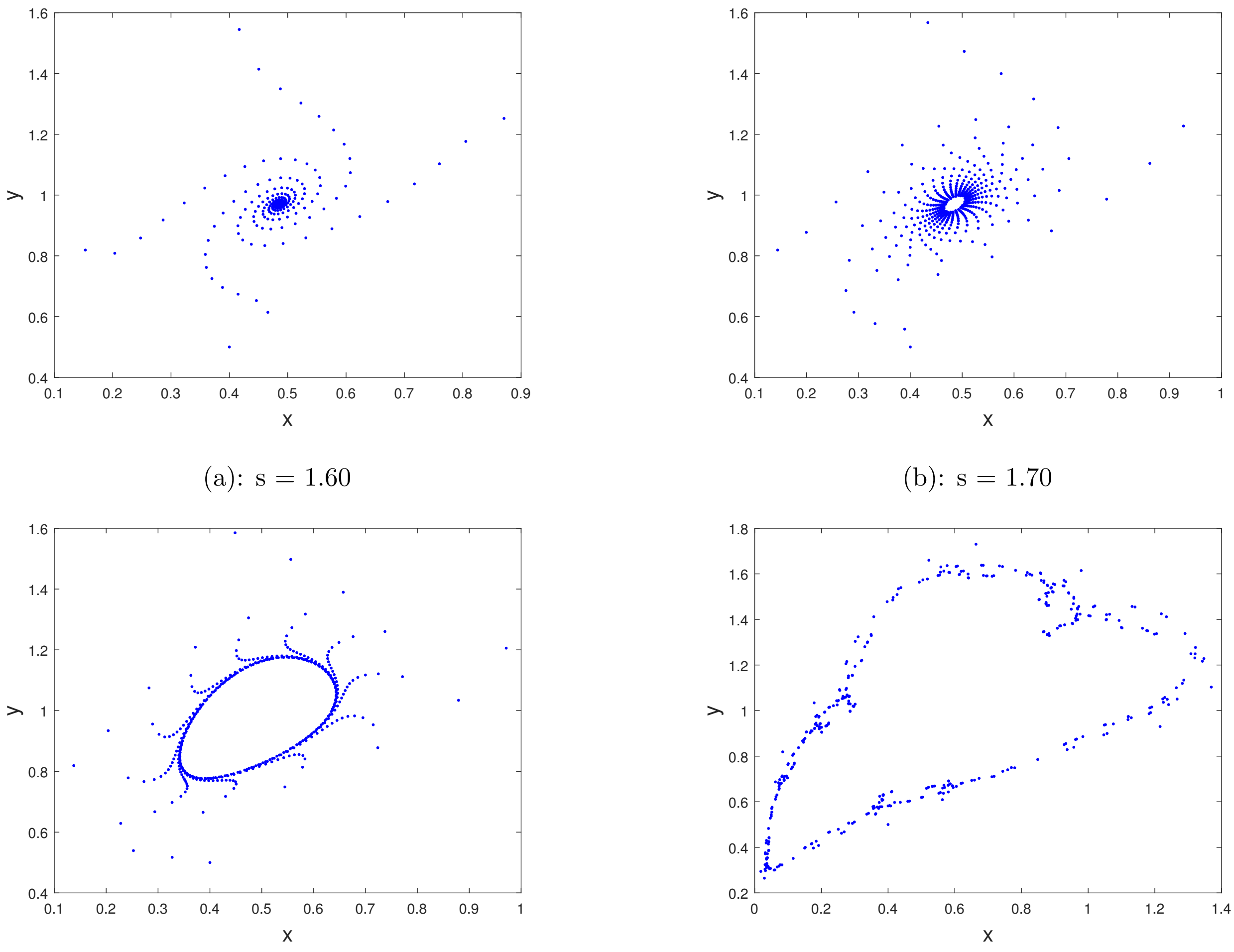
$$L_3 : \frac{x^2 m}{(A+x)(B+x)}u + \left(2 - x\left(\frac{r}{K} + \frac{my(AB-x^2)}{(A+x)^2(B+x)^2}\right)\right)v = \tag{5.10}$$

$$\left(2 - x\left(\frac{r}{K} + \frac{my(AB-x^2)}{(A+x)^2(B+x)^2}\right)\right)\left(2 - \widehat{s}\right) + \frac{x^2 m\widehat{s}}{h(A+x)(B+x)}.$$

The lines L_1, L_2, L_3 give the conditions for the eigenvalues to have absolute value less than one. The triangular region bounded by these lines accommodates stable eigenvalues. □

6 Numerical Simulations

To illustrate the theoretical results numerically, we choose the parameters as $(r, K, m, A, B, s, h) = (1.5, 2, 3, 0.5, 0.4, 2.1, 2)$ and the initial value $(x(0), y(0)) = (0.5, 0.4)$ for the system (1.3)-(1.4). The interior equilibrium $C(0.51, 1.02)$ is locally asymptotically stable, see Fig.(1)(a).



(a): s = 1.60

(b): s = 1.70

(c): s = 1.78

(d): s = 2.6

Figure 1: Phase portrait of system (1.3)-(1.4) for different values of s.

As shown in Fig.(1), by increasing the value of s from $s = 1.6$ to $s = 1.88$, the system (1.3)-(1.4) starts to lose its asymptotic stability. The existence of an attracting closed invariant curve implies that the discrete-time model (1.3)-(1.4) undergoes Neimark-Sacker bifurcation about $C(0.51, 1.02)$, the value $M = -0.114392726 < 0$ in the Theorem (3.1) proves theoretically the existence of an attracting Neimark-Sacker bifurcation. The full chaos induced by Neimark-Sacker bifurcation is drawn in Fig.(1)(d).

For exploring complexity in the system (1.3)-(1.4), the bifurcation diagram related to the period-doubling bifurcation of x and y is plotted with respect to s and m in Figs. (2)(a) and(3)(c). The chaotic behavior is justified by considering maximal Lyapunov exponent diagram given in Figs.(2)(b) and(3)(d). Biologically chaotic behavior means the species may go to extinction.

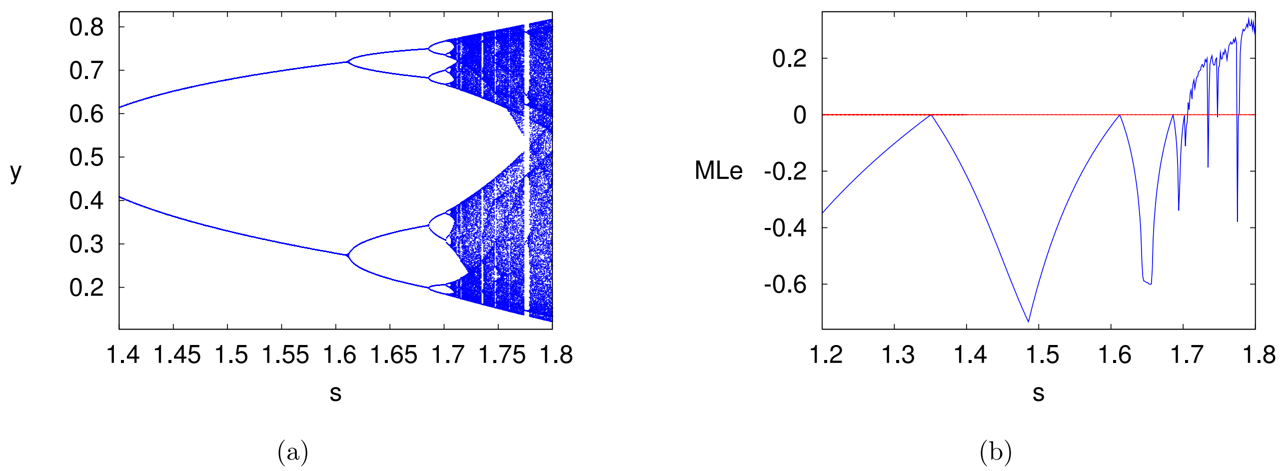


Figure 2: Bifurcation diagrams of y w.r.t s and the corresponding maximal Lyapunov exponent (MLe)

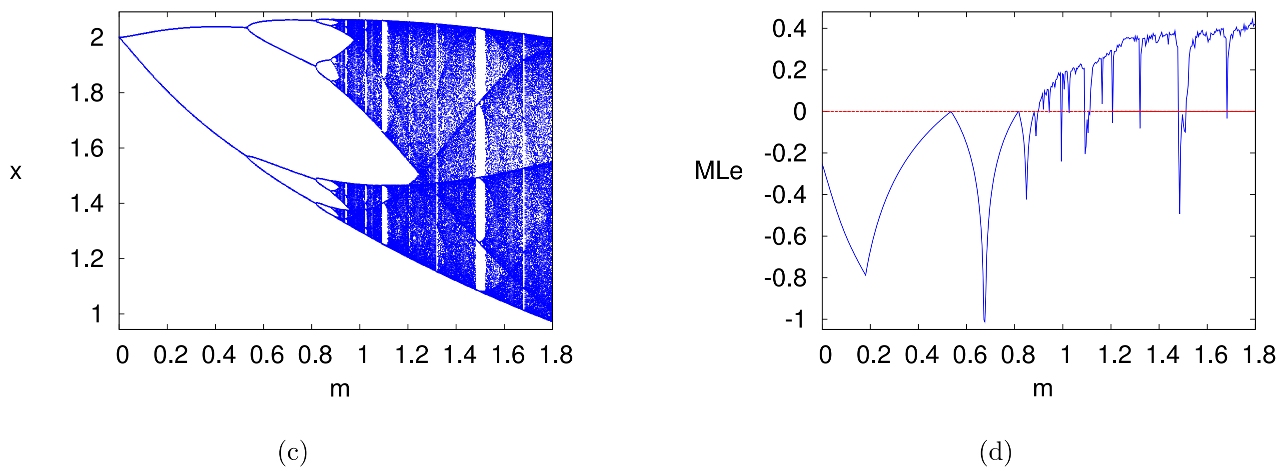


Figure 3: Bifurcation diagrams of x w.r.t m and the corresponding maximal Lyapunov exponent (MLe)

We use the SFC method to control the chaos produced by the Neimark-Sacker bifurcation. We choose the following parameter values: $(r, K, m, A, B, h) = (1.5, 2, 3, 0.4, 0.5, 2)$ and $\widehat{s} = 1.88$ see $(1)(d)$. Using lemma (5.1), one gets the following lines of marginal stability for the system (5.1)-(5.2).

$$l_1 : 1.4460711638u + 0.5165229356v = -0.028887365, \tag{6.1}$$

$$l_2 : 1.4460711638u - 0.483477064v = 2.702588328, \tag{6.2}$$

$$l_3 : 1.4460711638u + 1.5165229356v = 1.2556342. \tag{6.3}$$

The system (5.1)-(5.2) is stable in the domain bounded by the three lines (6.1), (6.2) and (6.3). Now, in order to stabilize the interior fixed point C, we consider the controlling force $P_n = u(x_n - 1.02) + v(y_n - 0.51)$ with feedback gains $u = 1$, $v = -1.6$, chosen, from the domain defined in Fig.(4). For these values, bifurcation diagrams are drawn for x and y with respect to the feedback gain u and v in Figs. (5) and (6), respectively.

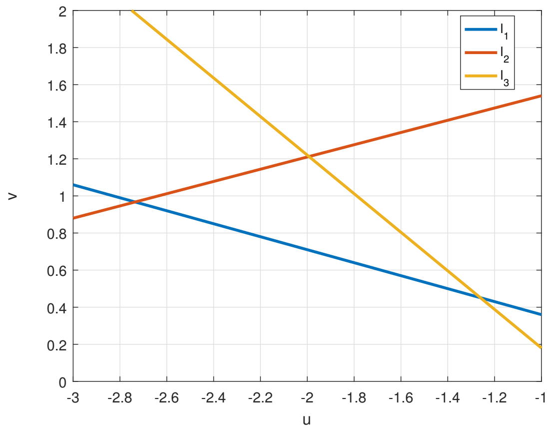


Figure 4: Stability region for the discrete model (5.1)-(5.2).

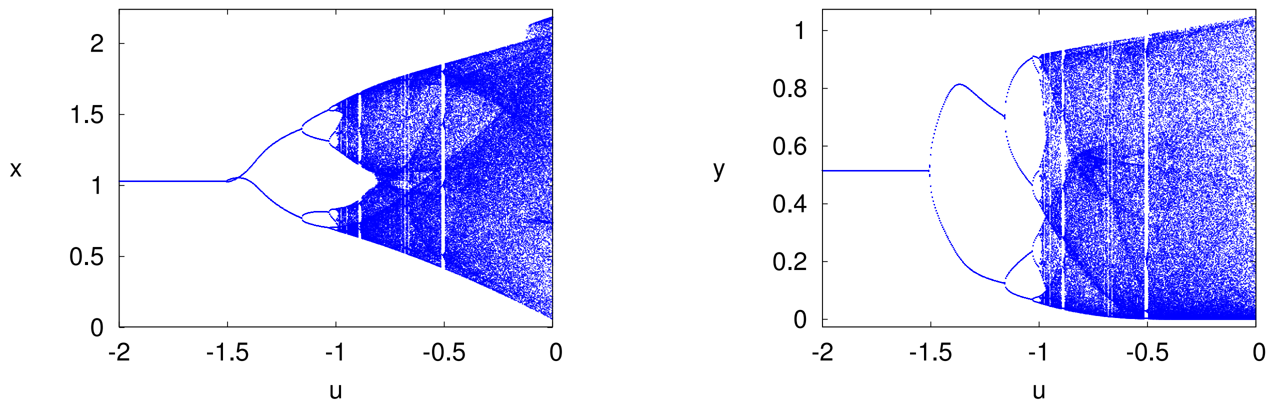


Figure 5: Bifurcation diagram for the controlled system (5.1)-(5.2) at $v = -2.7$ and $u \in (-2, 0.5)$;

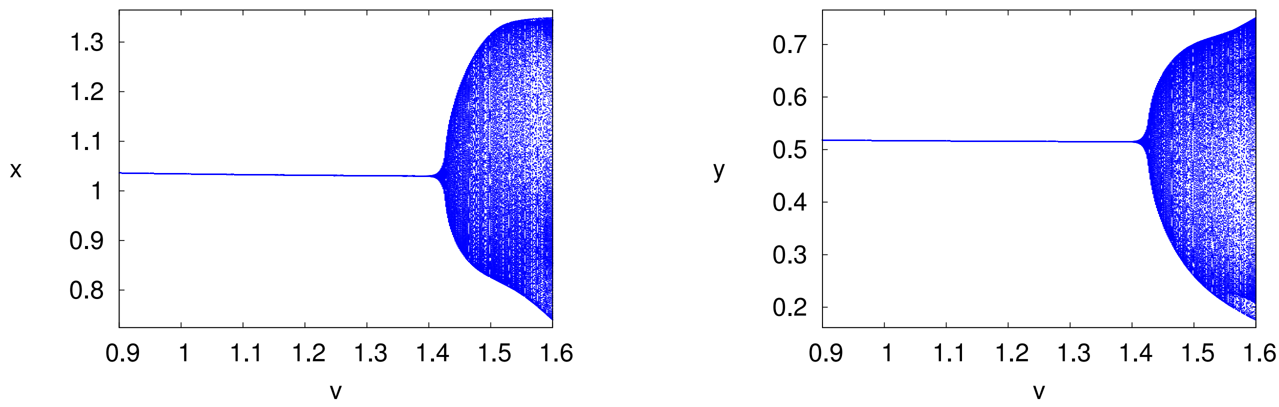


Figure 6: Bifurcation diagram for the controlled system (5.1)-(5.2) at $u = 1$ and $v \in (0, 2)$

7 Conclusion

In this paper, we have explored the dynamical properties of a discrete-time, two-dimensional prey-predator system. The proposed system exhibits various bifurcations of codimension 1, including Neimark-Sacker and period-doubling bifurcations, as the values of the parameters vary, as stated in Theorems 3.1 and 4.1. We observed the rich dynamics of the model when the intrinsic growth rate of the predator varies near 2.22 and 2.23. The Lyapunov exponents are numerically computed to further confirm the complexity of the dynamical behavior. The SFC method is implemented to achieve the asymptotic stability of the interior equilibrium, and numerical simulations give evidence of the successful implementation of the chaos control method. Bifurcation and chaos have always been regarded as unfavorable phenomena in biology. Because of their unpredictability, they can cause populations to run a higher risk of extinction, so they are harmful for the breeding of biological populations. Therefore, the use of the chaos control method makes prey and predator maintain stable dynamical behavior.

References

[1] H. Ben Ali, K. Mokni, and M. Ch-Chaoui, *Controlling chaos in a discretized prey-predator system*, Int. J. Nonlinear Anal. Appl. **14** (2023), no. 1, 1385–1398.

[2] M. Ch-Chaoui and K. Mokni, *A discrete evolutionary Beverton-Holt population model*, Int. J. Dynam. Control **11** (2023), 1060–1075.

[3] S. Elaydi, *Discrete Chaos, Applications in Science and Engineering*, Second Edition, Chapman and Hall / CRC London, 2008.

[4] S. Elaydi, *Global dynamics of discrete dynamical systems and difference equations*, Elaydi S et al. (eds) Differ. Equ. Discrete Dyn. Syst. Appl., ICDEA 2017, Springer Proc. Math. Statist. 2019.

[5] E. Elaydi, Y. Kang, and R. Luís, *The effects of evolution on the stability of competing species*, J. Bio. Dyn. **16** (2022), no. 1, 816.

[6] Q. Din, *Complexity and chaos control in a discrete-time prey–predator model*, Commun. Nonlin. Sci. Numer. Simulat. **49** (2017), 113—134.

[7] Z. He and X. Lai, *Bifurcation and chaotic behavior of a discrete-time predator–prey system*, Nonlinear Anal.: Real World Appl. **12** (2011), 403-417.

[8] S.B. Hsu and T.W, Huang, *Global stability for a class of predator-prey systems*, SIAM J. Appl. Math. **55** (1995), no. 13, 763–783.

[9] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, 3rd edn, Springer, New York, 2004.

[10] R. May, *Simple mathematical models with very complicated dynamics*, Nature **261** (1976), 459—467.

[11] K. Mokni and M. Ch-Chaoui, *Asymptotic stability, bifurcation analysis and chaos control in a discrete evolutionary Ricker population model with immigration*, Elaydi S, et al. (eds) Adv. Discrete Dyn. Syst. Differ. Equ. Appl., ICDEA 2021, Springer Proc. Mathe. Statist., 2023.

[12] K. Mokni and M. Ch-Chaoui, *Complex dynamics and Bifurcation analysis for a discrete evolutionary Beverton-Holt population model with Allee effect*, Int. J. Biomath. **16** (2023), no. 7, 2250127.

[13] K. Mokni, S. Elaydi, M. Ch-Chaoui, and A. Eladdadi, *Discrete evolutionary population models: A new approach*, J. Biol. Dyn. **14** (2020), no. 1, 454–478.

[14] M.D. Murry, *Mathematical Biology*, Springer, New York, 1989.

[15] P.K. Santra and G.S. Mahapatra, *Dynamical study of discrete-time prey-predator model with prey refugee under imprecise biological parameters*, J. Bio. Syst. **28** (2020), no. 3, 681–699.

[16] P.K. Santra, G.S. Mahapatra, and G.R. Phaijoo, *Bifurcation analysis and chaos control of discrete prey–predator model incorporating novel prey–refuge concept*, Comput. Math. Meth. **3** (2022), no. 6.

[17] H. Singh, J. Dhar, and H.S. Bhatti, *Discrete-time bifurcation behavior of a prey-predator system with generalized predator*, Adv. Differ. Equ. **2015** (2015), no. 1, 206.