

Fixed point for α_* - ψ - β_i -contractive set-valued mappings on Branciari S_b -metric space

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Abstract

In 1984, Khan et al. established some fixed point theorems in complete and compact metric spaces by altering distance functions. In 2020, Lotfy et al. introduced the α_* - ψ -common rational type mappings on generalized metric spaces applied to fractional integral equations. In 2022, Roy et al. described the notion of Branciari S_b -metric space and related fixed point theorems with an application. In this paper, we introduce the notion of fixed point theorems for α_* - ψ - β_i -contractive set-valued mappings on Branciari S_b -metric space with application to fractional integral equations.

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1 Introduction

We know that the fixed point theory has many applications and was extended by several authors from different views (see for example [1]-[22]). Samet et al [20] introduced the notion of α - ψ -contractive type mappings. Hassanzadeh Asl et al [11, 12] introduced the notion of common fixed point theorems for α_* - ψ -contractive multifunction. Farajzadeh et al [8] introduced the fixed point theorems for (ξ, α, η) -expansive mappings in complete metric spaces. Gungor et al established fixed point theorems on orthogonal metric spaces via altering distance functions. Lotfy et al [16] introduced the notion of α_* - ψ -common rational type mappings on generalized metric spaces with application to fractional integral equations. Roy et al [18] described the notion of Branciari S_b -metric space and related fixed point theorems with an application. This paper aims to introduce the notion of fixed point theorems for α_* - ψ - β_i -contractive set-valued mappings on Branciari S_b -metric space with application to fractional integral equations.

2 Preliminaries

In this section, we list some fundamental definitions that are useful tool in consequent analysis. Let 2^X denote the family of all nonempty subsets of X .

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Definition 2.1. ([15]) A function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

- (ψ_1) $\psi(0) = 0$ and $\psi(t) > 0$ for all $t \in (0, +\infty)$;
 - (ψ_2) ψ is continuous and non-decreasing;
 - (ψ_3) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$;
 - (ψ_4) $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$;
- for all $t_1, t_2 \in (0, +\infty)$.

These functions are known in the literature as (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) \leq t$ for all $t > 0$. We denote Ψ as the set of altering distance function ψ . The extended line is the ordered space $[-\infty; +\infty]$, considering of all points of the number line \mathbb{R} and two points, denoted by $-\infty, +\infty$ with the usual order relation for points of \mathbb{R} .

Definition 2.2. ([4, 7]) Let X be a nonempty set and $\rho : X \times X \rightarrow [0, \infty]$ be a mapping. Then ρ is said to be a rectangular metric if it satisfies the following conditions, for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y :

- (GMS1) $\rho(x, y) = 0$ if and if $x = y$;
- (GMS2) $\rho(x, y) = \rho(y, x)$ for any points $x, y \in X$;
- (GMS3) $\rho(x, y) \leq \rho(x, u) + \rho(u, v) + \rho(v, y)$ for any points $x, y, u, v \in X$ considering that if $d(x, u) = \infty$ or $\rho(u, v) = \infty$ or $d(v, y) = \infty$ then $\rho(x, u) + \rho(u, v) + d(v, y) = \infty$.

In this case the map ρ is called a generalized and abbreviated as *GM*. Here, the pair (X, ρ) is called a rectangular metric space and abbreviated as *GMS*. There are several rectangular metric spaces which are not usual metric spaces. Let us recall the following example.

In the above definition, if ρ satisfies only *GMS1* and *GMS2*, then it is called a semi-metric.

Example 2.3. ([13]) Let $U = \{0, 2\}$, $V = \{\frac{1}{n} : n \geq 1\}$ and $X = U \cup V$. Define $\rho : X^2 \rightarrow [0, \infty]$ by

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \text{ and either } x, y \in U \text{ or } x, y \in V, \\ y & \text{if } x \in U \text{ and } y \in V, \\ x & \text{if } x \in V \text{ and } y \in U. \end{cases}$$

Then ρ is a rectangular metric on X but not an usual metric space.

$$\rho(0, 2) = 1 > \rho(0, \frac{1}{3}) + \rho(\frac{1}{3}, 2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Sedghi et al. [21] introduced a new type of metric structure consisting of three variables known as S-metric. Subsequently Souayah and Mlaiki [22] investigated the notion of S_b -metric spaces which generalized the concept of S-metric spaces.

Definition 2.4. ([19, 21]) Let X be a nonempty set. An S-metric on X is a function $S : X^3 \rightarrow [0, \infty)$ that satisfies the following conditions, for all $x, y, z, t \in X$:

- (i) $S(x, y, z) = 0$ if and if $x = y = z$;
- (ii) $S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)$.

The pair (X, S) is called an S-metric space.

Example 2.5. ([21])

(1) Let \mathbb{R} be the real line and $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X . Then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X .

(2) Let \mathbb{R} be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S-metric on \mathbb{R} . This S-metric on \mathbb{R} is called the usual S-metric on \mathbb{R} .

Definition 2.6. ([17, 22]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $S_b : X^3 \rightarrow [0, \infty)$ is said to be S_b -metric if and if for all $x, y, z, t \in X$: the following conditions hold:

- (i) $S_b(x, y, z) = 0$ if and if $x = y = z$;
(ii) $S_b(x, y, z) \leq s[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)]$.

The pair (X, S_b) is called an S_b -metric space.

Example 2.7. ([22]) Let X be a nonempty set and $\text{card}(X) \geq 5$. suppose $X = X_1 \cup X_2$ a partition of X such that $\text{card}(X_1) \geq 4$. Let $s \geq 1$, then

$$S_b(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ 5 & \text{if } x = 1 = y \text{ and } z = 2, \\ \frac{1}{n+1} & \text{if } x = 1 = y \text{ and } z \geq 3, \\ \frac{1}{n+2} & \text{if } x = 2 = y \text{ and } z \geq 3, \\ 3 & \text{otherwise.} \end{cases}$$

for all $x, y, z, t \in X$. Then S_b is an S_b -metric on X with coefficient s .

Definition 2.8. ([18]) Let X be a nonempty set and $\lambda : X^3 \rightarrow \mathbb{R}_0^+$ be a function. Then λ is said to be Branciari S_b -metric if it satisfies the following condition:

- (i) $\lambda(x, y, z) = 0$ if and if $x = y = z$;
(ii) for any $x, y, z \in X$ and for $a, b \in X \setminus \{x, y, z\}$ with $a \neq b$ we have

$$\lambda(x, y, z) \leq k[\lambda(x, x, a) + \lambda(y, y, a) + \lambda(z, z, b) + \lambda(a, b, b)] \quad (2.1)$$

where $k \geq 1$. The pair (X, λ) is called Branciari S_b -metric space.

Definition 2.9. ([18]) A Branciari S_b -metric on a nonempty set X is said to be symmetric if $\lambda(x, x, y) = \sigma(y, y, x)$ for all $x, y \in X$.

Proposition 2.10. ([18]) (i) Let (X, S) be an S -metric spaces (see definition (2.4)). The X is also a Branciari S_b -metric space for $k = 2$.

(ii) Let (X, S_b) be an S_b -metric space with coefficient $s \geq 1$ (see definition (2.8)). The X is also a Branciari S_b -metric space for $k = 2s^2$.

Proposition 2.11. ([18]) Any S -metric space or S_b -metric space is also a Branciari S_b -metric space but there are several Branciari S_b -metric spaces which are neither S -metric spaces nor S_b -metric spaces.

Example 2.12. ([18]) Let $X = \mathbb{N}$ and $\lambda : X^3 \rightarrow \mathbb{R}_0^+$ be defined by

$$\lambda(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ 5 & \text{if } x = 1 = y \text{ and } z = 2, \\ \frac{1}{n+1} & \text{if } x = 1 = y \text{ and } z \geq 3, \\ \frac{1}{n+2} & \text{if } x = 2 = y \text{ and } z \geq 3, \\ 3 & \text{otherwise.} \end{cases}$$

for all $x, y, z, t \in X$. Also we take $\lambda(x, x, y) = \lambda(y, y, x)$ for all $x, y \in X$. Then λ is a symmetric S_b -metric space on X for $k = \frac{5}{3}$ but it is nether an S -metric nor an S_b -metric for any $k \geq 1$.

Definition 2.13. ([18]) Let (X, λ) be a Branciari S_b -metric space. Then

- (i) A sequence $\{x_n\}$ in X is said to be Branciari convergent to some $z \in X$ if $\lambda(x_n, x_n, z) \rightarrow 0$ as $n \rightarrow \infty$.
(ii) A sequence $\{x_n\}$ in X is said to be Branciari cauchy if $\lambda(x_n, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.
(iii) X is said to be Branciari complete if every Branciari cauchy sequence in X is Branciari convergent to some element in X .

Definition 2.14. We say that (X, λ) has the property α -regular Branciari S_b -metric space if, either

(i) $\{x_n\}$ is a monotone Branciari sequences in X such that $\alpha(x_n, x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then there exists a Branciari subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x_{n_k}, x) \geq 1$ for all k .

or

(ii) $\{x_n\}$ is a monotone Branciari sequences in X such that $\alpha(x_{n+1}, x_{n+1}, x_n) \geq 1$ for all n and $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then there exists a Branciari subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x, x, x_{n_k}) \geq 1$ for all k .

Definition 2.15. Let (X, λ) be a Branciari S_b -metric spaces. If $T : X \rightarrow 2^X$ is a set-valued mapping, then $x \in X$ is called fixed point for T if and only if $x \in F(x)$. The set

$$Fix(T) := \{x \in X \text{ such that } x \in Tx\}$$

is called the fixed point set of T .

Proposition 2.16. ([14, 7]) Suppose that $\{x_n\}$ is a Branciari Cauchy sequence in a (X, λ) be a Branciari S_b -metric space with $\lim_{n \rightarrow \infty} \lambda(x_n, x_n, u) = 0$ where $u \in X$. Then

$$\lim_{n \rightarrow \infty} \lambda(x_n, x_n, z) = \lambda(u, u, z)$$

for all $z \in X$. In particular, the Branciari sequence $\{x_n\}$ dose not Branciari converge to z if $z \neq u$.

Definition 2.17. Let (X, λ) be a Branciari S_b -metric space. A set-valued mapping $T : X \rightarrow 2^X$ is called Branciari order closed if for monotone Branciari sequences $x_n \in X$ and $y_n \in Tx_n$, with $x_n \rightarrow x$ and $y_n \rightarrow y$, implies $y \in Tx$.

Definition 2.18. Let (X, λ) be a Branciari S_b -metric space and $T : X \rightarrow 2^X$ with given set-valued, $\alpha : X \times X \times X \rightarrow [0, +\infty)$, $\alpha_* : 2^X \times 2^X \times 2^X \rightarrow [0, +\infty)$, $\alpha_*(A, A, B) = \inf\{\alpha(a, a, b) : a \in A, b \in B\}$, $\psi \in \Psi$, $\Lambda(s, s, Ts) = \inf\{\lambda(s, s, z)/z \in Ts\}$, H_λ is the Hausdorff metric

$$H_\lambda(Tx, Tx, Ty) = \max\{\sup_{a \in Tx} \Lambda(a, a, Ty), \sup_{b \in Ty} \Lambda(Tx, Tx, b)\}.$$

$\beta_i : \mathbb{R}^+ - \{0\} \rightarrow [0, 1)$ be four decreasing functions such that $\sum_{i=1}^4 \beta_i(t) \leq 1$ for every $t > 0$. One says that T is α_* - ψ - β_i -contractive set-valued mappings whenever

$$\begin{aligned} \alpha_*(Tx, Tx, Ty)\psi(H_\lambda(Tx, Tx, Ty)) \leq & \beta_1(\lambda(x, x, y))\psi(\lambda(x, x, y)) + \beta_2(\lambda(x, x, y))\psi(\Lambda(x, x, Tx)) \\ & + \beta_3(\lambda(x, x, y))\psi(\Lambda(y, y, Ty)) + \beta_4(\lambda(x, x, y)) \min\{\psi(\Lambda(x, x, Ty)), \psi(\Lambda(y, y, Tx))\}. \end{aligned} \quad (2.2)$$

One says that T are an α_* admissible if

$$\alpha(x, x, y) \geq 1 \Rightarrow \alpha_*(Tx, Tx, Ty) \geq 1 \quad (2.3)$$

for all $x, y \in X$.

Definition 2.19. A subset $B \subseteq X$ is said to be an approximation if for each given $y \in X$, there exists $z \in B$ such that $\Lambda(B, B, y) = \lambda(z, z, y)$.

Definition 2.20. A set-valued mapping $T : X \rightarrow 2^X$ is said to have an approximate values in X if Tx is an approximation for each $x \in X$.

3 Main result

Some fixed point theorems in symmetric Branciari S_b -metric space.

Theorem 3.1. Let (X, λ) be a complete symmetric Branciari S_b -metric space (not necessarily complete metric space), $T : X \rightarrow 2^X$ is α_* - ψ - β_i -Branciari contractive set-valued mappings satisfies the following conditions:

- (i) T is α_* -admissible;
- (ii) there exists $x_0 \in X$ such that

$$\alpha_*({x_0}, {x_0}, T{x_0}) \geq 1, \alpha_*({x_0}, {x_0}, T^2{x_0}) \geq 1;$$

- (iii) (X, λ) has the property α -regular Branciari S_b -metric space.

Then T has fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated Branciari sequences $\{x_n\}$ with $x_{n+1} \in Tx_n$ Branciari converges to the fixed point of T .

Proof . Let $x_0 \in X$ such that $\alpha_*(\{x_0\}, \{x_0\}, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 > 1$, then $x^* = x_{n_0}$ are a fixed point for T . So, we can assume that $x_n \notin Tx_n$ for all $n \in \mathbb{N}_0$. Since T is α_* -admissible, we have

$$\begin{aligned}\alpha(x_0, x_0, x_1) &\geq \alpha_*(\{x_0\}, \{x_0\}, Tx_0) \geq 1 \Rightarrow \alpha_*(Tx_0, Tx_0, Tx_1) \geq 1; \\ \alpha(x_1, x_1, x_2) &\geq \alpha_*(Tx_0, Tx_0, Tx_1) \geq 1 \Rightarrow \alpha_*(Tx_1, Tx_1, Tx_2) \geq 1; \\ \alpha(x_2, x_2, x_3) &\geq \alpha_*(Tx_1, Tx_1, Tx_2) \geq 1 \Rightarrow \alpha_*(Tx_2, Tx_2, Tx_3) \geq 1.\end{aligned}$$

Inductively, we have

$$\alpha(x_n, x_n, x_{n+1}) \geq 1 \Rightarrow \alpha_*(Tx_n, Tx_n, Tx_{n+1}) \geq 1$$

for all $n \in \mathbb{N}_0$. Similarly, we have

$$\begin{aligned}\alpha(x_0, x_0, x_2) &\geq \alpha_*(\{x_0\}, \{x_0\}, T^2x_0) \geq 1 \Rightarrow \alpha_*(Tx_0, Tx_0, Tx_2) \geq 1; \\ \alpha(x_1, x_1, x_3) &\geq \alpha_*(Tx_0, Tx_0, Tx_2) \geq 1 \Rightarrow \alpha_*(Tx_1, Tx_1, Tx_3) \geq 1; \\ \alpha(x_2, x_2, x_4) &\geq \alpha_*(Tx_1, Tx_1, Tx_3) \geq 1 \Rightarrow \alpha_*(Tx_2, Tx_2, Tx_4) \geq 1.\end{aligned}$$

Inductively, we have

$$\alpha(x_n, x_n, x_{n+2}) \geq 1 \Rightarrow \alpha_*(Tx_n, Tx_n, Tx_{n+2}) \geq 1$$

for all $n \in \mathbb{N}_0$. Without loss of generality, we may assume that $T : X \rightarrow 2^X$ be a α_* - ψ - β_i -contractive set-valued mappings. Consider equation (2.2), with $x = x_{2n+1}$ and $y = x_{2n+2}$. Clearly, we have

$$\begin{aligned}\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) &\leq \alpha_*(Tx_{2n}, Tx_{2n}, Tx_{2n+1})\psi(H_\lambda(Tx_{2n}, Tx_{2n}, Tx_{2n+1})) \\ &\leq \beta_1(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) + \beta_2(\Lambda(x_{2n}, x_{2n}, Tx_{2n}))\psi(\Lambda(x_{2n}, x_{2n}, Tx_{2n})) \\ &\quad + \beta_3(\Lambda(x_{2n+1}, x_{2n+1}, Tx_{2n+1}))\psi(\Lambda(x_{2n+1}, x_{2n+1}, Tx_{2n+1})) \\ &\quad + \beta_4(H_\lambda(Tx_{2n}, Tx_{2n}, Tx_{2n+1}))\min\{\psi(\Lambda(x_{2n}, x_{2n}, Tx_{2n+1})), \psi(\Lambda(x_{2n+1}, x_{2n+1}, Tx_{2n}))\} \\ &\leq \beta_1(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) + \beta_2(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) \\ &\quad + \beta_3(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2}))\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) \\ &\quad + \beta_4(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2}))\min\{\psi(\lambda(x_{2n}, x_{2n}, x_{2n+2})), \psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+1}))\} \\ &\leq \beta_1(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) + \beta_2(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) \\ &\quad + \beta_3(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2}))\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2}))\end{aligned}\tag{3.1}$$

Then

$$(1 - \beta_3(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2}))\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2}))) \leq \beta_1(\lambda(x_{2n}, x_{2n}, x_{2n+1})) + \beta_2(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\tag{3.2}$$

and

$$\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) \leq \frac{(\beta_1(\lambda(x_{2n}, x_{2n}, x_{2n+1})) + \beta_2(\lambda(x_{2n}, x_{2n}, x_{2n+1})))}{(1 - \beta_3(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})))}\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\tag{3.3}$$

Thus

$$\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) \leq \psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})).\tag{3.4}$$

for all $n \in \mathbb{N}_0$. Similarly,

$$\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) \leq \psi(\lambda(x_{2n-1}, x_{2n-1}, x_{2n})).\tag{3.5}$$

for all $n \in \mathbb{N}_0$. We have

$$\psi(\lambda(x_{n+1}, x_{n+1}, x_{n+2})) \leq \psi(\lambda(x_n, x_n, x_{n+1})) \leq \dots \leq \psi^n(\lambda(x_0, x_0, x_1)),\tag{3.6}$$

for all $n \in \mathbb{N}$. From the property of ψ , we conclude that

$$\lambda(x_n, x_n, x_{n+1}) \leq \lambda(x_{n-1}, x_{n-1}, x_n),\tag{3.7}$$

for all $n \in \mathbb{N}$, it is clear that

$$\lim_{n \rightarrow \infty} \lambda(x_{n+1}, x_{n+1}, x_{n+2}) = 0.\tag{3.8}$$

Consider equation (2.2), with $x = x_{2n-1}$ and $y = x_{2n+1}$. Clearly, we have

$$\begin{aligned}
\psi(\lambda(x_{2n-1}, x_{2n-1}, x_{2n+1})) &\leq \alpha_*(Tx_{2n-2}, Tx_{2n-2}, Tx_{2n})\psi(H_\lambda(Tx_{2n-2}, Tx_{2n-2}, Tx_{2n})) \\
&\leq \beta_1(\lambda(x_{2n-2}, x_{2n-2}, x_{2n}))\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n})) \\
&\quad + \beta_2(\Lambda(x_{2n-2}, x_{2n-2}, Tx_{2n-2}))\psi(\Lambda(x_{2n-2}, x_{2n-2}, Tx_{2n-2})) \\
&\quad + \beta_3(\Lambda(x_{2n}, x_{2n}, Tx_{2n}))\psi(\Lambda(x_{2n}, x_{2n}, Tx_{2n})) \\
&\quad + \beta_4(H_\lambda(Tx_{2n-2}, Tx_{2n-2}, Tx_{2n})) \min\{\psi(\Lambda(x_{2n-2}, x_{2n-2}, Tx_{2n})), \psi(\Lambda(x_{2n}, x_{2n}, Tx_{2n-2}))\} \\
&\leq \beta_1(\lambda(x_{2n-2}, x_{2n-2}, x_{2n}))\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n})) \\
&\quad + \beta_2(\lambda(x_{2n-2}, x_{2n-2}, x_{2n-1}))\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n-1})) \\
&\quad + \beta_3(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) \\
&\quad + \beta_4(\lambda(x_{2n-1}, x_{2n-1}, x_{2n+1})) \min\{\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n+1})), \psi(\lambda(x_{2n}, x_{2n}, x_{2n-1}))\}
\end{aligned} \tag{3.9}$$

Define $a_{2n} = \lambda(x_{2n-1}, x_{2n-1}, x_{2n+1})$ and $b_{2n} = \lambda(x_{2n}, x_{2n}, x_{2n+1})$. Then

$$\psi(a_{2n}) \leq \beta_1(a_{2n-1})\psi(a_{2n-1}) + \beta_2(b_{2n-1})\psi(b_{2n-1}) + \beta_3(b_{2n})\psi(b_{2n}) + \beta_4(a_{2n}) \min\{\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n+1})), \psi(b_{2n-1})\}. \tag{3.10}$$

From the (3.8) $\lim_{n \rightarrow \infty} b_{2n} = \lim_{n \rightarrow \infty} \lambda(x_{2n}, x_{2n}, x_{2n+1}) = 0$. We get

$$\psi(a_{2n}) \leq \beta_1(a_{2n-1})\psi(a_{2n-1}) \leq \psi(a_{2n-1}) \tag{3.11}$$

and hence,

$$\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} d(x_{2n-1}, x_{2n+1}) = 0 \Rightarrow \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \lambda(x_{n-1}, x_{n-1}, x_{n+1}) = 0.$$

Now, we shall prove that $x_n \neq x_m$ for all $n \neq m$. Assume on the contrary that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $n \neq m$. Since $d(x_p, x_{p+1}) > 0$ for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n + 1$, $m = 2k$ and $n = 2l$ for $k, l \in \mathbb{N}$. Substitute again $x = x_{2l} = x_{2k}$ and $y = x_{2l+1} = x_{2k+1}$ in (2.2), (3.7) which yields

$$\begin{aligned}
\psi(\lambda(x_{2l}, x_{2l}, x_{2l+1})) &= \psi(\lambda(x_{2k}, x_{2k}, x_{2k+1})) \\
&\leq \alpha_*(H_\lambda(Tx_{2k-1}, Tx_{2k-1}, Tx_{2k}))\psi(H_\lambda(Tx_{2k-1}, Tx_{2k-1}, Tx_{2k})) \\
&\leq \beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) \\
&\quad + \beta_2(\Lambda(x_{2k-1}, x_{2k-1}, Tx_{2k-1}))\psi(\Lambda(x_{2k-1}, x_{2k-1}, Tx_{2k-1})) \\
&\quad + \beta_3(\Lambda(x_{2k}, x_{2k}, Tx_{2k}))\psi(\Lambda(x_{2k}, x_{2k}, Tx_{2k})) \\
&\quad + \beta_4(H_\lambda(Tx_{2k}, Tx_{2k}, Tx_{2k-1})) \min\{\psi(\Lambda(x_{2k}, x_{2k}, Tx_{2k-1})), \psi(\Lambda(x_{2k-1}, x_{2k-1}, Tx_{2k}))\} \\
&\leq \beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) \\
&\quad + \beta_2(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) \\
&\quad + \beta_3(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\psi(\lambda(x_{2k}, x_{2k}, x_{2k+1})) \\
&\quad + \beta_4(\lambda(x_{2k+1}, x_{2k+1}, x_{2k})) \min\{\psi(\lambda(x_{2k}, x_{2k}, x_{2k})), \psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k+1}))\} \\
&\leq \beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) \\
&\quad + \beta_2(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) + \beta_3(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\psi(\lambda(x_{2k}, x_{2k}, x_{2k+1}))
\end{aligned} \tag{3.12}$$

which is impossible. From this it follows that $x_n \neq x_m$ for all n, m ($n \neq m$) $\in \mathbb{N}$.

Case I: Suppose that $S_n = \lambda(x_n, x_n, x_{n+1})$, $\psi(S_n) = \alpha_n S_n$ and $\alpha \in (0, \frac{1}{\sqrt{k}})$. Then

$$\begin{aligned}
S_n &= \lambda(x_n, x_n, x_{n+1}) \leq \psi(\lambda(x_{n-1}, x_{n-1}, x_n)) = \alpha_{n-1}\lambda(x_{n-1}, x_{n-1}, x_n) \\
&\leq \alpha_{n-1}\psi(\lambda(x_{n-2}, x_{n-2}, x_{n-1})) \leq \cdots \leq \alpha_{n-1}\alpha_{n-2} \cdots \alpha_1\alpha_0\lambda(x_0, x_0, x_1) = \alpha^n S_0
\end{aligned} \tag{3.13}$$

Similarly, we have

$$\begin{aligned}
S_n^* &= \lambda(x_n, x_n, x_{n+2}) \leq \psi(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) = \alpha_{n-1}\lambda(x_{n-1}, x_{n-1}, x_{n+1}) \\
&\leq \alpha_{n-1}\psi(\lambda(x_{n-2}, x_{n-2}, x_n)) \leq \cdots \leq \alpha_{n-1}\alpha_{n-2} \cdots \alpha_1\alpha_0\lambda(x_0, x_0, x_1) = \alpha^n S_0^*
\end{aligned} \tag{3.14}$$

for all $n \geq 1$ and $\alpha = \max_{0 \leq i \leq n-1} \{\alpha_i\}$. Now, we shall prove that $\{x_n\}$ is a Branciari Cauchy sequence, that is,

$$\lim_{n \rightarrow \infty} \lambda(x_n, x_n, x_{n+l}) = 0,$$

for all $l \in \mathbb{N}$. We have already proved the cases for $l = 1$ and $l = 2$ in (3.7) and (3.10), respectively. Now for $l = 2m + 1$, where $m \geq 1$. Using the inequality (2.1), we have

$$\begin{aligned} \lambda(x_n, x_n, x_{n+l}) &\leq k[\lambda(x_n, x_n, x_{n+1}) + \lambda(x_n, x_n, x_{n+1}) + \lambda(x_{n+l}, x_{n+l}, x_{n+2}) \\ &\quad + \lambda(x_{n+1}, x_{n+1}, x_{n+2})] \\ &= 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+l}, x_{n+l}, x_{n+2}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\stackrel{\text{Symmetric}}{=} 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) + k\lambda(x_{n+2}, x_{n+2}, x_{n+1}) \\ &\leq 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) + k(k[\lambda(x_{n+2}, x_{n+2}, x_{n+3}) \\ &\quad + \lambda(x_{n+2}, x_{n+2}, x_{n+3}) + \lambda(x_{n+l}, x_{n+l}, x_{n+4}) + \lambda(x_{n+3}, x_{n+3}, x_{n+4})]) \\ &\stackrel{\text{Symmetric}}{=} 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) + 2k^2\lambda(x_{n+2}, x_{n+2}, x_{n+3}) \\ &\quad + k^2\lambda(x_{n+3}, x_{n+3}, x_{n+4}) + k^2\lambda(x_{n+4}, x_{n+4}, x_{n+2m+1}) \\ &\leq \dots \\ &\vdots \\ &\leq 2k[\lambda(x_n, x_n, x_{n+1}) + \lambda(x_{n+1}, x_{n+1}, x_{n+2})] + 2k^2[\lambda(x_{n+2}, x_{n+2}, x_{n+3}) \\ &\quad + \lambda(x_{n+3}, x_{n+3}, x_{n+4})] \\ &\quad + \dots + 2k^m[\lambda(x_{n+2m-2}, x_{n+2m-2}, x_{n+2m-1}) + \lambda(x_{n+2m-1}, x_{n+2m-1}, x_{n+2m})] \\ &\quad + k^m\lambda(x_{n+2m}, x_{n+2m}, x_{n+2m+1}) \\ &\leq 2[\{k(\alpha_0^n + \alpha_0^{n+1}) + k^2(\alpha_0^{n+2} + \alpha_0^{n+3}) + \dots + k^m(\alpha_0^{n+2m-2} + \alpha_0^{n+2m-1})\} \\ &\quad + k^m\alpha_0^{n+2m}]S_0 = 2k(1 + \alpha_0)\alpha_0^n[1 + k\alpha_0^2 + \dots + k^m\alpha_0^{2m}]S_0 = \frac{2k(1+\alpha_0)}{1+k\alpha_0^2}\alpha_0^n S_0 \end{aligned} \quad (3.15)$$

for all $n \geq 1$. Also for $l = 2m$ we get

$$\lambda(x_n, x_n, x_{n+2m}) \leq \dots \leq \frac{2k(1+\alpha_0)}{1+k\alpha_0^2}\alpha_0^n S_0 + \alpha_0^n (k\alpha_0^2)^{m-1} S_0^* \quad (3.16)$$

for all $n \geq 1$. Thus we proved that $\{x_n\}$ is a Branciari Cauchy sequence in the complete metric space (X, λ) , there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} \lambda(x_n, x_n, x^*) = 0$$

by (X, λ) has the property α -regular Branciari S_b -metric space. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\alpha_*(\{x_{2n_k+1}\}, \{x_{2n_k+1}\}, \{x^*\}) \geq \alpha_*(Tx_{2n_k}, Tx_{2n_k}, Tx^*) \geq 1 \text{ for all } k. \quad (3.17)$$

Thus

$$\begin{aligned} \psi(\Lambda(x^*, x^*, Tx^*)) &\leq \psi(\lambda(x^*, x^*, x_{2n_k+1})) + \psi(\Lambda(x_{2n_k+1}, x_{2n_k+1}, Tx^*)) \\ &\leq \psi(\lambda(x^*, x^*, x_{2n_k+1})) + \alpha_*(Tx_{2n_k}, Tx_{2n_k}, Tx^*)\psi(H_\lambda(Tx_{2n_k}, Tx_{2n_k}, Tx^*)) \\ &\leq \psi(\lambda(x^*, x^*, x_{2n_k+1})) + \beta_1(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \\ &\quad + \beta_2(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx_{2n_k})) + \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*)) \\ &\quad \beta_4(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \min\{\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx^*), \psi(\Lambda(x^*, x^*, Tx_{2n_k}))\} \\ &\leq \psi(\lambda(x^*, x^*, x_{2n_k+1})) + \beta_1(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \\ &\quad + \beta_2(\lambda(x_{2n_k}, x_{2n_k}, x_{2n_k+1}))\psi(\lambda(x_{2n_k}, x_{2n_k}, x_{2n_k+1})) + \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*)) \\ &\quad \beta_4(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \min\{\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx^*), \psi(\lambda(x^*, x^*, x_{2n_k+1}))\} \\ &\leq \psi(0) + \beta_1(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(0) + \beta_2(\lambda(x_{2n_k}, x_{2n_k}, x_{2n_k+1}))\psi(0) \\ &\quad + \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*))\beta_4(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \min\{\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx^*), \psi(0))\} \\ &\leq \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*)) \\ &\leq \psi(\Lambda(x^*, x^*, Tx^*)) \end{aligned} \quad (3.18)$$

for all k , which is impossible. Hence, $\Lambda(x^*, x^*, Tx^*) = \Lambda(Tx^*, Tx^*, x^*) = 0$ and so $x^* \in Tx^*$.

Case-(II): $\alpha \in [\frac{1}{\sqrt{k}}, 1)$. Then there exists $N \in \mathbb{N}$ such that $\alpha^N \in (\frac{1}{\sqrt{k}}, 1)$. Now due to the contractive condition (2.2) we see that also satisfies the contractive condition (2.2) for the Lipschitz constant therefore by Case-(I) T^N has a fixed point in X and thus in this case also T has a fixed point. \square

Example 3.2. ([18]) Let $X = \mathbb{N}$ and $\lambda : X^3 \rightarrow \mathbb{R}_0^+$ be defined $\lambda(x, x, x) = 0$ and $\lambda(x, x, y) = \lambda(y, y, x)$ for all $x, y \in X$ with

$$\lambda(x, y, z) = \begin{cases} 10 & \text{if } x = 1 = y \text{ and } z = 2, \\ \frac{1}{2(n+1)} & \text{if } x = 1 = y \text{ and } z \geq 3, \\ \frac{1}{n+2} & \text{if } x = 2 = y \text{ and } z \geq 3, \\ 5 & \text{otherwise.} \end{cases}$$

Then λ is a complete symmetric Branciari S_b -metric space on X for $k = 4$ but it is nether an S -metric nor an S_b -metric for any $k \geq 1$. Let $T : X \rightarrow 2^X$ be

$$Tx = \begin{cases} \{3, 4\} & \text{if } x \in \{1, 2\}, \\ \{5, 6\} & \text{otherwise.} \end{cases}$$

Then T^2 satisfies the contractive condition (2.2) for any $\psi(x) = \frac{x^2}{1+x^2}$ and thus T^2 has a fixed point in X . Therefore T has a fixed point $x = 5$ in X .

Corollary 3.3. ([18]) (Analogue to Banach Contraction Theorem) Let (X, λ) be a complete symmetric Branciari S_b -metric space and $T : X \rightarrow X$ satisfies

$$\lambda(Tx, Tx, Ty) \leq \alpha(\lambda(x, x, y))$$

for all $x, y \in X$, where $\alpha \in (0, 1)$. Then T has a unique fixed point in X .

Example 3.4. Let $X = \mathbb{Z}$ and $Y \subseteq X$ be a finite set defined as $Y = \{1, 2, 4, 8\}$. Define $\lambda : Y \times Y \times Y \rightarrow [0, \infty)$ as:

$$\lambda(1, 1, 1) = \lambda(2, 2, 2) = \lambda(4, 4, 4) = \lambda(8, 8, 8) = 0,$$

$$\lambda(1, 1, 2) = \lambda(2, 2, 1) = 3,$$

$$\lambda(2, 2, 8) = \lambda(8, 8, 2) = \lambda(1, 1, 8) = \lambda(8, 8, 1) = 1 \text{ and}$$

$$\lambda(1, 1, 4) = \lambda(4, 4, 1) = \lambda(2, 2, 4) = \lambda(4, 4, 2) = \lambda(8, 8, 4) = \lambda(4, 4, 8) = \frac{1}{2}.$$

The function λ is not a metric on Y . Indeed, note

$$3 = \lambda(1, 1, 2) \geq \lambda(1, 1, 8) + \lambda(8, 8, 2) = 1 + 1 = 2,$$

that is, the triangle inequality is not satisfied. However, λ is a Branciari S_b -metric on Y and moreover (Y, λ) is a complete Branciari S_b -metric space. Define $T : Y \rightarrow 2^Y$ as: $T1 = T2 = T8 = \{2, 4\}$, $T4 = \{1, 8\}$ and $T1 = T2 = T4 = \{2, 8\}$, $T8 = \{1, 2\}$, $\alpha : Y \times Y \times Y \rightarrow [0, +\infty)$, $\alpha_* = \inf \alpha$ as $\alpha(x, x, y) = \alpha(y, y, x) = 1$ $\psi(t) = \frac{2}{3}t$. Clearly, T satisfies the conditions of Theorem (3.1) and has a fixed point $x = 2$.

3.1 Analogue to Kannan fixed point theorem

Theorem 3.5. (Analogue to Kannan fixed point theorem) Let (X, λ) be complete symmetric Branciari S_b -metric space and $T : X \rightarrow 2^X$ satisfies

$$\alpha_*(Tx, Tx, Ty)H_\lambda(Tx, Tx, Ty) \leq \beta_1(\lambda(x, x, y))\psi_1(\Lambda(x, x, Tx)) + \beta_2(\lambda(y, y, x))\psi_2(\Lambda(y, y, Ty)) \quad (3.19)$$

for all $x, y \in X$ where $\psi_i \in \Psi$ and $\sum_{i=1}^2 \beta_i(\lambda(x, x, y)) \in (0, \frac{1}{2})$. Then T has a fixed point in X .

Proof . Let $x_0 \in X$ be taken as arbitrary and let us construct the sequence $\{x_n\}$ in X by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 > 1$, then $x^* = x_{n_0}$ are a fixed point for T . So, we can assume that $x_n \notin Tx_n$ for all $n \in \mathbb{N}_0$. Here we show that $\{x_n\}$ is Cauchy sequence in X .

Case-I: $\sum_{i=1}^2 \beta_i(\lambda(x, x, y)) \in (0, \frac{1}{k+1})$. From the contraction condition (3.19), we get

$$\begin{aligned} \lambda(x_n, x_n, x_{n+1}) &\leq \alpha_*(Tx_{n-1}, Tx_{n-1}, Tx_n)H_\lambda(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq \beta_1(\lambda(x_{n-1}, x_{n-1}, x_n))\psi_1(\Lambda(x_{n-1}, x_{n-1}, Tx_{n-1})) + \beta_2(\lambda(x_n, x_n, x_{n-1}))\psi_2(\Lambda(x_n, x_n, Tx_n)) \\ &\leq \beta_1(\lambda(x_{n-1}, x_{n-1}, x_n))\psi_1(\lambda(x_{n-1}, x_{n-1}, x_n)) + \beta_2(\lambda(x_n, x_n, x_{n-1}))\psi_2(\lambda(x_n, x_n, x_{n+1})) \\ &\leq \beta_1(\lambda(x_{n-1}, x_{n-1}, x_n))\lambda(x_{n-1}, x_{n-1}, x_n) + \beta_2(\lambda(x_n, x_n, x_{n-1}))\lambda(x_n, x_n, x_{n+1}) \end{aligned}$$

for all $n \geq 1$. From which we get

$$S_n = \lambda(x_n, x_n, x_{n+1}) \leq \frac{\beta_1}{1 - \beta_2} \lambda(x_{n-1}, x_{n-1}, x_n) = \gamma \lambda(x_{n-1}, x_{n-1}, x_n) = \gamma S_{n-1} \leq \cdots \leq \gamma^n S_0$$

for all $n \in \mathbb{N}$, where $\gamma = \frac{\beta_1}{1 - \beta_2} < \frac{1}{k}$. Also we have,

$$\begin{aligned} S_n^* = \lambda(x_n, x_n, x_{n+2}) &\leq \alpha_*(Tx_{n-1}, Tx_{n-1}, Tx_{n+1}) H_\lambda(Tx_{n-1}, Tx_{n-1}, Tx_{n+1}) \\ &\leq \beta_1(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) \psi_1(\Lambda(x_{n-1}, x_{n-1}, Tx_{n+1})) \\ &\quad + \beta_2(\lambda(x_{n+1}, x_{n+1}, x_{n-1})) \psi_2(\Lambda(x_{n+1}, x_{n+1}, Tx_{n+1})) \\ &\leq \beta_1(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) \psi_1(\lambda(x_{n-1}, x_{n-1}, x_n)) \\ &\quad + \beta_2(\lambda(x_{n+1}, x_{n+1}, x_{n-1})) \psi_2(\lambda(x_{n+1}, x_{n+1}, x_{n+2})) \\ &\leq \beta_1(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) \lambda(x_{n-1}, x_{n-1}, x_n) + \beta_2(\lambda(x_{n+1}, x_{n+1}, x_{n-1})) \lambda(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\leq \beta_1(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) S_{n-1} + \beta_2(\lambda(x_{n+1}, x_{n+1}, x_{n-1})) S_{n+1} \\ &\leq \beta_1(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) \gamma^{n-1} S_0 + \beta_2(\lambda(x_{n+1}, x_{n+1}, x_{n-1})) \gamma^{n+1} S_0 \\ &\leq \beta(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) \gamma^{n-1} S_0 + \beta(\lambda(x_{n+1}, x_{n+1}, x_{n-1})) \gamma^{n+1} S_0 \\ &\leq \beta(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) [\gamma^{n-1} + \gamma^{n+1}] S_0 \\ &= \beta(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) [1 + \gamma^2] \gamma^{n-1} S_0 \end{aligned}$$

for all $n \in \mathbb{N}$, where

$$\beta(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) = \max\{\beta_1(\lambda(x_{n-1}, x_{n-1}, x_{n+1})), \beta_2(\lambda(x_{n+1}, x_{n+1}, x_{n-1}))\}.$$

Show that x_n is Cauchy sequence in X and therefore due to the completeness of X there exist a $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$. Now,

$$\begin{aligned} \Lambda(x_{n+1}, x_{n+1}, Tu) &\leq \alpha_*(Tx_n, Tx_n, Tu) H_\lambda(Tx_n, Tx_n, Tu) \\ &\leq \beta_1(\lambda(x_n, x_n, u)) \psi_1(\Lambda(x_n, x_n, Tx_n)) + \beta_2(\lambda(u, u, x_n)) \psi_2(\Lambda(u, u, Tu)) \\ &= \beta_1(\lambda(x_n, x_n, u)) \psi_1(\lambda(x_n, x_n, x_{n+1})) + \beta_2(\lambda(u, u, x_n)) \psi_2(\Lambda(u, u, Tu)) \\ &\leq \beta_1(\lambda(x_n, x_n, u)) \lambda(x_n, x_n, x_{n+1}) + \beta_2(\lambda(u, u, x_n)) \Lambda(u, u, Tu) \\ &\leq \beta_1(\lambda(x_n, x_n, u)) \lambda(x_n, x_n, x_{n+1}) + \beta_2(\lambda(u, u, x_n)) k [2\lambda(u, u, x_n) + \lambda(Tu, Tu, x_{n+1}) + \lambda(x_n, x_n, x_{n+1})] \\ &\leq \beta(\lambda(x_n, x_n, u)) \lambda(x_n, x_n, x_{n+1}) + \beta(\lambda(u, u, x_n)) k [2\lambda(u, u, x_n) + \lambda(Tu, Tu, x_{n+1}) + \lambda(x_n, x_n, x_{n+1})], \end{aligned}$$

for all $n \geq 1$. Therefore

$$\Lambda(x_{n+1}, x_{n+1}, Tu) \leq \frac{\beta(\lambda(x_n, x_n, u))(1 + k)\lambda(x_n, x_n, x_{n+1}) + 2k\beta(\lambda(x_n, x_n, u))\lambda(x_n, x_n, u)}{1 - k\beta(\lambda(x_n, x_n, u))} \rightarrow 0$$

as $n \rightarrow \infty$ and $\beta(\lambda(x_n, x_n, u)) = \max\{\beta_1(\lambda(x_n, x_n, u)), \beta_2(\lambda(x_n, x_n, u))\}$. Hence $u \in Tu$ and u is a fixed point of T .

Case-II: $\beta = \max\{\beta_1, \beta_2\} \in [\frac{1}{k+1}, \frac{1}{2})$. Then there exists $N \in \mathbb{N}$ such that $\beta\gamma^{N-1} \in (0, \frac{1}{k+1})$. \square

Example 3.6. Let $X = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ and $\lambda : X^3 \rightarrow [0, \infty)$ be defined by $\lambda(x, x, x) = 0$ and $\lambda(x, x, y) = \lambda(y, y, x)$ for all $x, y \in X$ with

$$\lambda(x, y, z) = \begin{cases} |n - m| & \text{if } x = \frac{1}{n} = y, z = \frac{1}{m} \text{ and } |n - m| > 1, \\ \frac{1}{3} & \text{if } x = \frac{1}{n} = y, z = \frac{1}{m} \text{ and } |n - m| = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then λ is a complete symmetric Branciari S_b -metric space for $k = 3$ but not an S -metric, since

$$\lambda\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right) = 2 > 2\lambda\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}\right) + \lambda\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{3}\right) = 1.$$

Let $T : X \rightarrow 2^X$ be given by

$$Tx = \begin{cases} \{\frac{1}{3}, \frac{1}{4}\} & \text{if } x = \frac{1}{2}, \\ \{\frac{1}{5}, \frac{1}{6}\} & \text{if } x \leq \frac{1}{3}. \end{cases}$$

Then T satisfies the contractive condition (3.19) for $\sum_{i=1}^2 \beta_i = \frac{1}{6}$ and thus T has a fixed point $x = \frac{1}{5}$ in X .

In this section we give some consequences of the main results presented above. Specifically, we apply our results to complete symmetric Branciari S_b -metric space endowed with a partial order.

3.2 Fixed point theorems for weakly increasing on X has the property α -regular Branciari S_b -metric space

In the following we provide set-valued versions of the preceding theorem. The results are related to those in ([9]). Let X be a topological space and \preceq be a partial order on X .

Definition 3.7. ([3]). Let A, B be two nonempty subsets of X , the relations between A and B are defines follows:
 (r₁) If for every $a \in A$, there exists $b \in B$ such that $a \preceq b$, then $A \prec_1 B$.
 (r₂) If for every $b \in B$ there exists $a \in A$, such that $a \preceq b$, then $A \prec_2 B$.
 (r₃) If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Definition 3.8. ([5], [6]). Let (X, \preceq) be a partially ordered set. Two mappings $f, g : X \rightarrow X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ hold for all $x \in X$.

Definition 3.9. ([2]) Let (X, \preceq) be a partially ordered set. Two mapping $F, G : X \rightarrow 2^X$ are said to be weakly increasing with respect to \prec_1 if for any $x \in X$ we have $Fx \prec_1 Gy$ for all $y \in Fx$ and $Gx \prec_1 Fy$ for all $y \in Gx$. Similarly two maps $F, G : X \rightarrow 2^X$ are said to be weakly increasing with respect to \prec_2 if for any $x \in X$ we have $Gy \prec_2 Fx$ for all $y \in Fx$ and $Fy \prec_2 Gx$ for all $y \in Gx$.

Now we give some examples.

Example 3.10. ([2]) Let $X = [1, \infty)$ and \leq be usual order on X . Consider two mappings $F, G : X \rightarrow 2^X$ defined by $Fx = [1, x^2]$ and $Gx = [1, 2x]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to \prec_2 but not \prec_1 . Indeed, since

$$Gy = [1, 2y] \prec_2 [1, x^2] = Fx \text{ for all } y \in Fx$$

and

$$Fy = [1, y^2] \prec_2 [1, 2x] = Gx \text{ for all } y \in Gx$$

so F and G are weakly increasing with respect to \prec_2 but $F2 = [1, 4] \not\prec_1 [1, 2] = G1$ for $1 \in F2$, so F and G are not weakly increasing with respect to \prec_1 .

Example 3.11. ([2]) Let $X = [1, \infty)$ and \leq be usual order on X . Consider two mappings $F, G : X \rightarrow 2^X$ defined by $Fx = [0, 1]$ and $Gx = [x, 1]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to \prec_1 but not \prec_2 . Indeed, since

$$Fx = [0, 1] \prec_1 [y, 1] = Gy \text{ for all } y \in Fx$$

and

$$Gx = [x, 1] \prec_1 [0, 1] = Fy \text{ for all } y \in Gx$$

so F and G are weakly increasing with respect to \prec_1 but $G1 = 1 \not\prec_2 0, 1 = F1$ for $1 \in F1$, so F and G are not weakly increasing with respect to \prec_2 .

Theorem 3.12. Let (X, \preceq, λ) be a partially ordered complete symmetric Branciari S_b -metric space. Suppose that $T : X \rightarrow 2^X$ are set-valued mappings and satisfies the following conditions:

(i)

$$H_\lambda(Tx, Tx, Ty) \leq \beta_1(\lambda(x, x, y))\psi(\lambda(x, x, y)) + \beta_2(\lambda(x, x, y))\psi(\Lambda(x, x, Tx)) + \beta_3(\lambda(x, x, y))\psi(\Lambda(y, y, Ty)) + \beta_4(\lambda(x, x, y)) \min\{\psi(\Lambda(x, x, Ty), \psi(\Lambda(y, y, Tx)))\}. \tag{3.20}$$

- (ii) T and i_x be a weakly increasing pair on X w.r.t \prec_1 ;
- (iii) there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$ and $\{x_0\} \prec_1 T^2x_0$;
- (iv) X has the property α -regular generalized metric space.

Then T has fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ converges to the fixed point of T .

Proof . Define the sequence x_n in X by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then $x^* = x_n$ is a fixed point for T . Using that the pair of set-valued mappings T and i_x is weakly increasing and by define $\alpha : X \times X \times X \rightarrow [0, +\infty)$

$$\alpha(x, x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{if } x \succ y. \end{cases}$$

It can be easily shown that the sequence x_n is nondecreasing w.r.t, \preceq i.e; and

$$\alpha_*(\{x_0\}, \{x_0\}, Tx_0) \geq 1 \Rightarrow \exists x_1 \in Tx_0, \text{ such that } \alpha(x_0, x_0, x_1) \geq 1 \Rightarrow x_0 \preceq x_1.$$

Now since T and i_x is weakly increasing with respect to \prec_1 , we have $x_1 \in Tx_0 \prec_1 Tx_1$. Thus there exist some $x_2 \in Tx_1$ such that $x_1 \preceq x_2$. Again since T and i_x is weakly increasing with respect to \prec_1 , we have $x_2 \in Tx_1 \prec_1 Tx_2$. Thus there exist some $x_3 \in Tx_2$ such that $x_2 \preceq x_3$. Continue this process, we will get a nondecreasing sequence $\{x_n\}_{n=1}^\infty$ which satisfies $x_{n+1} \in Tx_n, n = 0, 1, 2, 3, \dots$ We get

$$x_0 \preceq x_1 \preceq x_2 \preceq \dots \preceq x_n \preceq x_{n+1} \preceq x_{n+2} \preceq \dots .$$

In particular x_n, x_{n+j} are comparable for all $j \in \mathbb{N}$. $\alpha(x_n, x_{n+j}) \geq 1$ for all $n \in \mathbb{N}_0$ and by equation (2.1) and (2.3) we have $\lim_{n \rightarrow \infty} \lambda(x_n, x_n, x_{n+j}) = 0$. Following the proof of Theorem (3.1), we know that $\{x_n\}$ is a Cauchy sequence in the partially ordered complete symmetric Branciari S_b -metric space (X, \preceq, λ) . There exists $x^* \in X$ such that $\lim_{n \rightarrow +\infty} \lambda(x_n, x_n, x^*) = 0$. and condition (iv), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\alpha(x_{n_j+1}, x_{n_j+1}, x^*) \geq \alpha_*(Tx_{n_j}, Tx_{n_j}, Tx^*) \geq 1$ for all j . Thus,

$$\begin{aligned} \Lambda(x^*, x^*, Tx^*) &\leq k[\lambda(x^*, x^*, x_{n_j+1}) + \lambda(x^*, x^*, x_{n_j+1}) + \Lambda(Tx^*, Tx^*, x_{n_j+2}) + \lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2})] \\ &= 2k\lambda(x^*, x^*, x_{n_j+1}) + k\Lambda(Tx^*, Tx^*, x_{n_j+2}) + k\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) \\ &= 2k\lambda(x^*, x^*, x_{n_j+1}) + j\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) + k\Lambda(Tx^*, Tx^*, Tx_{n_j+1}) \\ &\leq 2k\lambda(x^*, x^*, x_{n_j+1}) + k\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) + kH_\lambda(Tx^*, Tx^*, Tx_{n_j+1}) \\ &\leq 2k\lambda(x^*, x^*, x_{n_j+1}) + k\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) + k[\beta_1(\lambda(x^*, x^*, x_{n_j+1}))\psi(\lambda(x^*, x^*, x_{n_j+1})) \\ &\quad + \beta_2(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) \\ &\quad + \beta_4(\lambda(x^*, x^*, x_{n_j+1}))\min\{\psi(\Lambda(x^*, x^*, Tx_{n_j+1})), \psi(\Lambda(x_{n_j+1}, x_{n_j+1}, Tx^*))\}] \\ &\leq 2k\lambda(x^*, x^*, x_{n_j+1}) + k\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) + k[\beta_1(\lambda(x^*, x^*, x_{n_j+1}))\psi(\lambda(x^*, x^*, x_{n_j+1})) \\ &\quad + \beta_2(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) \\ &\quad + \beta_4(\lambda(x^*, x^*, x_{n_j+1}))\min\{\psi(\lambda(x^*, x^*, x_{n_j+1})), \psi(\Lambda(x_{n_j+1}, x_{n_j+1}, Tx^*))\}] \\ &\leq k[\beta_2(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*))] \\ &\leq k[\beta_2(\lambda(x^*, x^*, x_{n_j+1}))\Lambda(x^*, x^*, Tx^*) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))\Lambda(x^*, x^*, Tx^*)] \\ &< k[\beta_2(\lambda(x^*, x^*, x_{n_j+1})) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))]\Lambda(x^*, x^*, Tx^*) \end{aligned}$$

for all $j \in \mathbb{N}$ and $k \geq 1$. Hence, $\Lambda(x^*, x^*, Tx^*) = 0$ and so $x^* \in Tx^*$. \square

Theorem 3.13. Let (X, \preceq, λ) be a partially ordered complete symmetric Branciari S_b -metric space. Suppose that $T : X \rightarrow 2^X$ are set-valued mappings and satisfies the following conditions:

(i)

$$H_\lambda(Tx, Tx, Ty) \leq \beta_1(\lambda(x, x, y))\psi(\lambda(x, x, y)) + \beta_2(\lambda(x, x, y))\psi(\Lambda(x, x, Tx)) + \beta_3(\lambda(x, x, y))\psi(\Lambda(y, y, Ty)) + \beta_4(\lambda(x, x, y))\min\{\psi(\Lambda(x, x, Ty)), \psi(\Lambda(y, y, Tx))\}. \quad (3.21)$$

(ii) T and i_x be a weakly increasing pair on X w.r.t \prec_2 ;

(iii) there exists $x_0 \in X$ such that $Tx_0 \prec_2 \{x_0\}$ and $T^2x_0 \prec_2 \{x_0\}$;

(iv) X has the property α -regular generalized metric space.

Then T has fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ converges to the fixed point of T .

Proof . Define the sequence x_n in X by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then $x^* = x_n$ is a fixed point for T . Using that the pair of set-valued mappings T and i_x is weakly increasing and by define $\alpha : X \times X \times X \rightarrow [0, +\infty)$

$$\alpha(x, x, y) = \begin{cases} 1 & \text{if } x \succeq y \\ 0 & \text{if } x \prec y. \end{cases}$$

It can be easily shown that the sequence x_n is non-increasing w.r.t, \preceq i.e; and

$$\alpha_*(Tx_0, Tx_0, \{x_0\}) \geq 1 \Rightarrow \exists x_1 \in Tx_0, \text{ such that } \alpha(x_1, x_1, x_0) \geq 1 \Rightarrow x_1 \preceq x_0;$$

Now since T and i_x are weakly increasing with respect to \prec_2 , we have $Tx_1 \prec_2 Tx_0$. Thus there exist some $x_2 \in Tx_1$ such that $x_2 \preceq x_1$. Again since T and i_x are weakly increasing with respect to \prec_2 , we have $Tx_2 \preceq_2 Tx_1$. Thus there exist some $x_3 \in Tx_2$ such that $x_3 \preceq x_2$. Continue this process, we will get a non-increasing sequence $\{x_n\}_{n=1}^\infty$ which satisfies $x_{n+1} \in Tx_n$ and $x_{n+2} \in Tx_{n+1}$, $n = 0, 1, 2, 3, \dots$ We get

$$x_0 \succeq x_1 \succeq x_2 \succeq \dots \succeq x_n \succeq x_{n+1} \succeq x_{n+2} \succeq \dots .$$

In particular x_{n+j}, x_n are comparable for all $k \in \mathbb{N}$, $\alpha(x_{n+j}, x_n) \geq 1$ for all $j \in \mathbb{N}_0$ and by equation (2.1) and (2.3) we have $\lim_{n \rightarrow \infty} \lambda(x_{n+j}, x_{n+j}, x_n) = 0$. Following the proof of Theorem (3.1), we know that $\{x_n\}$ is a Cauchy sequence in the partially ordered complete symmetric Branciari S_b -metric space. (X, \prec, λ) . There exists $x^* \in X$ such that $\lim_{n \rightarrow +\infty} \lambda(x_n, x_n, x^*) = 0$. Following the proof of Theorem (3.1), we know that $\{x_n\}$ is a Cauchy sequence in the partially ordered complete symmetric Branciari S_b -metric space (X, \preceq, λ) . There exists $x^* \in X$ such that $\lim_{n \rightarrow +\infty} \lambda(x_n, x_n, x^*) = 0$. and condition (iv), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\alpha(x_{n_j+1}, x_{n_j+1}, x^*) \geq \alpha_*(Tx_{n_j}, Tx_{n_j}, Tx^*) \geq 1$ for all j . Thus,

$$\begin{aligned} \Lambda(x^*, x^*, Tx^*) &\leq k[\lambda(x^*, x^*, x_{n_j+1}) + \lambda(x^*, x^*, x_{n_j+1}) + \Lambda(Tx^*, Tx^*, x_{n_j+2}) + \lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2})] \\ &= 2k\lambda(x^*, x^*, x_{n_j+1}) + k\Lambda(Tx^*, Tx^*, x_{n_j+2}) + k\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) \\ &= 2k\lambda(x^*, x^*, x_{n_j+1}) + j\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) + k\Lambda(Tx^*, Tx^*, Tx_{n_j+1}) \\ &\leq 2k\lambda(x^*, x^*, x_{n_j+1}) + k\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) + kH_\lambda(Tx^*, Tx^*, Tx_{n_j+1}) \\ &\leq 2k\lambda(x^*, x^*, x_{n_j+1}) + k\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) + k[\beta_1(\lambda(x^*, x^*, x_{n_j+1}))\psi(\lambda(x^*, x^*, x_{n_j+1})) \\ &\quad + \beta_2(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) \\ &\quad + \beta_4(\lambda(x^*, x^*, x_{n_j+1})) \min\{\psi(\Lambda(x^*, x^*, Tx_{n_j+1})), \psi(\Lambda(x_{n_j+1}, x_{n_j+1}, Tx^*))\}] \\ &\leq 2k\lambda(x^*, x^*, x_{n_j+1}) + k\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) + k[\beta_1(\lambda(x^*, x^*, x_{n_j+1}))\psi(\lambda(x^*, x^*, x_{n_j+1})) \\ &\quad + \beta_2(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) \\ &\quad + \beta_4(\lambda(x^*, x^*, x_{n_j+1})) \min\{\psi(\lambda(x^*, x^*, x_{n_j+2})), \psi(\Lambda(x_{n_j+1}, x_{n_j+1}, Tx^*))\}] \\ &\leq k[\beta_2(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*))] \\ &\leq k[\beta_2(\lambda(x^*, x^*, x_{n_j+1}))\Lambda(x^*, x^*, Tx^*) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))\Lambda(x^*, x^*, Tx^*)] \\ &< k[\beta_2(\lambda(x^*, x^*, x_{n_j+1})) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))]\Lambda(x^*, x^*, Tx^*) \end{aligned}$$

for all $j \in \mathbb{N}$ and $k \geq 1$. Hence, $\Lambda(x^*, x^*, Tx^*) = 0$ and so $x^* \in Tx^*$. \square

3.3 Coupled fixed point

Definition 3.14. ([10]) Let $F : X \times X \rightarrow X$ be a mapping, where (X, λ) is a symmetric Branciari S_b -metric space. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if

$$x = F(x, y) \quad y = F(y, x).$$

Note that if (x, y) is a coupled fixed point of F then (y, x) are coupled fixed points of F too. Our results are based on the following simple lemma.

Lemma 3.15. ([20]) Let $F : X \times X \rightarrow X$ be a given mapping. Define the mapping $T_F : X \times X \rightarrow X \times X$ by $T_F(x, y) = (F(x, y), F(y, x))$ for all $(x, y) \in X \times X$. Then, (x, y) is a coupled fixed point of F if and only if (x, y) is a fixed point of T_F .

Theorem 3.16. Let (X, λ) be a complete symmetric Branciari S_b -metric space and $F : X \times X \rightarrow X$ be a given mapping. Assume there are exist nondecreasing functions $\psi_i : [0, +\infty) \rightarrow [0, +\infty)$, $i = 1, 2$, such that $\psi = \psi_1 + \psi_2$ is convex, $\psi(0) = 0$, $\lim_{n \rightarrow +\infty} \psi^n(t) = 0$ for all $t > 0$, a function $\alpha : X^2 \times X^2 \times X^2 \rightarrow [0, +\infty)$ and satisfies the following conditions:

(i) for all $(x, y), (u, v) \in X \times X$,

$$\alpha((x, y), (x, y), (u, v))\lambda(F(x, y), F(x, y), F(u, v)) \leq \psi_1(\lambda(x, x, u)) + \psi_2(\lambda(y, y, v));$$

(ii) if for all $(x, x, y), (u, u, v) \in X \times X \times X$,

$$\alpha((x, x, y), (u, u, v)) \geq 1 \Rightarrow \alpha(T_F(x, y), T_F(x, y), T_F(u, v)) \geq 1;$$

(iii) there exists $(x_0, x_0, y_0) \in X \times X \times X$ such that

$$\alpha((x_0, x_0, y_0), T_F(x_0, x_0, y_0)) \geq 1 \quad \text{and} \quad \alpha(T_F(y_0, x_0), T_F(y_0, x_0), (y_0, x_0)) \geq 1; \quad \text{or}$$

(iii)* there exists $(x_0, x_0, y_0) \in X \times X \times X$ such that

$$\alpha(T_F(x_0, x_0, y_0), (x_0, x_0, y_0)) \geq 1 \quad \text{and} \quad \alpha((y_0, y_0, x_0), T_F(y_0, y_0, x_0)) \geq 1;$$

(iv) if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\alpha(x_n, x_{n+1}) \geq 1$, $\alpha(y_n, y_{n+1}) \geq 1$, for all n , $x_n \rightarrow x \in X$, $y_n \rightarrow y \in X$ as $n \rightarrow \infty$, then there are exist subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(x_{n_k}, x_{n_k}, x) \geq 1$ and $\alpha(y_{n_k}, y_{n_k}, y) \geq 1$ for all k ; or

(iv)* if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\alpha(x_{n+1}, x_{n+1}, x_n) \geq 1$, $\alpha(y_{n+1}, y_{n+1}, y_n) \geq 1$, for all n , $x_n \rightarrow x \in X$, $y_n \rightarrow y \in X$ as $n \rightarrow \infty$, then there are exist subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(x, x, x_{n_k}) \geq 1$ and $\alpha(y, y, y_{n_k}) \geq 1$ for all k .

Then, F has a coupled fixed point, that is, there exists $(x^*, x^*, y^*) \in X \times X \times X$ such that $x^* = F(x^*, x^*, y^*)$ and $y^* = F(y^*, y^*, x^*)$.

Proof . The idea consists in transporting the problem to the complete symmetric Branciari S_b -metric space (Y, δ) , where $Y = X \times X$ and $\delta((x, y), (x, y), (u, v)) = \lambda(x, x, u) + \lambda(y, y, v)$, for all $(x, y), (u, v) \in X \times X$. From condition (i), we have

$$\alpha((x, y), (x, y), (u, v))\lambda(F(x, y), F(x, y), F(u, v)) \leq \psi_1(\lambda(x, x, u)) + \psi_2(\lambda(y, y, v)) \quad (3.22)$$

and

$$\alpha((v, u), (v, u), (y, x))\lambda(F(v, u), F(v, u), F(y, x)) \leq \psi_1(\lambda(v, v, y)) + \psi_2(\lambda(u, u, x)) \quad (3.23)$$

for all $x, y, u, v \in X$. Adding (3.22) to (3.23), we get (note that ψ is super-additive)

$$\begin{aligned} \beta(\xi, \xi, \eta)\delta(T_F\xi, T_F\xi, T_F\eta) &\leq \psi_1(\lambda(\xi_1, \xi_1, \eta_1)) + \psi_2(\lambda(\xi_2, \xi_2, \eta_2)) + \psi_1(\lambda(\eta_2, \eta_2, \xi_2)) + \psi_2(\lambda(\eta_1, \eta_1, \xi_1)) \\ &\leq \psi_1(\lambda(\xi_1, \xi_1, \eta_1) + \lambda(\eta_2, \eta_2, \xi_2)) + \psi_2(\lambda(\xi_2, \xi_2, \eta_2) + \lambda(\eta_1, \eta_1, \xi_1)) \\ &= \psi(\lambda(\xi_1, \xi_1, \eta_1) + d(\eta_2, \eta_2, \xi_2)) \\ &= \psi(\delta(\xi, \xi, \eta)) \end{aligned} \quad (3.24)$$

for all $\xi = (\xi_1, \xi_1, \xi_2)$, $\eta = (\eta_1, \eta_1, \eta_2) \in Y$, where $\beta : Y \times Y \rightarrow [0, +\infty)$ is the function defined by

$$\beta((\xi_1, \xi_1, \xi_2), (\eta_1, \eta_1, \eta_2)) = \min\{\alpha((\xi_1, \xi_1, \xi_2), (\eta_1, \eta_1, \eta_2)), \alpha((\eta_2, \eta_2, \eta_1), (\xi_2, \xi_2, \xi_1))\} \quad (3.25)$$

and $T_F : Y \rightarrow Y$ is given by lemma (3.15). Let $\{(x_n, x_n, y_n)\}$ be a sequence in $Y = X \times X \times X$ such that

$$\beta((x_n, x_n, y_n), (x_{n+1}, x_{n+1}, y_{n+1})) \geq 1 \quad \text{and} \quad (x_n, x_n, y_n) \rightarrow (x, x, y)$$

as $n \rightarrow +\infty$. Using the condition (iv), we obtain easily there exists a subsequence $\{(x_{n_k}, x_{n_k}, y_{n_k})\}$ of $\{(x_n, x_n, y_n)\}$ such that $\beta((x_{n_k}, x_{n_k}, y_{n_k}), (x, x, y)) \geq 1$ for all k . Then all the hypotheses of Theorem (3.1) are satisfied. We deduce the existence of a fixed point of T_F that gives us from Lemma (3.15) the existence of a coupled fixed point of F . \square

3.4 Application

In this section, an existence result for a fractional integral equation

$$y(t) = \frac{f(t, x(t), y(t))}{\Gamma(\alpha)} \int_0^t \frac{h'(s)g(s, x(s), y(s))}{(h(t) - h(s))^{1-\alpha}} ds, \quad t \in [0, T], \quad (3.26)$$

where $T > 0$, $\alpha \in (0, 1)$ and $h : [0, T] \rightarrow \mathbb{R}$. We suppose that the following conditions are satisfied.

(i) The function $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

(ii) There exists an upper semi-continuous function $\psi_i : [0, +\infty) \rightarrow [0, +\infty)$, $i = 1, 2$, are nondecreasing functions such that $\psi = \psi_1 + \psi_2$ is convex, $\psi(0) = 0$, and $\lim_{n \rightarrow \infty} \psi^n(t) = 0$ for each $t > 0$,

$$|f(t, x(t), y(t)) - f(t, u(t), v(t))| \leq \psi_1(x - u) + \psi_2(y - v), \quad (3.27)$$

for all $(t, x(t), y(t))$ and $(t, u(t), v(t)) \in [0, T] \times \mathbb{R} \times \mathbb{R}$.

(iii) The function $h : [0, T] \rightarrow \mathbb{R}$ is C^1 and nondecreasing.

(iv) The function $g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exists a nondecreasing function $\omega : [0, \infty) \rightarrow [0, \infty)$ such that

$$|g(t, x(t), y(t))| \leq \omega(|(x(t), y(t))|) \quad (t, x(t), y(t)) \in [0, T] \times \mathbb{R} \times \mathbb{R}.$$

(v) There exists $r_0 > 0$ such that

$$(\psi(r_0) + F_0)\omega(r_0)(g(T) - g(0))^\alpha \leq r_0\Gamma(\alpha + 1) \text{ and } \frac{\omega(r_0)}{\Gamma(\alpha + 1)} \times (g(T) - g(0))^\alpha \leq 1 \quad (3.28)$$

where $F_0 = \frac{1}{2} \max\{|f(t, 0, 0)| : t \in [0, T]\}$.

Example 3.17. Let $X = C([0, T], \mathbb{R})$, $\lambda : X^3 \rightarrow \mathbb{R}_0^+$ and $\lambda(x, y, z) = |x(t) - y(t)| + |x(t) - z(t)| + |y(t) - z(t)|$ is a complete symmetric Branciari S_b -metric space for all $x, y, z \in X$ and $t \in [0, T]$.

$$\lambda(x, x, y) = |x(t) - x(t)| + |x(t) - y(t)| + |x(t) - y(t)| = \lambda(y, y, x) \quad (3.29)$$

$$|x - y| \leq |x - a| + |a - b| + |b - y| \quad (3.30)$$

$$|x - y| \leq |x - b| + |b - a| + |a - y|. \quad (3.31)$$

Adding (3.30) to (3.31), we get

$$\lambda(x, x, y) = |x - y| + |x - y| \leq |x - a| + |a - b| + |b - y| + |x - b| + |b - a| + |a - y| \quad (3.32)$$

$$\begin{aligned} &\leq 4k|x - a| + 2k|y - b| + 2k|a - b| + |x - b| \\ &= k[\lambda(x, x, a) + \lambda(x, x, a) + \lambda(y, y, b) + \lambda(a, a, b)] \end{aligned} \quad (3.33)$$

for all $x, y, z \in X$ and $a, b \in X \setminus \{x, y, z\}$, $a \neq b$, $k \geq 1$.

Theorem 3.18. Consider fractional integral equation (3.26) with $g \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is C^1 and nondecreasing in the third variables. Suppose that for $x \geq u$ and $y \geq v$, we have

$$0 \leq g(t, x, y) - g(t, u, v) \leq \frac{\Gamma(\alpha + 1)}{F_0(h(t) - h(s))^\alpha} (\psi_1(x - u) + \psi_2(y - v)). \quad (3.34)$$

Then the fractional integral equation (3.26) with the assumptions (i-v) has at least one solution $y^* \in C([0, T], \mathbb{R})$.

Proof . Let $X = C([0, T], \mathbb{R})$ is partially ordered if we define the following order relation in X :

$$x, y \in X, \quad x \leq y \Leftrightarrow x(t) \leq y(t), \text{ for all } t \in [0, T].$$

It is well-known that (X, λ) is a complete symmetric Branciari S_b -metric space with the metric

$$\lambda(x, y, z) = |x(t) - y(t)| + |x(t) - z(t)| + |y(t) - z(t)|.$$

Suppose $\{x_n\}$ is a nondecreasing sequence in X that converges to $x \in X$. Then for every $t \in [0, T]$, the sequence of the real numbers

$$x_1(t) \leq x_2(t) \leq \dots \leq x_n(t) \leq \dots,$$

converges to $x(t)$. Therefore, for all $t \in I$ and $n \in \mathbb{N}$, we have $x_n(t) \leq x(t)$. Hence $x_n \leq x$, for all $n \in \mathbb{N}$. Also, $X \times X$ is a partially ordered set if we define the following order relation in $X \times X$:

$$(x, y) \leq_r (u, v) \Leftrightarrow x(t) \leq u(t) \text{ and } y(t) \leq v(t), \text{ for all } t \in [0, T],$$

for all $(x, y), (u, v) \in X \times X$. For any $x, y \in X$, $\max\{x(t), u(t)\}$ for all $t \in [0, T]$ is in X and is the upper bound of x, u . Therefore, for every (x, y) and $(u, v) \in X \times X$ $\max\{x(t), u(t)\}, \max\{y(t), v(t)\}$, in $X \times X$ for all $t \in [0, T]$ is comparable to (x, y) and (u, v) . Define $F : X \times X \rightarrow X$ by

$$F(x, y)(t) = \frac{f(t, x(t), y(t))}{\Gamma(\alpha)} \int_0^t \frac{h'(s)g(s, x(s), y(s))}{(h(t) - h(s))^{1-\alpha}} ds, \text{ for all } t \in [0, T].$$

Since f is nondecreasing in the second and third of its variables then F is nondecreasing in each of its variables. Now, for $x \geq u$, $y \geq v$, that is, $x(t) \geq u(t)$, $y(t) \geq v(t)$ for all $t \in [0, T]$. we have

$$\begin{aligned}
 \lambda(F(x, y), F(x, y), F(u, v)) &= |F(x, y)(t) - F(x, y)(t)| + |F(x, y)(t) - F(u, v)(t)| + |F(x, y)(t) - F(u, v)(t)| \\
 &= 2 \left\{ \frac{f(t, x(t), y(t))}{\Gamma(\alpha)} \int_0^t \frac{h'(s)g(s, x(s), y(s))}{(h(t) - h(s))^{1-\alpha}} ds \right\} \\
 &\leq 2 \left\{ \frac{F_1}{\Gamma(\alpha)} \int_0^t \frac{h'(s)}{(h(t) - h(s))^{1-\alpha}} (g(s, x(s), y(s)) - g(s, u(s), v(s))) ds \right\} \\
 &\leq \left\{ \frac{F_0}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha + 1)}{F_0(h(t) - h(s))^\alpha} (\psi_1(x - u) + \psi_2(y - v)) \int_0^t \frac{h'(s)}{(h(t) - h(s))^{1-\alpha}} ds \right\} \\
 &\leq \left\{ \frac{F_0}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha + 1)}{F_1(h(t) - h(s))^\alpha} (\psi_1(x - u) + \psi_2(y - v)) \frac{(h(t) - h(0))^\alpha}{\alpha} \right\} \\
 &\leq \left\{ \frac{F_0}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha + 1)}{F_1(h(t) - h(s))^\alpha} \times \frac{(h(t) - h(0))^\alpha}{\alpha} (\psi_1(x - u) + \psi_2(y - v)) \right\} \\
 &\leq \psi_1(d(x, u)) + \psi_2(d(y, v)). \tag{3.35}
 \end{aligned}$$

Thus F satisfies the condition of Theorem (3.16). Now, let (x^*, y^*) be a coupled lower solution of the fractional integral equation problem (3.26) then we have $x^* \leq F(x^*, y^*)$ and $y^* \leq F(y^*, x^*)$. Then, Theorem (3.16) gives that F has a unique coupled fixed point (x^*, y^*) with $x^* = y^*$. Then $x^*(t)$ is the solution of the integral equation (3.26). \square

References

- [1] M. Abbas, T. Nazir, and S. Radenovic, *Common fixed points of four maps in partially ordered metric spaces*, Appl. Math. Lett. **24** (2011), 1520–1526.
- [2] I. Altun and V. Rakocevic, *Ordered cone metric spaces and fixed point results*, Comput. Math. Appl. **60** (2010), no. 5, 1145–1151.
- [3] M. Asadi, H. Soleimani, and S.M. Vaezpour, *An order on subsets of cone metric spaces and fixed points of set-valued contractions*, Fixed Point Theory Appl. **2009** (2009), Article ID 723203, 8 pages.
- [4] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Pub. Math. Debrecen **57** (2000), no. 1-2, 31–37.
- [5] B.C. Dhage, *Condensing mappings and applications to existence theorems for common solution of differential equations*, Bull. Korean Math. Soc. **36** (2000), no. 3, 565–578.
- [6] B.C. Dhage, D. Oregan, and R.P. Agarwal, *Common fixed theorems for a pair of countably condensing mappings in ordered Banach spaces*, J. Appl. Math Stoch. Anal. **16** (2003), no. 3, 243–248.
- [7] M. Eshaghi Gordji, M. Ramezani, M. De La Sen, and Y.J. Cho, *On orthogonal sets and Banach fixed point theorem*, Fixed Point Theory **18** (2017), no. 2, 569–578.
- [8] A. Farajzadeh, A. Kaewcharoen, and P. Lahawech, *On Fixed point theorems for (ξ, α, η) -Expansive mappings in complete metric spaces*, Int. J. Pure Appl. Math. **102** (2015), no. 1, 129.
- [9] Y. Feng and S. Liu, *Fixed point theorems for multi-valued increasing operators in partially ordered spaces*, Soochow J. Math. **30** (2004), no. 4, 461–469.
- [10] D. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal.: Theory Meth. Appl. **11** (1987), 623–632.
- [11] J. Hassanzadeh Asl, *Common fixed point theorems for α - ψ -contractive type mappings*, Int. J. Anal. **2013** (2013), 1- -7.
- [12] J. Hassanzadeh Asl, Sh. Rezapour, and N. Shahzad, *On fixed points of α - ψ -contractive multifunctions*, Fixed Point Theory Appl. **2012** (2012), 1–6.
- [13] Z. Kadelburg and S. Radenović, *On generalized metric spaces, A survey*, TWMS J. Pure Appl. Math. **5** (2014), 3–13.

-
- [14] W.A. Kirk and N. Shahzad, *Generalized metrics and Caristi's theorem*, Fixed Point Theory Appl. **2013** (2013), Article ID 129.
- [15] M.S. Khan, M. Swaleh, and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc. **30** (1984), no. 1, 1–9.
- [16] F. Lotfy and J. Hassanzadeh Asl, *Some fixed point theorems for α_* - ψ -common rational type mappings on generalized metric spaces with application to fractional integral equations*, Int. J. Nonlinear Anal. Appl. **12** (2021), no. 1, 245–260.
- [17] Y. Rohen, T. Došenović, and S. Radenović, *A fixed point theorems in S_b -metric spaces*, Filomat **31** (2017), no. 11, 3335–3346.
- [18] K. Royy and M. Saha, *Branciari S_b -metric space and related fixed point theorems with an application*, Appl. Math. E-Notes **22** (2022), 8–17.
- [19] S. Sedghi and N.V. Dung, *Fixed point theorems on S -metric spaces*, Mat. Vesnik **66** (2014), 113–124.
- [20] B. Samet, C. Vetro, and P. Vetro, *Fixed point theorems for α - ψ -contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165.
- [21] S. Sedghi, N. Shobe, and A. Aliouche, *A generalization of fixed point theorems in S -metric spaces*, Mat. Vesnik **64** (2012), 258–266.
- [22] N. Souayah and N. Mlaiki, *A fixed point theorem in S_b -metric spaces*, J. Math. Comput. Sci. **16** (2016), 131–139.