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Research Article

The Behavior of Heat Flow and Temperature under Quantum-Relativistic Conditions

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ABSTRACT

In this article, the effect and behaviour of ultra-high-velocity heat motion are described. The ultra-high velocity of heat conduction in a system composed of particles in gas form is viewed as the motion of particles that aligns with the principles of thermodynamics, the theory of relativity, and quantum physics theory. An alternative method for ultra-high velocity heat conduction has been developed and explained. This method has been achieved by using the Lowrance invariant of the microscopic environment in Makowski spacetime, hence both quantum and relativistic concepts are used, presenting a quantum-relativistic environment. The average number of field quanta has been obtained based on the relativistic effect, which is connected with the constituent mass of particles and determines the density matrix of a quantum oscillator. The presented relativistic heat conduction model is theoretically consistent with many important laws of physics and provides a more accurate representation of heat conduction in many technologically important situations.

1. Introduction

In fundamental particle physics, heat flow transfer rate σ refers to the amount of heat Q that is transferred per unit of time between two particles, objects, or regions, which are at different temperatures. It is a fundamental parameter in the study of heat transfer and is important in many scientific applications. Thermal radiation from gamma rays to radio waves, conduction, and convection are the main forms of heat flow [1]. Thermal radiation with the majority in the infrared region is the heat transfer through electromagnetic waves such as infrared radiation. It refers to the process by which thermal energy is emitted from the fundamental particles in the form of photonic waves. Thermal radiation has important applications in astrophysics, nanotechnology, and engineering [2]. During 2005-2017 by Zhang [2-5] and then in 2023 Vistman [6] this important subject, has been described under relativistic conditions. Based on their result, *"The law for the black-body radiation in the whole interval of its (black-body) movement speed has been obtained, i.e., from zero up to the speed of light in vacuum."* [5]. Hence, generally radiative heat flow transfer rates are proportional to differences in temperature to the higher power or fourth power as follows:

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$$\sigma \sim {T_1}^4 - {T_2}^4 \tag{1}$$

where, T_1 is temperature, T_2 is a reference temperature and σ is the conducted heat flow rate in the medium. Conduction is the heat flow transfer through a material by the movement of its molecules or electrons. It occurs in a material due to the direct contact between constituent particles at different temperatures, and from the particle with a higher temperature to the one with a lower temperature is transferred until both particles reach thermal equilibrium. The heat flow transfer rate to the temperature gradient in material (particle) using Fourier's law can be calculated. Mathematically heat flow transfer rate can be expressed in $1D$ space in the x-direction as

$$\sigma_x = -kA\nabla_x T \tag{2}$$

where T is temperature, $\nabla_x T$ is the thermal gradient, parameter A is the cross-section area through which heat flow is transferred, σ_x is the conducted heat flow rate in the x-direction, k is the thermal conductivity of the medium and would be constant or in more cases could depend on position and time. Equation (2) by experimental data and is widely used in engineering and physics to model heat transfer in physical systems has been validated. Convention is the heat flow transfer by the movement of a fluid, such as hadronic fluid, air or water, due to differences in density and temperature. Heat conduction between two particles with different temperatures, T_1 and T_2, in the direction from the particle at T_1 to the particle at T_2. Particles have an important role in thermal prosses; Hence, the physical behavior of particles and their effect on these processes have needed to be described. It may usually be calculated as follows

$$\sigma = h(T_1 - T_2) \tag{3}$$

where T_2 is a reference temperature and h is the convective heat flow transfer coefficient. Parameters h and k may additionally depend upon T_1 being very susceptible. While there may be cases wherein the connection between conductive and convective heat flow transfer rates and temperature differences $(T_1 - T_2)$ is approximately linear, this isn't normally proper for all applications and conditions. In many cases, the relationship between σ and T_1 variations may be quite complex. It can rely upon a selection of factors, including of the substances concerned, the geometry of the system, and the presence of different modes of heat flow transfers. For example, the convective heat flow transfer coefficient, which describes the heat flow rates between a fluid and a solid surface, is usually nonlinear and depends on factors v fluid speed and T_1. Particles, as recognized by theoretical and experimental scientists, are considered the smallest harmonic oscillating quantum system from both a physical and a mathematical perspective. The quantum harmonic oscillator is a theoretical version utilized in quantum mechanics to explain the behavior of a particle. The quantum harmonic oscillator is subject to a restoring force proportional to its displacement from a fixed point. The model assumes that the particle is moving in a $1D$ space and transferring heat flow. In this model, the energy of the quantum harmonic oscillator is quantized, meaning that the particle can have a certain discrete energy eigenvalue. The eigenenergy of the particle is associated with its natural frequency ω. It is a useful model for understanding the behavior of quantum systems that exhibit thermal motion. In quantum mechanics, the behavior of a particle as a quantum harmonic oscillator is described using the Schrödinger equation $H\Psi = E\Psi$, which is a fundamental equation of quantum mechanics that describes the evolution of the wavefunction of a particle over time or the evolution of a system within heat flow transfer. The heat flow is related to the particles which transfer temperature. Here, the relativistic thermal relation of heat flow can be defined. Therefore, the heat flow conduct based on all physical properties of systems (particles) has needed to be explained, i.e., we need quantum mechanics and special relativity theories and principles [3]. The main focus of a quantum-relativistic thermal system is the changes in the states that occur due to heat, energy, and mass transformation across its boundary conditions. To understand the properties and behavior of a quantum-relativistic thermal system, it is essential to discuss the relativistic and quantum characteristics of motion and heat flow. The relativity theory is concerned with particles moving at speeds comparable to the speed of light, while most of these physical systems have microscopic dimensions and are described by quantum mechanics theory. Therefore, quantum-relativistic effects cannot be ignored. The main quantum-relativistic properties of particles that propagate into the environment are described by photons with the $c = 3 \cdot 10^8 (m/s)$ speed of light in a vacuum (electromagnetic waves) or phonon with the speed of sound in the heat flow conduction. During the period 1959-1979, Stewart, Müller, and Israel developed a thermo-relativistic theory of heat flux as a part of the stress–energy-momentum tensor of order 2, ${T^{\mu\nu}}_{\mu\nu=0,1,2,3}$ in the Makowski spacetime. Where the contravariant components of the stress-energy tensor are: the time-time component $(T^{00} = \rho)$ is the density of relativistic mass

$E = mc^2$, the components $(T^{0\mu} = T^{\mu 0})_{\mu=1,2,3}$ are the momentum, and the components $(T^{\upsilon\mu})_{(\mu=\upsilon)=1,2,3}$ are the pressure [5]. In 2021 based on mass-energy relation in relativity limit, the heat flow behavior is physically described by a dual form. Heat flow in the relativistic presentation can be as bulk with the mass during its motion, and its conversion can be wide-spreading as energy. This duality manner of heat flow is named thermo-mass theory. A systematic reconstruction of thermo-mass theory within quantum mechanics theory can give us a quantum-relativistic relation for heat flow. So, the quantum-relativistic nano and multi-femto particle systems such as gas flux or hadronic flow can be described or approximated. The paper is organized as follows:

- In the present section, a quantum-relativistic thermal system has been considered.
- In Section 2, the general remarks about the relativistic transformation law for temperature and heat flux have been explained.
- In the next section, Section 3, the quantum mechanics relation of heat flow has been considered and linked to the relativistic behavior of a system.
- In the last section, the conclusion has been presented.

2. Relativistic Temperature Transformation

All The usual Cartesian Euclidian space and time around us is made of a *3D* x, y, z Euclidean space with the 1-dimensional t time. In this space and time coordinate is named after Newtonian mechanics, all physical law and interactions between objects are required to be invariant concerning transformation (to any Galilean invariant or Galilean relativity (Fig. 1)).

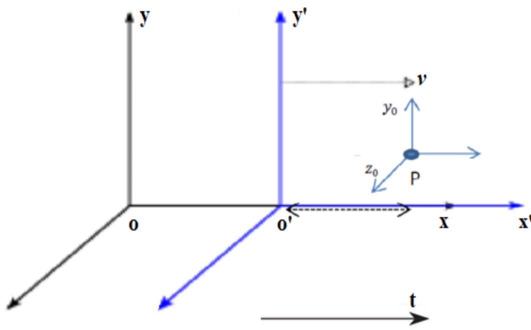

Fig. 1. Newtonian space-time 3D continuum

As we know in Newtonian mechanics where $v \ll c$, where c is speed of light, the motion of subject hold in all frames related to one another by a Galilean transformation i.e.:

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

For all inertial frames of reference, physical quantities remain the same within a space and time, and the distance, ds, between two points, defined by the Pythagorean theorem $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$.

As we know, ds is invariant i.e. ds remains unchanged irrespective of any transformation and rotation of the $3D$ coordinate system. In relativistic physics, when the speed of particles is comparable to the speed of light, the behavior of space and time are changed. In this situation, we link to the field of relativistic physics, in which particles move at speeds comparable to the speed of light. Einstein presented a new theory, which describes the interconnected of coordinates and time (space-like time) [5]. The Cartesian Euclidian spacetime that describes this property is made of a $4D$ t,x,y,z pseudo-Euclidean spacetime (Fig. 2) with the length dimension, and in 1908 it was named after Hermann Minkowski spacetime. Hence, as we know in relativistic mechanics where $v \approx c$, the motion of subject hold in all frames related to one another by a Lorentz invariance i.e.

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$y' = y$$
$$z' = z$$
$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Frames of reference (x, y, z, t) and (x',y',z',t') can be divided into two groups: inertial (relative motion with constant velocity) and non-inertial (accelerating, moving in curved paths, rotational motion with constant angular velocity, etc.). The term "Lorentz transformations" only refers to transformations between inertial frames, usually in the context of special relativity (Fig. 2).

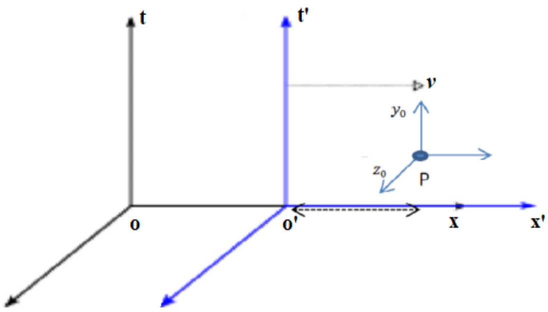

Fig. 2. Minkowski spacetime 4D continuum

A Minkowski spacetime continuum is a geometric representation of motions in a $4D$ t,x,y,z pseudo-Euclidean continuum, and all events in this continuum are called a world point, and the movement of some particles re called a world line (Fig. 3).

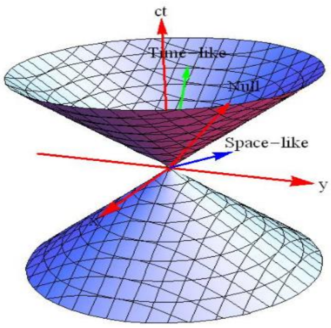

Fig. 3. The light cone, the absolute future, the absolute past, and elsewhere based on the Minkowski spacetime continuum.

The distance, ds, between two events of the Minkowski spacetime must be invariant in all inertial coordinates based on Lorentz invariant and transformation of two related inertial reference frames S and $S'$with the v relative velocity between the two frames.

$$dx^2 - c^2dt^2 = dx'^2 - c^2dt'^2 \quad (4)$$

The Lorentz invariant is as follows [5]

$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$
$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad (5)$$

Hence, the distance between two points in the Minkowski spacetime continuum is defined by the Pythagorean theorem is given by the formula

$$ds = \sqrt{(d\tau)^2 + (dx)^2 + (dy)^2 + (dz)^2} \quad (6)$$

where $d\tau = icdt$ is the imaginary component of real-time, c is the speed of light, and $i = \sqrt{-1}$ is the imaginary unit. This transformation results in a space-like time component $d\tau$, which is used in the Minkowski spacetime coordinate to describe the relation of time and space. In this theme, time and space are parts of the $4D$ spacetime, and one cannot distinguish between time and space components in the Minkowski spacetime. Time and space relations are described by the ds interval in the relativistic framework. According to the principles of Einstein's special theory of relativity, when two reference frames are moving relative to each other near the speed of light, they will measure different values for the length and the duration of events. This is due to the effects of time dilation $dt' = \sqrt{1-\frac{v^2}{c^2}}dt$ and length contraction, $dx = \sqrt{1-\frac{v^2}{c^2}}dx'$, which arises from the fact that the speed of light is constant in all inertial reference frames. As a particle moves with velocity v of an object comparable to the speed of light, its mass and heat transfer (energy) also increase by relation $Q = mc^2$. Also, the temperature T and heat flow transfer Q undergo relativistic changes. Hence, the relativistic corrections to the temperature heat flow transfer must be described. The thoughts of relativistic thermal transformation behavior have been explored in lots of research from 1905 to 1963. The relativistic temperature refers back to the temperature of a particle or object with the inertial reference frame S. as measured by observer who's in a different inertial reference frame S'. According to the principle of relativity, the laws of physics should be the same in all inertial reference frames. However, we know that temperature and heat flow have to measure by observer's relative velocity v, like the results of length contraction and time dilation. In 1907 Einstein, and Plank proposed that if a body of temperature T in its initial rest reference frame S, and a moving observer with velocity $v = const$ and close to the speed of light in its rest inertial reference frame S', the observer detects the temperature of the body in the S, $T' \prec T$, $Q' \prec Q$ by the formula [7]

$$T' = T' = T\sqrt{1-\frac{v^2}{c^2}}, \qquad Q' = Q\sqrt{1-\frac{v^2}{c^2}} \quad (7)$$

where T and Q are the temperature and heat flow transfer of the body in its rest reference S, T' and Q' are the temperature and heat flow transfer in the inertial reference S' as measured by the observer, and v is the relative velocity between S and S'[5]. Equation (7) predicts that the inertial reference S would appear colder, and the heat flow transfer would reduce to a moving observer in the inertial reference S'. In 1966 Landsberg determined that the temperature is a scalar value in the Lowrance invariant scale in both inertial reference S and S'. Hence, he describes that the temperature and the heat flow transfer are equivalent in both inertial reference S and S', i.e.,

$$T' = T, \qquad Q' = Q \quad (8)$$

In 1963, German physicist Heinrich Ott analyzed and verified the thermal relativistic transformation problems and defined the inverse of Planck's transformation laws. He explained that if an observer with a velocity v measures temperature and heat flow transfer of the continuity inertial reference frame outside itself, the observer would measure temperature with a hotter temperature and the heat flow with a larger value. Generally, the thermal relativistic transformation of a body or event inside the inertial reference frame S as measured by an observer in a different inertial reference S' is given temperature $T' \succ T$, $Q' \succ Q$ by the formula [7]

$$T' = \frac{T}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad Q' = \frac{Q}{\sqrt{1-\frac{v^2}{c^2}}} \tag{9}$$

where T and Q are the temperature and heat flow transfer of the body in its rest reference S, T' and Q' are the temperature and heat flow transfer in the inertial reference S' as measured by the observer, and v is the relative velocity between S and S'. Equation (9) presents the relativistic effect of temperature. The key result is the transformation laws for the thermal transfer and the heat flow, predicting that a body or system would appear hotter to a moving observer. It is an essential idea inside the theoretical framework of special relativity. It has implications for our understanding and knowledge of the thermal characteristics of particles and events in the relativistic limit.

3. Thermal Quantum-Relativistic Relation

In this section, a relation between a quantum system at its reference frame S with temperature T and the environment as an observer S' with the relative velocity v between them have been provided. The issue of quantum-relativistic transformation of temperature and heat flow using a moving quantum system in the inertial reference environment with relativistic velocity have been defined. Then, the quantum system (body) weakly interacting with the quantum scalar field φ in the thermal environment at the finite temperature $T = \frac{1}{k_b \beta}$ has been investigated. The total Hamiltonian of the system is $H = H_p + H_\varphi + H_I$, where H_p is the Hamiltonian of particles, H_φ is the Hamiltonian of free massless scalar field, and H_I is the interaction Hamiltonian and also the system as a simple quantum oscillator has been supposed [8].

The quantum oscillator is an important model that is used to explain the behavior of particles and physical phenomena. The total Hamiltonian of the quantum oscillator system is divided into the zeroth approximation $H_0\Psi = E_0\Psi$ (Hamiltonian of the ground state or vacuum for a free particle $H_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}x^2$) with mass m, oscillator frequency ω and charge q, that obtained analytically and perturbative parts H_I, to the zeroth approximation. H_I should be small corrections to the Zeroth approximation. We propose that the quantum oscillator system's wavefunction has a Gaussian behavior for short and large distances. This condition has been leaded us to define the expansion of these wavefunctions over the oscillator basis in Hilbert space. Based on this representation, the variables in the Schrodinger equation and properties and behavior of particles should be modified. Now for describing the temperature and heat flow transfer of the system and its relation in both inertial reference S and S', we have to define the master equation for the quantum harmonic oscillator due to describe the time evolution of the density matrix (density operator) $\hat{\rho} = \sum_i p_i |\varphi_i\rangle\langle\varphi_i| \rightarrow \int \varphi_i^*(x)\hat{\rho}\varphi_j(x)dx$, where p_i is the probability of finding the system in the i state, $|\varphi_i\rangle$ and $\langle\varphi_i|$ are the wave function for the i state. We know that $\hat{\rho} = \hat{\rho}^2$, $tr\hat{\rho} = 1$ [8]. The master equation is a fundamental tool in studding quantum oscillator systems, which can exchange energy and heat with the environment. The general form of this equation for the quantum harmonic oscillator without spin-orbit interactions, based on the annihilation and creation operators, reads

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \sum_i \gamma(\omega)\left(L_i\,\hat{\rho}L_i^+ - \frac{1}{2}\{L_i\,L_i^+,\hat{\rho}\}\right) \tag{10}$$

where L_i is a set of jump operators and describes how the environment acts on the system, $\{\hat{a},\hat{b}\} = \hat{a}\hat{b} + \hat{b}\hat{a}$ is anticommutator, $[\hat{a},\hat{b}] = \hat{a}\hat{b} - \hat{b}\hat{a}$ is commutator, and $\gamma(\omega)$ is the positive and constant decay rate $\gamma(\omega) = \omega^2\frac{2q^2}{3mc^3}$.

The Hamiltonian of a free harmonic oscillator in $1D$ spacetime coordinates is $H_0 = h\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) = h\omega\left(n + \frac{1}{2}\right)$ where $n = 0,1,2,\cdots$, and canonical variables in the form of annihilation $\hat{a}$ and creation $\hat{a}^+$ operators read [9]

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} - \hat{a}^+), \quad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} + \hat{a}^+) \tag{11}$$

and transition operators are related to the annihilation $\hat{a}$ and creation $\hat{a}^+$ operators as follows

$$\hat{A} = \sqrt{\frac{\hbar}{2m\omega}}\,\hat{a}, \qquad \hat{A}^+ = \sqrt{\frac{\hbar}{2m\omega}}\,\hat{a}^+ \tag{12}$$

Now, using equation (12)and insert into equation (13), the master equation becomes [11]

$$\frac{d\hat{\rho}}{dt}=-\frac{i}{\hbar}[\hat{H},\hat{\rho}]+\sum_{\omega>0}\gamma(\omega)\,(N(\omega)++1)$$
$$\times\left(\hat{A}\hat{\rho}\hat{A}^{+}-\frac{1}{2}\hat{A}^{+}\hat{A}\hat{\rho}-\frac{1}{2}\hat{\rho}\hat{A}^{+}\hat{A}\right)\tag{13}$$
$$+\sum_{\omega>0}\gamma(\omega)N(\omega)\,\hat{A}^{+}\hat{A}\hat{\rho}-\frac{1}{2}\hat{A}\hat{A}^{+}\hat{\rho}-\frac{1}{2}\hat{\rho}\hat{A}\hat{A}^{+}$$

where $N(\omega)$ is the average number of field quanta and reads

$$N(\omega)=\frac{k_bT\sqrt{1-\frac{v^2}{c^2}}}{2v\hbar\omega}log\left(\frac{1-exp\left(-\frac{\hbar\omega}{k_bT}\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\right)}{1-exp\left(-\frac{\hbar\omega}{k_bT}\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\right)}\right)\tag{14}$$

and, in the non-relativistic limit $v\to 0$, $N(\omega)=\frac{1}{e^{-\frac{\hbar\omega}{k_bT}}-1}$.

After a series of mathematical changes and the replacement of the annihilation $\hat{a}$ and creation $\hat{a}^{+}$ operators from equation (11), the master equation defines as follows [11]

$$\frac{d\hat{\rho}}{dt}=-i\omega[\hat{a}^{+}\hat{a}\;+H,\hat{\rho}]+\frac{\gamma(\omega)}{2m\omega}(N(\omega)+1)$$
$$\times\left(\hat{a}\hat{\rho}\hat{a}^{+}-\frac{1}{2}\hat{a}^{+}\hat{a}\hat{\rho}-\frac{1}{2}\hat{\rho}\hat{a}^{+}\hat{a}\right)\tag{15}$$
$$+\frac{\gamma(\omega)}{2m\omega}N(\omega)\left(\hat{a}^{+}\hat{a}\hat{\rho}-\frac{1}{2}\hat{a}\hat{a}^{+}\hat{\rho}-\frac{1}{2}\hat{\rho}\hat{a}a^{+}\right)$$

So, the master equation for the quantum oscillator system has been presented. The relativistic rule for the thermal behavior of a moving quantum oscillating system in the thermal environment will be defined in the next section. Therefore, the connection between quantum system and thermal environment behavior in relativistic limits will be presented.

4. Frequency and Thermal Relation of Quantum Harmonic Oscillator

We work and deal in quantum mechanics and quantum field theory with the perturbative behavior of Hamiltonian [10]. In most cases, we have to describe heat flow transformation and relativistic effects of quantum systems. When we try to begin calculation, the most important problem is the ground state of a system under heat flow transform and thermal interaction with the environment. Therefore, we need to present calculations based on the properties of thermodynamics, quantum mechanics, and quantum field theory [14]. One of the most methods in quantum field theory is the Normal ordering product of canonical and field operators. This method lets us define the ground and excited states, and then based on the master equation, we can approximate the quantum-relativistic effect of a system that exists of bound states of two particles under thermal and heat flow transformation. The Hamiltonian of two particles, which we choose as a quantum harmonic oscillator system $1D$ reference frame under the thermal environment within spherical symmetry potential $U(x)$ and supposed existence of a bound state, read [13]

$$H=\frac{\hat{p}^2}{2m}+U(x)\tag{16}$$

then rewrite it in the form [12]

$$H=\left(\frac{\hat{p}^2}{2m}+\frac{m\omega^2}{2}x^2\right)+\left(U(x)-\frac{m\omega^2}{2}x^2\right)\tag{17}$$

The equation (17) based on (11) can be defined as follows [11]

$$H=H_0+\varepsilon_0+H_I\tag{18}$$

where $\varepsilon_0=\frac{\omega}{2}$ is the vacuum state or ground state of Hamiltonian H, $H_0=\omega\hat{a}^{+}\hat{a}$ is the free oscillator Hamiltonian and

$$H_I=U(x)-\frac{m\omega^2}{2}x^2$$
$$=\int\left(\frac{dk}{2\pi}\right)\tilde{U}(k)e^{-\frac{k^2}{4\omega}}:e^{ikx}:-\frac{\omega^2}{2}\left(:q^2:+\frac{1}{2\omega}\right)$$

is interaction Hamiltonian and :∗: is the Normal ordering symbol. The wavefunction of the vacuum state with the normalization condition $\langle 0|0\rangle=1$, reads

$$|0\rangle=\left(\frac{\omega}{\pi}\right)^{1/4}e^{-\frac{\omega^2}{2}x^2}\tag{19}$$

This presentation of Hamiltonian in the Normal ordering form has a requirement that the interaction Hamiltonian should not contain terms in x^2. This condition can present the equation for the quantum harmonic oscillator frequency ω, which we will use to calculate of the master equation (15)[15]. To define frequency ω, we use the bound state Hamiltonian within the radial Schrödinger equation [13] function $H\Psi(r,\theta,\varphi)=E(\mu)\,\Re(r)\Theta(\theta)\Phi(\varphi)$ as follows ($\hbar=c=1$)

$$\int_0^\infty d^3r\Re(r)\left(-\frac{1}{2m_1}\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right]-\frac{1}{2m_2}\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right]+\frac{\ell(\ell+1)}{2\mu r^2}+U(x)-E(\mu)\right)\Re(r)=0\tag{20}$$
$$\frac{1}{m_1}+\frac{1}{m_2}=\frac{1}{\mu}$$

ℓ is the angular momentum quantum number, μ is the reduced mass of the bound state. by changing $r=q^{2\rho}$, $\Re(r)\to\Phi(q^{2\rho})$ and using (11), equation (20) reads

$$\left[\frac{1}{2\mu}(\hat{p}^2(q)) + 4\rho^2 q^{4\rho-2}(U(q) - E(\mu))\right]\Phi = 0 \qquad (21)$$

where $D = 2 + 2\rho + 4\rho\ell$, $\rho \succ 0$ and determines from the potential model and wavefunction at large or short distances. Now using a series of mathematical transformations from (20) and (18), one can define

$$\varepsilon_0 = \frac{d\omega}{4} + 4\mu\rho^2 q^{4\rho-2}(U(q) - E(\mu)) = 0 \qquad (22)$$

Parameter μ presents relativistic mass correction in the system based on quantum field theory and Feynman path integral form [14], and reads

$$\frac{1}{\mu} = \frac{1}{\sqrt{m_1^2 - 2\mu^2 \frac{dE(\mu)}{d\mu}}} + \frac{1}{\sqrt{m_2^2 - 2\mu^2 \frac{dE(\mu)}{d\mu}}} \qquad (23)$$

$E(\mu)$ is the eigenvalue of the radial Schrödinger equation (20), and $\mu_i = \sqrt{m_i^2 - 2\mu^2 \frac{dE(\mu)}{d\mu}}$ is the relativistic mass of particle constituent particles that moves with velocity v. Applying the main quantum oscillating condition, i.e. $\frac{d\varepsilon_0(E_n)}{d\omega} = 0$, (the bound state exists at the minimum of oscillator frequency and energy eigenvalue), we determine the oscillator frequency in R^N space [12]

$$\omega^2 + \int \left(\frac{dk}{2\pi}\right)^N \left(\frac{k^2}{N}\right) e^{\frac{-k^2}{4\omega}} \breve{U}(k^2) = 0 \qquad (24)$$

after a series of mathematical representations, we determine the frequency as follows

$$\omega = 2\int_0^\infty du \frac{u^D e^{-u}}{\Gamma(d+1)} \frac{d}{du} U\left(\left(\frac{u}{\omega}\right)^{1/2}\right) \qquad (25)$$

Our work provides a link between the relativistic motion of quantum systems in the thermal heat environment. We will define the thermal and relativistic form of the master equation concerning the heat flow transfer [15]. Now we determine the master equation [16] of a quantum system in the heat flow transfer environment at the relativistic limit for the Coulomb potential type $\rho = 1$ and equation (21) one can define

$$\varepsilon_0 = \frac{D\omega}{2} + 4\mu q^2\left(-\frac{\alpha_s}{q^2} - E(\mu)\right) = 0 \qquad (26)$$

and then the ground state ($\ell = 0$) frequency for the bound states of particles with the rest masses $m_1 = m_2 = m$, defined $\omega = 2\mu\alpha_s$, hence the master equation for the moving quantum bound state system reads

$$\frac{d\hat{\rho}}{dt} = -i2\mu\alpha_s[\hat{a}^+\hat{a} + \varepsilon_0, \hat{\rho}] + \frac{\gamma(\omega)}{4\mu^2\alpha_s}(N(\omega) + 1)\left(\hat{a}\hat{\rho}\hat{a}^+ - \frac{1}{2}\hat{a}^+\hat{a}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{a}^+\hat{a}\right) + \frac{\gamma(\omega)}{4\mu^2\alpha_s}N(\omega)\left(\hat{a}^+\hat{a}\hat{\rho} - \frac{1}{2}\hat{a}\hat{a}^+\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{a}a^+\right) \qquad (27)$$

The parameter $N(\omega)$ and μ present the quantum-relativistic behavior of the system based on heat flow transfer in the inertial reference S' as measured by the observer, and v is the relative velocity between S and S'. Then $N(\omega)$ and μ, ($\mu_1 = \mu_2 = 2\mu$) is determined as follows

$$N(\omega) = \frac{k_b T\sqrt{1-v^2}}{2\mu_1\alpha_s v} log\left(\frac{1 - exp\left(-\frac{\mu_1\alpha_s}{k_b T}\sqrt{\frac{1+v}{1-v}}\right)}{1 - exp\left(-\frac{\mu_1\alpha_s}{k_b T}\sqrt{\frac{1-v}{1+v}}\right)}\right) \qquad (28)$$

Using equation (27) for a free quantum oscillator with the frequency ω_0 and mass m, one can calculate diagonal matrix elements of $\hat{\rho}$ in the nonrelativistic limit $v \to 0$, as follows [14]

$$\hat{\rho} = \begin{pmatrix} \rho_{00} & \cdots & \rho_{03} \\ \vdots & \ddots & \vdots \\ \rho_{10} & \cdots & \rho_{33} \end{pmatrix} = \left(e^{-\frac{\hbar\omega_0}{k_b T}} - 1\right) e^{\frac{\hbar\omega_0}{k_b T}n} \delta_{nn'}; \quad n, n' = 0,1,2,3 \qquad (29)$$

5. Conclusions

The link between two systems, in which one of the two systems with respect to the other moves, with relativistic velocity u has been studied. One of the systems is the bound state of two equal particles, and the other system can be supposed as an environment with an observer. The thermal relativistic transformation of a system as a quantum harmonic oscillator in the inertial reference frame S as measured by an observer in a different inertial reference S' that moves with relativistic velocity u concerning each other $T' = \gamma T > T$. In conclusion, a helpful method for defining the master equation of a quantum harmonic oscillator that describes heat flow transfer and temperature relation of two inertial systems with relativistic relative velocity v have been presented. Then the quantum harmonic frequency ω based on the quantum field theory presentation and the master equation due to calculate the matrix density have been explained. The quantum harmonic oscillator as a body with a finite temperature has been considered and then the bound state system (body) of two equal particles at a relatively strong coupling environment under Otto's formula (temperature and heat transport) has been explained and defined. The relativistic effect directly includes the constituent mass of particles $\mu_1$and in the number of field quanta $N(\omega)$ has been determined and based on this result the equation of density matrix of a quantum oscillator has been explained. Based on the relativistic heat conduction model, Eq. (28-29), that are consistent with many laws of physics, and is a more accurate representation of heat conduction in many technologically important situations such as astrophysics, cosmology, condensed matter physics, reactor engineering, thermal engineering, and high-energy laser engineering, the achievements claimed by this paper are:

1. Proper the relativistic corrections of mass in the of heat conduction theory.
2. Determination the frequency of system under heat conduction at ultra-high velocities.
3. Calculation the matrix elements with the relativistic conditions of heat conduction.
4. Assimilation with thermodynamics, quantum mechanics and relativity.

Nomenclature

T	Temperature [K]
H	Hamiltonian [MeV]
m	Mass [MeV]
E	Energy eigenvalue [MeV]
ω	Quantum oscillator frequency [MeV]
μ	The reduced mass of system [MeV]
μ_i	Constituent mass of particle [MeV]
$N(\omega)$	The average number of field quanta
α_s	Coupling constant
ℓ	Orbital quantum number
$\Gamma(x)$	Gamma function
$\hat{a}$	Annihilation
a^+	Creation operator
$\hbar$	Reduced Planck constant [J.s]
k_b	Boltzmann constant [J/K]
$:*:$	Normal ordering symbol
m	Rest mass [MeV]
Q	Heat [J]

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Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this article.

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