Int. J. Nonlinear Anal. Appl. 16 (2025) 6, 1–8
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.33464.4987

Chebyshev-type fractional inequalities via (k, ψ)-Hilfer operator

Bouharket Benaissa[a,*], Noureddine Azzouz[b,c]

[a] Laboratory of Informatics and Mathematics, Faculty of Material Sciences, University of Tiaret, Algeria
[b] Faculty of Sciences, University Center Nour Bachir, El Bayadh, Algeria
[c] University Belhadj Bouchaib Ain Temouchent, Ain Temouchent, Algeria

(Communicated by Mugur Alexandru Acu)

Abstract

In this paper, we use the (k, ψ)-Hilfer fractional integral of functions with respect to another function to generalize Chebyshev-type fractional integral inequalities. Some inequalities involving (k, ψ)-Hilfer fractional integrals are also to be proved.

Keywords: Chebyshev inequality, Hilfer operator, Fractional operator
2020 MSC: 26D10, 26A33, 26D15

1 Introduction

In applied sciences, integral inequalities are incredibly important. Furthermore, the study of integral inequalities using fractional integration theory has become extremely important; for specific applications, see ([6], [11]). In this paper, we will examine the Chebyshev inequality.

$$T(f,\, g)(x) \geq 0, \tag{1.1}$$

introduced in [3] for the following so-called Chebyshev functional

$$T(f,\, g)(x) = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\,\frac{1}{b-a}\int_a^b g(x)\,dx, \tag{1.2}$$

where f and g are two integrable functions and synchronous on $[a, b]$, that is, for all $x, y \in [a, b]$

$$(f(x) - f(y))(g(x) - g(y)) \geq 0. \tag{1.3}$$

Over the previous decade, several authors established different new integral inequalities of type 1.1 using various fractional integral operators, See ([10, 5, 8, 4, 1, 12, 13]). In particular, Belarbi and Dahmani [2] developed the following results about Chebyshev inequality using the Riemann-Liouville fractional integral operator defined by

$$\mathrm{I}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt.$$

*Corresponding author
Email addresses: bouharket.benaissa@univ-tiaret.dz (Bouharket Benaissa), n.azzouz@cu-elbayadh.dz (Noureddine Azzouz)

Received: March 2024 *Accepted:* April 2024

Theorem 1.1. Let f and g be two synchronous functions on $[0,+\infty[$, then for all $x>0$, $\alpha>0$.

$$\mathrm{I}^{\alpha}(f\,g)(x)\geq\frac{\Gamma(\alpha+1)}{x^{\alpha}}\mathrm{I}^{\alpha}f(x)\,\mathrm{I}^{\alpha}g(x). \tag{1.4}$$

Theorem 1.2. Let f and g be two synchronous functions on $[0,+\infty[$, then for all $x>0$, $\alpha>0$ and $\alpha>0$, we have

$$\frac{x^{\beta}}{\Gamma(\beta+1)}\,\mathrm{I}^{\alpha}(f\,g)(x)+\frac{x^{\alpha}}{\Gamma(\alpha+1)}\,\mathrm{I}^{\beta}(f\,g)(x)\geq\,\mathrm{I}^{\alpha}f(x)\,{}_{a^{+}}\mathrm{I}^{\beta}g(x)+\,\mathrm{I}^{\beta}f(x)\,\mathrm{I}^{\alpha}g(x). \tag{1.5}$$

Theorem 1.3. Let $i=1,2,...,m$ and f_i be n positive and increasing on $[a,b]$, then for all integer $m\geq 1$ we have

$$\mathrm{I}^{\alpha}\left(\prod_{i=1}^{m}f_i\right)(x)\geq\left(\frac{\Gamma(\alpha+1)}{x^{\alpha}}\right)^{m-1}\prod_{i=1}^{m}\mathrm{I}^{\alpha}\left(f_i\right)(x). \tag{1.6}$$

On the other hand, the (k,ψ)-Hilfer integral fractional operators are defined as follows [7]:

Definition 1.4. Let $k>0$ and ψ be an increasing positive monotone function on $[a;b]$ such that ψ' is continuous on (a,b). The left and right-sided (k,ψ)-Hilfer fractional integral operators of a function f with respect to the function ψ on $[a,b]$ are defined respectively as:

$$_{a^{+}}\mathrm{J}_{k}^{\alpha,\psi}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}f(t)dt,\quad a<x\leq b. \tag{1.7}$$

$$_{b^{-}}\mathrm{J}_{k}^{\alpha,\psi}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b\psi'(t)(\psi(t)-\psi(x))^{\frac{\alpha}{k}-1}f(t)dt,\quad a\leq x<b.$$

Our aim in this study is to establish Chebyshev fractional inequalities involving the (k,ψ)-Hilfer integral fractional operator defined in (1.7) with two parameters. Chebyshev fractional inequalities will be derived according to specific choices of the function ψ. This paper is organized as follows: in Section 2, we present some preliminary results; in Section 3, the main results are stated and proved; and in Section 4, some derived Chebyshev fractional inequalities are given.

2 Preliminaries

The space $L_p^W[a,b]$ of all real-valued Lebesgue measurable functions $f\neq 0$ on $[a,b]$ with norm condition :

$$\parallel f\parallel_p^W=\left(\int_a^b\mid f(x)\mid^p W(x)dx\right)^{\frac{1}{p}}<\infty,\ p\geq 1,$$

is known as weighted Lebesgue space, where W is a weighted function (positive and measurable).

1. Put $p=1$ and $W\equiv 1$, the space $L_p^W[a,b]$ reduces to the classical Lebesgue space $L\left([a,b]\right)$.
2. Choose $p=1$ and $W(x)=\psi\,'(x)$, we get

$$L_{X_{\psi}\,'}\left([a,b]\right)=\left\{f:\parallel f\parallel_{X_{\psi}\,'}=\int_a^b\mid f(x)\mid\psi\,'(x)dx\ <\infty\right\}. \tag{2.1}$$

In the next theorem, we show that the (k,ψ)-Hilfer integral fractional operators are well defined on $L_{X_{\psi}\,'}\left([a,b]\right)$.

Theorem 2.1. For all functions $f\in L_{X_{\psi}\,'}\left([a,b]\right)$ we have ${}_{a^{+}}\mathrm{J}_{k}^{\alpha,\psi}f\ \in L_{X_{\psi}\,'}\left([a,b]\right)$ and ${}_{b^{-}}\mathrm{J}_{k}^{\alpha,\psi}f\ \in L_{X_{\psi}\,'}\left([a,b]\right)$. Moreover the operators ${}_{a^{+}}\mathrm{J}_{k}^{\alpha,\psi}$ and ${}_{b^{-}}\mathrm{J}_{k}^{\alpha,\psi}$ are bounded on $L_{X_{\psi}\,'}\left([a,b]\right)$. Explicitly

$$\left\|{}_{a^{+}}\mathrm{J}_{k}^{\alpha,\psi}f\right\|_{X_{\psi}\,'}\leq C\,\|f\|_{X_{\psi}\,'}\,,\qquad\left\|{}_{b^{-}}\mathrm{J}_{k}^{\alpha,\psi}f\right\|_{X_{\psi}\,'}\leq C\,\|f\|_{X_{\psi}\,'}\,,$$

where $C=\dfrac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}$.

Proof . Let $f \in L_{X_{\psi'}}([a,b])$ then, using Fubini's Theorem we get

$$\begin{aligned}\left\|{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f\right\|_{X_{\psi'}} &= \int_a^b |{}_{a^+}\,\mathrm{J}_k^{\alpha,\psi} f(x) \mid \psi'(x)\,dx \\ &\leq \frac{1}{k\,\Gamma_k(\alpha)} \int_a^b \int_a^x |f(s)|\,\psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1}\psi'(x)\,ds\,dx \\ &= \frac{1}{k\,\Gamma_k(\alpha)} \int_a^b |f(s)| \left(\int_s^b (\psi(x)-\psi(s))^{\frac{\alpha}{k}-1}\psi'(x)\,dx\right)\psi'(s)\,ds \\ &= \frac{1}{\alpha\,\Gamma_k(\alpha)} \int_a^b |f(s)|\,(\psi(b)-\psi(s))^{\frac{\alpha}{k}}\,\psi'(s)\,ds \\ &\leq \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \int_a^b |f(s)|\,\psi'(s)\,ds \\ &= C\,\|f\|_{X_{\psi'}}\,.\end{aligned}$$

Similarly, we establish that

$$\left\|{}_{b^-}\mathrm{J}_k^{\alpha,\psi} f\right\|_{X_{\psi'}} \leq \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\ \|f\|_{X_{\psi'}}\,.$$

□

Remark 2.2. If the function f is continuous, then

$$\begin{aligned}\| f \|_{X_{\psi'}} &= \int_a^b |\, f(x) \mid \psi'(x)dx \\ &\leq \max_{a\leq x\leq b} |\, f(x) \mid \int_a^b \psi'(x)dx \\ &\leq (\psi(b)-\psi(a))\ \max_{a\leq x\leq b} |\, f(x) \mid \quad < \infty.\end{aligned}$$

Thus, continued functions belong to the space $L_{X_{\psi'}}([a,b])$. In all that follows, we will assume that the considered functions are in $L_{X_{\psi'}}([a,b])$.

The (k,ψ)-Hilfer integral fractional operators are notable for their ability to generate specific types of k-fractional integrals depending on the choice of the function ψ.

1. Taking $\psi(\tau)=\tau$, the (k,ψ)-Hilfer yields to the k- Riemann-Liouville fractional integral operator of order $\alpha>0$

$$_{a^+}\mathcal{RL}_k^{\alpha} f(x) = \frac{1}{k\,\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt,, \quad x>a,$$

$$_{b^-}\mathcal{RL}_k^{\alpha} f(x) = \frac{1}{k\,\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt,, \quad x<b.$$

2. Using $\psi(\tau)=\ln\tau$, the (k,ψ)-Hilfer reduces to the k-Hadamard fractional integral operator of order $\alpha>0$

$$_{a^+}\mathcal{H}_k^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x \left(\ln\frac{x}{t}\right)^{\frac{\alpha}{k}-1} f(t)\frac{dt}{t}, \quad x>a>1,$$

$$_{b^-}\mathcal{H}_k^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b \left(\ln\frac{t}{x}\right)^{\frac{\alpha}{k}-1} f(t)\frac{dt}{t}, \quad 1<x<b.$$

3. Putting $\psi(\tau)=\frac{\tau^{\rho+1}}{\rho+1}$ where $\rho>0$, the (k,ψ)-Hilfer makes it similar to the k-Katugompola fractional integral operator of order $\alpha>0$

$$_{a^+}\mathcal{K}_k^{\alpha} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)} \int_a^x \left(x^{\rho+1}-t^{\rho+1}\right)^{\frac{\alpha}{k}-1} f(t)t^{\rho}dt,\ x>a, \tag{2.2}$$

$$_{b^-}\mathcal{K}_k^{\alpha} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)} \int_x^b \left(t^{\rho+1}-x^{\rho+1}\right)^{\frac{\alpha}{k}-1} f(t)t^{\rho}dt,\ x<b.$$

4. Setting $\psi(\tau)=\frac{(\tau-a)^\theta}{\theta}$ ($\psi(\tau)=-\frac{(b-\tau)^\theta}{\theta}$) respectively where $\theta>0$, the left sided (right sided) (k,ψ)-Hilfer respectively is reduced to the k-fractional conformable integral operator of order $\alpha>0$ [9].

$$
{}_{a^+}\mathcal{C}_k^\alpha f(x)=\frac{\theta^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_a^x\left((x-a)^\theta-(t-a)^\theta\right)^{\frac{\alpha}{k}-1}\frac{f(t)}{(t-a)^{1-\theta}}dt,\ x>a, \tag{2.3}
$$

$$
{}_{b^-}\mathcal{C}_k^\alpha f(x)=\frac{\theta^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_x^b\left((b-x)^\theta-(b-t)^\theta\right)^{\frac{\alpha}{k}-1}\frac{f(t)}{(b-t)^{1-\theta}}dt,\ x<b.
$$

3 Main results

The Chebyshev-type inequalities are presented below with the (k,ψ)-Hilfer operator.

Theorem 3.1. Let $\alpha,\beta,\,k>0$, f and g be two synchronous functions on $[a,b]$ and ψ be a positive increasing function on $[a,b]$ having a continuous derivative ψ' on (a,b), then for $x>a$ the following inequalities hold:

$$
\begin{aligned}
&\frac{(\psi(x)-\psi(a))^{\frac{\beta}{k}}}{\Gamma_k(\alpha+k)}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(f\,g)(x)+\frac{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_{a^+}\mathrm{J}_k^{\beta,\psi}(f\,g)(x)\\
&\geq\ {}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_k^{\beta,\psi}g(x)+\ {}_{a^+}\mathrm{J}_k^{\beta,\psi}f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}g(x).
\end{aligned} \tag{3.1}
$$

and

$$
{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(f\,g)(x)\geq\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}g(x). \tag{3.2}
$$

Proof . Let f and g be two synchronous functions on $[a,b]$, then according to (1.3) we have for all $t,s\in[a,b]$

$$
f(t)g(t)+f(s)g(s)\geq f(t)g(s)+f(s)g(t). \tag{3.3}
$$

Multiplying by $\frac{\psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}}{k\,\Gamma_k(\alpha)}$ and integrating with respect to t over (a,x), we get

$$
{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)g(x)+f(s)g(s)\,\frac{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\geq g(s)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)+f(s)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}g(x). \tag{3.4}
$$

Multiplying inequality (3.4) by $\frac{\psi'(s)(\psi(x)-\psi(s))^{\frac{\beta}{k}-1}}{k\,\Gamma_k(\beta)}$ and integrating with respect to s over (a,x), we get the desired inequality (3.1).

Putting $\beta=\alpha$ in the inequality (3.1), we'll get the required inequality (3.2). □

Remark 3.2. We present some special cases of the above Theorem 3.1.

1. By putting $k=1$, $b=+\infty$ and $a=0$, we obtain Theorem 7 and Theorem 6 in [10].
2. If we choose $\psi(x)=\ln x$, we obtain Theorem 3 in [5].
3. Putting $k=1$, $a=0$ and $\psi(\tau)=\frac{(\tau)^{\xi+\eta}}{\xi+\eta}$ gives Theorems 2.2 and 2.1 in [8].
4. Taking $k=1$, $a=0$ and $\psi(\tau)=\frac{(\tau)^\theta}{\theta}$ yields Theorems 6 and 5 in [13].

Corollary 3.3. Let $i=1,2,...,m$ and $f,\ f_i\in L_{X^1_{\psi'}}[a,b]$. If the functions f and f_i are positive and increasing on $[a,b]$, then for all integer $m\geq1$ we have

$$
{}_{a^+}\mathrm{J}_k^{\alpha,\psi}\prod_{i=1}^m f_i(x)\geq\left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1}\left(\prod_{i=1}^m {}_{a^+}\mathrm{J}_k^{\alpha,\psi}f_i(x)\right), \tag{3.5}
$$

and

$$
{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f^m(x)\geq\left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1}\left({}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)\right)^m. \tag{3.6}
$$

Proof . For $m=1$ the equality holds. Let $m\neq 1$, by using the inequality (3.2) with $g=\prod_{i=2}^{m} f_i$, we obtain

$$\begin{aligned}
{}_{a^+}\mathrm{J}_k^{\alpha,\psi}\prod_{i=1}^{m} f_i(x) \geq& \frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f_1(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}\prod_{i=2}^{m} f_i(x)\\
\geq& \frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f_1(x)\left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f_2(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}\prod_{i=3}^{m} f_i(x)\right)\\
=& \left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^2\left(\prod_{i=1}^{2}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f_i(x)\right){}_{a^+}\mathrm{J}_k^{\alpha,\psi}\prod_{i=3}^{m} f_i(x)\\
\geq& \left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^3\left(\prod_{i=1}^{3}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f_i(x)\right){}_{a^+}\mathrm{J}_k^{\alpha,\psi}\prod_{i=4}^{m} f_i(x)\\
&\vdots\\
\geq& \left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1}\left(\prod_{i=1}^{m-1}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f_i(x)\right){}_{a^+}\mathrm{J}_k^{\alpha,\psi} f_m(x)\\
=& \left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1}\left(\prod_{i=1}^{m}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f_i(x)\right),
\end{aligned}$$

which yields to the desired inequality (3.5). Putting $f_1=f_2=\cdots=f_m=f$ in the inequality (3.5), we get the inequality (3.6). □

Remark 3.4. We present some particular cases of Corollary 3.3.

1. By putting $k=1$, $b=+\infty$, and $a=0$ in the above Corollary, we obtain Theorem 8 and Corollary 2 in [10].
2. Setting $k=1$, $a=0$ and $\psi(\tau)=\frac{(\tau)^{\xi+\eta}}{\xi+\eta}$ gives Theorem 2.3 in [8].
3. Taking $k=1$, $a=0$ and $\psi(\tau)=\frac{(\tau)^{\theta}}{\theta}$ yields Theorem 7 in [13].

Corollary 3.5. Let f and g be two functions defined on $L_{X_\psi}{}'([a,b])$, such that f is monotone, g is differentiable on (a,b) and there exists a real numbers $m:=\inf_{x\in[a,b]} g'(x)$ and $M:=\sup_{x\in[a,b]} g'(x)$.

- If f is an increasing function, then

$$\begin{aligned}
{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(f\,g)(x)\geq& \frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} g(x) - m\,\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} I_d(x)\\
&+ m\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(I_d f)(x). \qquad (3.7)
\end{aligned}$$

- If f is a decreasing function, then

$$\begin{aligned}
{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(f\,g)(x)\geq& \frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} g(x) - M\,\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} I_d(x)\\
&+ M\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(I_d f)(x), \qquad (3.8)
\end{aligned}$$

where $I_d(x)$ is the identity function.

Proof . Taking $G(x)=g(x)-m\,x$, thus G is differentiable and increasing on (a,b). By using the inequality (3.2), we get

$$\begin{aligned}
{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(f\,G)(x)\geq& \frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(g-m\,x)(x)\\
=& \frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} g(x) - m\,\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} I_d(x), \qquad (3.9)
\end{aligned}$$

we also have

$$ {}_{a^+}\mathrm{J}_k^{\alpha,\psi} f(g - mI_d)(x) = {}_{a^+}\mathrm{J}_k^{\alpha,\psi}(fg)(x) - m\, {}_{a^+}\mathrm{J}_k^{\alpha,\psi}(I_d f)(x). \tag{3.10} $$

Combining inequalities (3.9) and (3.10), we obtain the required inequality (3.7). Considering the situation of a decreasing function f and taking $G(x) = g(x) - M\,x$, we will obtain the inequality (3.8) by sketching the proof of the inequality (3.7). □

Remark 3.6. We give some specific results of Corollary 3.5.

1. Using $k = 1$, $b = +\infty$ and $a = 0$, we obtain Theorem 10 in [10].
2. Taking $k = 1$, $a = 0$ and $\psi(\tau) = \frac{(\tau)^{\xi+\eta}}{\xi+\eta}$ yields Theorem 2.4 in [8].
3. Setting $k = 1$, $a = 0$ and $\psi(\tau) = \frac{(\tau)^{\theta}}{\theta}$ gives Theorem 8 in [13].

4 Applications

The previously mentioned result is now applied to Chebyshev inequalities involving two specific operators: the k-Katugompola operator and the k-fractional conformable integral operator.

4.1 Chebyshev-type inequalities via k-Katugompola operator

Putting $\psi(\tau) = \frac{\tau^{\rho+1}}{\rho+1}$ where $\rho > 0$, the left side (k,ψ)-Hilfer ${}_{a^+}\mathrm{J}_k^{\alpha,\psi}$ reduces to the left side k-Katugompola fractional integral operator (2.2) of order $\alpha > 0$ and the following results hold.

Corollary 4.1. Let f and g be two synchronous functions on $[a, b]$ and let $\alpha, \beta,\ k > 0$, then for $x > a$ we have

$$ \frac{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}\, {}_{a^+}\mathcal{K}_k^{\alpha}(f\,g)(x) + \frac{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}\, {}_{a^+}\mathcal{K}_k^{\alpha}(f\,g)(x) \geq {}_{a^+}\mathcal{K}_k^{\alpha} f(x)\, {}_{a^+}\mathrm{J}_k^{\beta,\psi} g(x) + {}_{a^+}\mathcal{K}_k^{\alpha} f(x)\, {}_{a^+}\mathrm{J}_k^{\alpha,\psi} g(x), $$

and

$$ {}_{a^+}\mathcal{K}_k^{\alpha}(f\,g)(x) \geq \frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}\, {}_{a^+}\mathcal{K}_k^{\alpha} f(x), {}_{a^+}\mathcal{K}_k^{\alpha} g(x). $$

Corollary 4.2. Let $i = 1, 2, ..., m$ and $f,\ f_i \in L_{X^1_\psi}, [a, b]$. If the functions f and f_i are positive increasing functions on $[a, b]$, then for all integer $m \geq 1$ we have

$$ {}_{a^+}\mathcal{K}_k^{\alpha} \prod_{i=1}^{m} f_i(x) \geq \left(\frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}} \right)^{m-1} \left(\prod_{i=1}^{m} {}_{a^+}\mathcal{K}_k^{\alpha} f_i(x) \right), $$

and

$$ {}_{a^+}\mathcal{K}_k^{\alpha} f^m(x) \geq \left(\frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}} \right)^{m-1} \left({}_{a^+}\mathcal{K}_k^{\alpha} f(x) \right)^m. $$

Corollary 4.3. Let f and g be two functions defined on $L_{X_\psi}, ([a, b])$, such that f is monotone and g is differentiable on $]a, b[$ and there exists a real numbers $m := \inf_{x\in[a,b]} g'(x)$ and $M := \sup_{x\in[a,b]} g'(x)$.

- If f is an increasing function, then

$$ \begin{aligned} {}_{a^+}\mathcal{K}_k^{\alpha}(f\,g)(x) \geq & \frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}\, {}_{a^+}\mathcal{K}_k^{\alpha} f(x)\, {}_{a^+}\mathrm{J}_k^{\alpha,\psi} g(x) - m \frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}\, {}_{a^+}\mathcal{K}_k^{\alpha} f(x) {}_{a^+}\mathcal{K}_k^{\alpha} I_d(x) \\ & + m\, {}_{a^+}\mathcal{K}_k^{\alpha}(I_d f)(x). \end{aligned} $$

- If f is a decreasing function, then

$$ \begin{aligned} {}_{a^+}\mathcal{K}_k^{\alpha}(f\,g)(x) \geq & \frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}\, {}_{a^+}\mathcal{K}_k^{\alpha} f(x)\, {}_{a^+}\mathcal{K}_k^{\alpha} g(x) - M \frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}\, {}_{a^+}\mathcal{K}_k^{\alpha} f(x)\, {}_{a^+}\mathcal{K}_k^{\alpha} I_d(x) \\ & + M\, {}_{a^+}\mathcal{K}_k^{\alpha}(I_d f)(x), \end{aligned} $$

where $I_d(x)$ is the identity function.

4.2 Chebyshev-type inequalities via k-fractional conformable integral operator

Set $\psi(\tau)=\frac{(\tau-a)^{\theta}}{\theta}$ where $\theta>0$, the left side (k,ψ)-Hilfer ${}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}$ reduces to the left side k-fractional conformable integral operator (2.3) of order $\alpha>0$ and we derive what follows.

Corollary 4.4. Let f and g be two synchronous functions on $[a,b]$ and let $\alpha,\beta,\ k>0$, then for $x>a$ we have

$$\frac{(x-a)^{\frac{\theta\beta}{k}}}{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathcal{C}_k^{\alpha}(f\,g)(x)+\frac{(x-a)^{\frac{\theta\beta}{k}}}{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathcal{C}_k^{\alpha}(f\,g)(x)\geq\,{}_{a^+}\mathcal{C}_k^{\alpha}f(x)\,{}_{a^+}\mathrm{J}_k^{\beta,\psi}g(x)+\,{}_{a^+}\mathcal{C}_k^{\alpha}f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}g(x),$$

and

$${}_{a^+}\mathcal{C}_k^{\alpha}(f\,g)(x)\geq\frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\,{}_{a^+}\mathcal{C}_k^{\alpha}f(x),{}_{a^+}\,\mathcal{C}_k^{\alpha}g(x).$$

Corollary 4.5. Let $i=1,2,...,m$ and $f,\ f_i\in L_{X^1_{\psi}},[a,b]$. If the functions f and f_i are positive increasing functions on $[a,b]$, then for all integer $m\geq1$ we have

$${}_{a^+}\mathcal{C}_k^{\alpha}\prod_{i=1}^{m}f_i(x)\geq\left(\frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\right)^{m-1}\left(\prod_{i=1}^{m}\,{}_{a^+}\mathcal{C}_k^{\alpha}f_i(x)\right),$$

and

$${}_{a^+}\mathcal{C}_k^{\alpha}f^m(x)\geq\left(\frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\right)^{m-1}\left({}_{a^+}\mathcal{C}_k^{\alpha}f(x)\right)^m.$$

Corollary 4.6. Let f and g be two functions defined on $L_{X_{\psi}},([a,b])$, such that f is monotone and g is differentiable on $]a,b[$ and there exists a real numbers $m:=\inf\limits_{x\in[a,b]}g'(x)$ and $M:=\sup\limits_{x\in[a,b]}g'(x)$.

- If f is an increasing function, then

$${}_{a^+}\mathcal{C}_k^{\alpha}(f\,g)(x)\geq\frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\,{}_{a^+}\mathcal{C}_k^{\alpha}f(x)\,{}_{a^+}\mathrm{C}_k^{\alpha,\psi}g(x)-m\,\frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\,{}_{a^+}\mathcal{C}_k^{\alpha}f(x)\,{}_{a^+}\mathcal{C}_k^{\alpha}I_d(x)+\,m\,{}_{a^+}\mathcal{C}_k^{\alpha}(I_df)(x).$$

- If f is a decreasing function, then

$${}_{a^+}\mathcal{C}_k^{\alpha}(f\,g)(x)\geq\frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\,{}_{a^+}\mathcal{C}_k^{\alpha}f(x)\,{}_{a^+}\mathrm{C}_k^{\alpha,\psi}g(x)-M\,\frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\,{}_{a^+}\mathcal{C}_k^{\alpha}f(x)\,{}_{a^+}\mathcal{C}_k^{\alpha}I_d(x)+\,M\,{}_{a^+}\mathcal{C}_k^{\alpha}(I_df)(x),$$

where $I_d(x)$ is the identity function.

References

[1] A.O. Akdemir, S.I. Butt, M. Nadeem, and M.A. Ragusa, *New general variants of Chebyshev type inequalities via generalized fractional integral operators*, Mathematics **9** (2021), no. 2, 122.

[2] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities*, J. Inequal. Pure Appl. Math. **10** (2009), no. 3, 1–12.

[3] P.L. Chebyshev, *Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites*, Proc. Math. Soc. Charkov **2** (1882), 93–98.

[4] M. Houas, Z. Dahmani, and M.Z. Sarikaya, *Some integral inequalities for (k,s)-Riemann-Liouville fractional operators*, J. Interdiscip. Math. **21** (2018), 7-8, 1575–1585.

[5] S. Iqbal, S. Mubeen, and M. Tomar, *On Hadamard k-fractional integrals*, J. Fractional Calc. Appl. **9** (2018), no. 2, 255–267.

[6] V. Kiryakova, *Generalized Fractional Calculus and Applications*, vol. 301 of Pitman Research Notes in Mathematics Series, Longman Scientific et Technical, Harlow, UK, 1994.

[7] Y.C. Kwun, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, *Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities*, IEEE Access **6** (2018), 64946–64953.

[8] K.S. Nisar, G. Rahman, and K. Mehrez, *Chebyshev type inequalities via generalized fractional conformable integrals*, J. Inequal. Appl. **2019** (2019), 245.

[9] F. Qi, S. Habib, S. Mubeen, and M.N. Naeem, *Generalized k-fractional conformable integrals and related inequalities*, AIMS Math. **4** (2019), no. 3, 343–358.

[10] A. Senouci and M. Sofrani, *Generalizations of some integral inequalities for Riemann-Liouville operator*, Chebyshevskii Sbornik **23** (2022), no. 2, 161–169.

[11] E. Set and A. Gozpnar, *Some new inequalities involving generalized fractional integral operators for several class of functions*, AIP Conf. Proc. **1833** (2017), no. 1.

[12] E. Set, I. Mumcu, and S. Demirbas, *Conformable fractional integral inequalities of Chebyshev type*, Rev. Real Acad. Cien. Exact. Fis. Natur. Ser. A. Mate. **113** (2019), 2253–2259.

[13] E. Set, M.E. Özdemir, and S. Demirbas, *Chebyshev type inequalities involving extended generalized fractional integral operators*, AIMS Math. **5** 2020, no. 4, 3573–3583.