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Regularity properties for convex-like $C(T)$-valued functions on Hilbert spaces

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Abstract

In this paper, we introduce several regularity properties for the non-differentiable convex-like $C(T)$-valued functions which are defined on a Hilbert space. The relationships with various regularity properties are investigated. All results are given in terms of the convex subdifferential. Non-trivial numerical examples are incorporated to demonstrate the validity of the results established in this paper. To the best of our knowledge, this paper is the first to investigate the regularity properties for the $C(T)$-valued functions, even in the differentiable case of finite-dimensional spaces.

Keywords: $C(T)$-valued function, Regularity property, Hilbert space, Convex subdifferential
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1 Introduction

In this paper, we study the regularity property for convex-like $C(T)$-valued functions $f : \mathcal{H} \to C(T)$, where $\mathcal{H}$ is a Hilbert space and $C(T)$ denotes the set of real-valued continuous functions on a (not necessarily compact) metric space T. Note that the $C(T)$-valued function $f : \mathcal{H} \to C(T)$ is said to be convex-like if for all $t \in T$ the function $f(\cdot)(t) : \mathcal{H} \to \mathbb{R}$ is convex, i.e.,

$$f\big(\lambda x_1 + (1-\lambda)x_2\big)(t) \leq \lambda f(x_1)(t) + (1-\lambda)f(x_2)(t), \quad \forall x_1, x_2 \in \mathcal{H},\ \forall \lambda \in [0,1],\ \forall t \in T.$$

If we write $f \leq g$ for two $C(T)$-valued functions $f, g : \mathcal{H} \to C(T)$, it means $f(\cdot)(t) \leq g(\cdot)(t)$ for all $t \in T$, i.e.,

$$f(x)(t) \leq g(x)(t), \qquad \forall x \in \mathcal{H},\ \forall t \in T.$$

Given a convex-like $C(T)$-valued functions $f : \mathcal{H} \to C(T)$, we consider the following subset of $\mathcal{H}$,

$$S := \big\{x \in \mathcal{H} \mid f \leq \mathbf{0}\big\},$$

where $\mathbf{0}$ denotes the zero $C(T)$-valued function. In other words, S can be rewritten as

$$S = \big\{x \in \mathcal{H} \mid f(x)(t) \leq 0, \quad \forall t \in T\big\}. \tag{1.1}$$

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According to

$$S = \bigcap_{t \in T} \{x \in \mathcal{H} \mid f(x)(t) \leq 0\} = \bigcap_{t \in T} f^{-1}\big((-\infty, 0]\big)(t),$$

and regarding the convexity of level sets of real-valued convex functions [14, p. 41], we conclude that S is an intersection of convex sets, and consequently is itself convex (Here, $f^{-1}(\cdot)(t)$ denotes the inverse relation of $f(\cdot)(t)$). In what follows we shall assume that $S \neq \emptyset$.

If a convex-like $C(T)$-valued function $f : \mathcal{H} \to C(T)$ and a vector $\hat{x} \in S$ are given, the following condition is very important in many theoretical and applied problems (see, e.g., [13, 15]):

$$N(S, \hat{x}) = cone\left(\bigcup_{t \in T(\hat{x})} \partial f(\hat{x})(t)\right), \tag{1.2}$$

where the normal cone of S at $\hat{x}$ is denoted by $N(S, \hat{x})$, the subdifferential of convex function $f(\cdot)(t)$ at $\hat{x}$ is denoted by $\partial f(\hat{x})(t)$, and $T(\hat{x})$ is defined as

$$T(\hat{x}) := \{t \in T \mid f(\hat{x})(t) = 0\}.$$

Any property that is a sufficient condition for the equality (1.2) is called a `regularity property`.

It should be noted that if T is a finite set, $C(T)$ can be considered as same as $\mathbb{R}^{|T|}$ (see, e.g., [1]), and then the $C(T)$-valued functions $f : \mathcal{H} \to C(T)$ are reduced to the vector-valued functions $f : \mathcal{H} \to \mathbb{R}^{|T|}$. The regularity properties of this special type of $C(T)$-valued functions can be seen in [1, 4, 12]. Also, if $\mathcal{H} = \mathbb{R}^n$ and T is a compact space, then the $C(T)$-valued function $f : \mathbb{R}^n \to C(T)$ is said to be a semi-infinite function. The regularity properties of semi-infinite functions are studied in [7] for linear case, in [3] for differentiable case, in [2, 11] for convex case, in [6] for DC (difference of convex functions) case, in [8] for quasiconvex case, and in [5, 9, 10] for nonsmooth case. Recently, it has been studied in [13, 15] the cases where $\mathcal{H}$ is a Banach space and T is a compact space.

Since in this article, we do not consider any of the limitations of the above papers (even compactness of T), the results of this article can be considered as a generalization of all the above papers.

2 Preliminaries

In this section, we describe our natation and present preliminary results. Throughout the paper, the inner product of two vectors x and y in the Hilbert space $\mathcal{H}$ will be denoted by $\langle x, y\rangle$, and the null vector of $\mathcal{H}$ will be denoted by $0_{\mathcal{H}}$. Given a set $A \subseteq \mathcal{H}$, we denote by $\overline{A}$, the closure of A. Also, the convex hull and the convex cone generated by A are denoted, respectively, by $conv(A)$ and $cone(A)$, defined as

$$conv(A) := \begin{cases} \bigcap\big\{B \mid B \text{ is convex and } A \subseteq B\big\}, & \text{if } A \neq \emptyset \\ \emptyset, & \text{if } A = \emptyset \end{cases},$$

$$conv(A) := \begin{cases} \bigcup\big\{r\, conv(A) \mid r \geq 0\big\}, & \text{if } A \neq \emptyset \\ \{0_{\mathcal{H}}\}, & \text{if } A = \emptyset \end{cases}.$$

The negative polar cone and the strictly negative set of $A \subseteq \mathcal{H}$ are, respectively, defined as

$$A^{\odot} := \begin{cases} \big\{u \in \mathcal{H} \mid \langle u, x\rangle \leq 0,\ \forall\, x \in A\big\}, & \text{if } A \neq \emptyset \\ \{0_{\mathcal{H}}\}, & \text{if } A = \emptyset \end{cases},$$

$$A^{\ominus} := \begin{cases} \big\{u \in \mathcal{H} \mid \langle u, x\rangle < 0,\ \forall\, x \in A\big\}, & \text{if } A \neq \emptyset \\ \emptyset, & \text{if } A = \emptyset \end{cases}.$$

We can see ([1]) that $A^{\odot}$ is always a closed convex cone in $\mathcal{H}$, $A^{\odot} = \big(\overline{A}\big)^{\odot} = \big(conv(A)\big)^{\odot} = \big(cone(A)\big)^{\odot}$, and $A^{\odot} = \overline{A^{\ominus}}$ if $A^{\ominus} \neq \emptyset$. The bipolar theorem (see [1, 14]) states that

$$A^{\odot\odot} := \big(A^{\odot}\big)^{\odot} = \overline{cone}(A) := \overline{cone(A)}.$$

If Ω is an arbitrary index and $B_\gamma \subseteq \mathcal{H}$ is a nonempty convex set as $\gamma \in \Omega$, then [14, p. 34]

$$conv\Big(\bigcup_{\gamma\in\Omega} B_\gamma\Big) = \left\{ \sum_{\gamma\in\Omega_*} \alpha_\gamma b_\gamma \mid \alpha_\gamma \geq 0,\ \sum_{\gamma\in\Omega_*} \alpha_\gamma = 1,\ b_\gamma \in B_\gamma,\ \Omega_* \subseteq \Omega,\ |\Omega_*| < \infty \right\}. \tag{2.1}$$

Theorem 2.1. [14, p. 15] Let $A \subset \mathcal{H}$ be a compact set. Then, $cone(A)$ is closed if $0_{\mathcal{H}} \notin conv(A)$.

Let $B \subseteq \mathcal{H}$ be a closed convex set, the normal cone of B at $x_0 \in B$ is defined as

$$N(B, x_0) := \big\{u \in \mathcal{H} \mid \langle u, x - x_0\rangle \leq 0, \quad \text{for all } x \in B\big\}.$$

The negative polar cone of $N(B, x_0)$ is called the tangent cone of B at x_0, i.e., $\Gamma(B, x_0) := \big(N(B, x_0)\big)^{\odot}$, and the feasible directions cone of B at x_0 is defined by

$$D(B, x_0) := \big\{v \in \mathcal{H} \mid \exists\delta > 0\ :\ x_0 + \varepsilon v \in B\ \ \forall\varepsilon \in (0, \delta)\big\}.$$

We can see ([4, 14]) $\Gamma(B, x_0)$ and $N(B, x_0)$ are always closed convex cones in $\mathcal{H}$, $N(B, x_0) = \big(\Gamma(B, x_0)\big)^{\odot}$,

$$\Gamma(B, x_0) = \big\{v \in \mathcal{H} \mid \exists r_n \downarrow 0,\ \exists v_n \to v\ :\ x_0 + r_n v_n \in B\ \ \forall n \in \mathbb{N}\big\},$$

and $\Gamma(B, x_0) = \overline{D(B, x_0)}$. Suppose that $\varphi : \mathcal{H} \to \mathbb{R}$ is a convex function. The Fenchel (or convex) subdifferential of φ at $x_0 \in \mathcal{H}$ is defined as

$$\partial\varphi(x_0) := \{\xi \in \mathcal{H} \mid \varphi(x) - \varphi(x_0) \geq \langle \xi, x - x_0\rangle, \quad \forall x \in \mathcal{H}\}.$$

As the final point of this section, in the following theorem we summarize some important properties of the convex subdifferential from [14] which are widely used in what follows.

Theorem 2.2. Let $\varphi_1, \cdots, \varphi_m$ be convex functions from $\mathcal{H}$ to $\mathbb{R}$. For a given $x_0 \in \mathcal{H}$, the following assertions hold:

(i) $\partial\varphi_1(x_0)$ is a nonempty convex compact subset of $\mathcal{H}$.

(ii) For all non-negative real numbers $\alpha_1, \cdots, \alpha_m$, one has

$$\partial\Big(\sum_{i=1}^{m} \alpha_i\varphi_i\Big)(x_0) = \sum_{i=1}^{m} \alpha_i \partial\varphi_i(x_0).$$

(iii) If $\Phi : \mathcal{H} \to \mathbb{R}$ is defined by $\Phi(x) := \max\big\{\varphi_i(x) \mid i = 1, \cdots, m\big\}$, then

$$\partial\Phi(x_0) \subseteq conv\Big(\bigcup_{i\in I} \partial\varphi_i(x_0)\Big),$$

where $I := \{i \mid \Phi(x_0) = \varphi_i(x_0)\}$.

(iv) One has

$$\partial\varphi_1(x_0) = \{\xi \in \mathcal{H} \mid \varphi_1'(x_0; d) \geq \langle \xi, d\rangle, \quad \forall d \in \mathcal{H}\},$$
$$\varphi_1'(x_0; d) = \max\big\{\langle \xi, d\rangle \mid \xi \in \partial\varphi_1(x_0)\big\}, \quad \forall d \in \mathcal{H},$$

where $\varphi_1'(x_0; d)$ denotes the directional derivative of φ_1 at x_0 in the direction $d \in \mathcal{H}$, i.e.,

$$\varphi_1'(x_0; d) := \lim_{h\downarrow 0} \frac{\varphi_1(x_0 + hd) - \varphi_1(x_0)}{h}.$$

3 Main Results

At the first point of this section, we note that if the convex-like $C(T)$-valued function $f : \mathcal{H} \to C(T)$ and a vector $\hat{x} \in S$ are given, we will always assume that

$$\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t) \neq \emptyset. \tag{3.1}$$

It should be noted that the assumption (3.1) is not restrictive at all, because otherwise we have $T(\hat{x}) = \emptyset$, and as a result, both sides of (1.2) are equal to $\{0_{\mathcal{H}}\}$.

Theorem 3.1. Assume that the convex-like $C(T)$-valued function $f : \mathcal{H} \to C(T)$ and a vector $\hat{x} \in S$ are given. Then

$$cone\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right) \subseteq N(S,\hat{x}).$$

Proof . Suppose that $\xi \in \bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)$ is given. Thus, $\xi \in \partial f(\hat{x})(t_0)$ for some $t_0 \in T(\hat{x})$, and so $f(\hat{x})(t_0) = 0$. If $d \in D(S,\hat{x})$, then $\hat{x} + \delta d \in S$ for some $\delta > 0$, and hence $f(\hat{x} + \delta d)(t_0) \leq 0$. Now, by the definition of subdifferential we get

$$0 \geq \frac{1}{\delta}\Big(f(\hat{x}+\delta d)(t_0) - f(\hat{x})(t_0)\Big) \geq \frac{1}{\delta}\langle \xi, \delta d\rangle = \langle \xi, d\rangle.$$

Since the above inequality holds for all $d \in D(S,\hat{x})$, we have $\xi \in \big(D(S,\hat{x})\big)^{\odot}$. This inclusion and the fact that

$$(D(S,\hat{x}))^{\odot} = \left(\overline{D(S,\hat{x})}\right)^{\odot} = (\Gamma(S,\hat{x}))^{\odot} = N(S,\hat{x}),$$

conclude that $\xi \in N(S,\hat{x})$, and so

$$\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t) \subseteq N(S,\hat{x}).$$

Taking convex hulls in both sides of the above inclusion, we get

$$cone\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right) \subseteq cone\,(N(S,\hat{x})) = N(S,\hat{x}),$$

and the proof is complete. □

Remark 3.2. The above theorem shows that to prove equality (1.2), it is enough to show

$$N(S,\hat{x}) \subseteq cone\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right). \tag{3.2}$$

The following lemma provides an equivalent for inclusion (3.2).

Theorem 3.3. If a convex-like $C(T)$-valued function $f : \mathcal{H} \to C(T)$ and a vector $\hat{x} \in S$ are given, then (3.2) holds if and only if

$$\begin{cases} \left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\odot} \subseteq \Gamma(S,\hat{x}), \\ cone\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right) \text{ is closed.} \end{cases} \tag{3.3}$$

Proof . If $N(S,\hat{x}) \subseteq cone\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)$, then

$$\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\odot} = \left(cone\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)\right)^{\odot} \subseteq (N(S,\hat{x}))^{\odot} = \Gamma(S,\hat{x}),$$

and regarding to Theorem 3.1 and closedness of $N(S,\hat{x})$, we conclude that $cone\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)$ is closed. Conversely, if (3.3) holds, the bipolar Theorem implies that

$$N(S,\hat{x}) = (\Gamma(S,\hat{x}))^{\odot} \subseteq \left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\odot\odot} = \overline{cone\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)} = cone\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right).$$

□

The above theorem leads us to the following definition.

Definition 3.4. Let $f : \mathcal{H} \to C(T)$ be a convex-like $C(T)$-valued function and $\hat{x} \in S$. We say that f satisfies the `Abadie regularity property` (ARP, briefly) at $\hat{x}$ if

$$\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\odot} \subseteq \Gamma(S,\hat{x}).$$

Theorem 3.3 and Remark 3.2 show that any sufficient condition for establishing ARP is a regularity property for f, which can lead us to (1.2). We associate with the convex-like $C(T)$-valued function $f : \mathcal{H} \to C(T)$ the marginal function $\psi : \mathcal{H} \to (-\infty, 0]$ as follows:

$$\psi(x) := \sup\{f(x)(t) \mid t \in T\}, \quad \forall x \in S.$$

It is easy to see that ψ is a convex function (see [14, p. 97]). One reason for difficulty of extending the results finite T to infinite one is that in the finite case we have (see Theorem 2.2)

$$\partial\psi(\hat{x}) \subseteq conv\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right), \qquad \forall \hat{x} \in \mathcal{H}, \tag{3.4}$$

but in general, (3.4) does not hold if T is infinite. We are thus led to the following definition.

Definition 3.5. We say that the `Ioffe-Tikhomirov property` (ITP, briefly) is satisfied at $\hat{x} \in \mathcal{H}$, if (3.4) holds.

Now, we introduce a Kuhn-Tucker type regularity property for convex-like $C(T)$-valued functions.

Definition 3.6. Let $f : \mathcal{H} \to C(T)$ be a convex-like $C(T)$-valued function and $\hat{x} \in S$. We say that f satisfies the `Kuhn-Tucker regularity property` (KTRP, briefly) at $\hat{x}$ if

$$\left\{d \in \mathcal{H} \mid \psi'(\hat{x}; d) \leq 0\right\} \subseteq \Gamma(S,\hat{x}).$$

Notice, the ARP and the KTRP are named *geometric regularity properties*, since they are depended to the tangent cone of S at $\hat{x}$.

Theorem 3.7. Let a convex-like $C(T)$-valued function $f : \mathcal{H} \to C(T)$ be given. If KTRP and ITP are satisfied at $\hat{x} \in S$, then ARP holds at $\hat{x}$.

Proof . According to Theorem 2.2, we have

$$\begin{aligned} d^* \in \{d \in \mathcal{H} \mid \psi'(\hat{x}; d) \leq 0\} &\iff \psi'(\hat{x}; d^*) \leq 0 \\ &\iff \max\left\{\langle \xi, d^* \rangle \mid \xi \in \partial\psi(\hat{x})\right\} \leq 0 \\ &\iff \langle \xi, d^* \rangle \leq 0, \quad \forall \xi \in \partial\psi(\hat{x}) \\ &\iff d^* \in (\partial\psi(\hat{x}))^{\odot}. \end{aligned}$$

Thus,

$$\{d \in \mathcal{H} \mid \psi'(\hat{x}; d) \leq 0\} = (\partial\psi(\hat{x}))^{\odot},$$

which, together with ITP and KTRP, yields

$$\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\odot} = \left(conv\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)\right)^{\odot} \subseteq \big(\partial\psi(\hat{x})\big)^{\odot} = \big\{d \in \mathcal{H} \mid \psi'(\hat{x}; d) \leq 0\big\} \subseteq \Gamma(S, \hat{x}),$$

as required. □

Now, we introduce some *algebraic regularity properties*, that are not depended to the tangent cone of S at $\hat{x}$. For a given $\hat{x} \in S$ and a $\varepsilon \geq 0$, put

$$T_\varepsilon(\hat{x}) := \{t \in T \mid f(\hat{x})(t) \geq -\varepsilon\}.$$

Since $f(\hat{x})(t) \leq 0$ for all $t \in T$, we have clearly $T(\hat{x}) = T_0(\hat{x})$.

Definition 3.8. Let $f : \mathcal{H} \to C(T)$ be a convex-like $C(T)$-valued function and $\hat{x} \in S$. We say that f satisfies

- the `Slater regularity property` (SRP) when there is a Slater point, i.e., There exists $x_0 \in \mathcal{H}$ such that

$$f(x_0)(t) < 0, \qquad \forall t \in T.$$

- the `strong Slater regularity property` (SSRP) when there is a strong Slater point, i.e.,there exist $x_0 \in \mathcal{H}$ and $\varepsilon > 0$ such that

$$f(x_0)(t) \leq -\varepsilon, \qquad \forall t \in T.$$

- the `local Slater regularity property` (LSRP) at $\hat{x}$ when

$$\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\ominus} \neq \emptyset.$$

- the `perturbed Mangasarian-Fromovitz regularity property` (PMFRP) at $\hat{x}$ if there exists $d_* \in \mathcal{H}$ such that

$$\inf_{\varepsilon>0} \sup \left\{\langle \xi, d_*\rangle \mid \xi \in \bigcup_{t\in T_\varepsilon(\hat{x})} \partial f(\hat{x})(t)\right\} < 0.$$

  In other words, PMFRP holds at $\hat{x}$ if the following inequality holds for some $d_* \in \mathcal{H}$:

$$\inf_{\varepsilon>0} \sup_{t\in T_\varepsilon(\hat{x})} f'(\hat{x}; d_*)(t) < 0,$$

  where $f'(\hat{x}; d_*)(t)$ denotes the directional derivative of the function $f(\cdot)(t)$ at $\hat{x}$ in the direction d_*.

Theorem 3.9. Let a convex-like $C(T)$-valued function $f : \mathcal{H} \to C(T)$ be given. The following implications are established among the regularity properties introduced in Definition 3.8 at $\hat{x} \in S$:

$$\begin{array}{ccccc} & & & & \text{PMFRP} \\ & & & & \Downarrow \\ \text{SSRP} & \Longrightarrow & \text{SRP} & \Longrightarrow & \text{LSRP}. \end{array}$$

Proof .

[SSRP $\Longrightarrow$ SRP]: It is a straightforward consequence of Definition 3.6.

[SRP $\Longrightarrow$ LSRP]: Assume that $t \in T(\hat{x})$ and $\xi_t \in \partial f(\hat{x})(t)$ are given. Let $d := x_0 - \hat{x}$, where x_0 is the Slater point. Then, by the definition of subdifferential we have

$$\langle \xi_t, d\rangle = \langle \xi_t, x_0 - \hat{x}\rangle \leq \overbrace{f(x_0)(t)}^{<0} - \overbrace{f(\hat{x})(t)}^{=0} < 0.$$

So,

$$d \in \left(\bigcup_{t \in T(\hat{x})} \partial f(\hat{x})(t) \right)^{\ominus},$$

as required.

[PMFRP $\Longrightarrow$ LSRP]: Assume that $\varepsilon > 0$ is such that

$$\sup \left\{ \langle \xi, d_* \rangle \mid \xi \in \bigcup_{t \in T_\varepsilon(\hat{x})} \partial f(\hat{x})(t) \right\} < 0.$$

Since $T(\hat{x}) \subseteq T_\varepsilon(\hat{x})$,

$$\left(\bigcup_{t \in T(\hat{x})} \partial f(\hat{x})(t) \right) \subseteq \left(\bigcup_{t \in T_\varepsilon(\hat{x})} \partial f(\hat{x})(t) \right),$$

and hence,

$$\langle \xi, d_* \rangle < 0, \qquad \forall \xi \in \bigcup_{t \in T_\varepsilon(\hat{x})} \partial f(\hat{x})(t).$$

This means that

$$d_* \in \left(\bigcup_{t \in T(\hat{x})} \partial f(\hat{x})(t) \right)^{\ominus},$$

as required. □

The following example shows that the ITP is not implied by the defined geometric and algebraic regularity properties.

Example 3.10. Let $\mathcal{H} = \mathbb{R}$, $T = \mathbb{N} \cup \{0\}$, $\hat{x} = 0$, and

$$f(x)(t) = \begin{cases} 2x, & \text{if} \quad t = 0, \\ x - \frac{2}{t+1}, & \text{if} \quad t \in \{1, 3, 5, \ldots\}, \\ 3x - \frac{2}{t}, & \text{if} \quad t \in \{2, 4, 6, \ldots\}. \end{cases}$$

It is easy to see that

- $S = \Gamma(S, \hat{x}) = (-\infty, 0]$, and hence $N(S, \hat{x}) = [0, +\infty)$.
- $T(\hat{x}) = \{0\}$, and hence $\bigcup_{t \in T(\hat{x})} \partial f(\hat{x})(t) = \{2\}$. So

$$\left(\bigcup_{t \in T(\hat{x})} \partial f(\hat{x})(t) \right)^{\odot} = (-\infty, 0], \qquad \text{and} \qquad \left(\bigcup_{t \in T(\hat{x})} \partial f(\hat{x})(t) \right)^{\ominus} = (-\infty, 0).$$

- -1 is a strong Slater point.
- $\psi(x) = \begin{cases} x, & \text{if } x < 0, \\ 3x, & \text{if } x \geq 0 \end{cases}$, and hence

$$\psi'(\hat{x}; d) = \begin{cases} -1, & \text{if } d < 0, \\ 0, & \text{if } d = 0, \\ 3, & \text{if } d > 0 \end{cases}, \qquad \text{and} \qquad \partial \psi(\hat{x}) = [1, 3].$$

- for $d_* = 1$ and for all $\varepsilon > 0$ we have

$$\sup\left\{\langle \xi, d_*\rangle \mid \xi \in \bigcup_{t\in T_\varepsilon(\hat{x})} \partial f(\hat{x})(t)\right\} = \sup_{t\in T_\varepsilon(\hat{x})} f'(\hat{x}; d_*)(t) = \sup_{t\in T_\varepsilon(\hat{x})} \langle f'(\hat{x})(t), d_*\rangle = \sup\{1,2,3\} = 3.$$

Thus, SSRP, PMFRP, ARP, and KTRP hold at $\hat{x}$ while ITP fails at that point.

Now, we present the implications among the introduced geometric and algebraic regularity properties.

Theorem 3.11. Suppose that $f : \mathcal{H} \to C(T)$ is a convex-like $C(T)$-valued function. If the LSRP and ITP hold at $\hat{x} \in S$, then ARP holds at $\hat{x}$.

Proof . Let $d \in \left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\ominus}$. Since

$$\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\ominus} = \left(conv\Big(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\Big)\right)^{\ominus},$$

the ITP leads to $d \in \big(\partial\psi(\hat{x})\big)^{\ominus}$. Hence, $\langle \xi, d\rangle < 0$ for all $\xi \in \partial\psi(\hat{x})$, and so $\psi'(\hat{x}; d) < 0$. Consequently, there exists a scalar $\delta > 0$ such that $\psi(\hat{x} + \beta d) < \psi(\hat{x}) \le 0$, for all $\beta \in (0, \delta]$. Therefore, we have $\hat{x} + \beta d \in S$ for all $\beta \in (0, \delta]$, which implies $d \in D(S, \hat{x})$. We have thus proved the inclusion

$$\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\ominus} \subseteq D(S, \widehat{x}),$$

and hence

$$\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\odot} = \overline{\left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\ominus}} \subseteq \overline{D(S, \widehat{x})} = \Gamma(S, \hat{x}).$$

The proof is complete. □

Theorem 3.12. Suppose that $f : \mathcal{H} \to C(T)$ is a convex-like $C(T)$-valued function. If the LSRP and ITP are satisfied at $\hat{x} \in S$, then KTRP holds at $\hat{x}$.

Proof . If $d \in \left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\ominus}$, by the same argument as in the proof of Theorem 3.11 we have $\psi'(\hat{x}; \widehat{d}) < 0$, and so

$$\big\{d \in \mathcal{H} \mid \psi'(\hat{x}; d) < 0\big\} \neq \emptyset.$$

If $\hat{d} \in \big\{d \in \mathcal{H} \mid \psi'(\hat{x}; d) < 0\big\}$ is given arbitrarily, then $\psi'(\hat{x}; \hat{d}) < 0$. By the same argument as in the proof of Theorem 3.11, we obtain a $\delta > 0$ such that $\hat{x} + \delta\hat{d} \in S$. Thus, regarding to the definition of subdifferential, for each $t \in T(\hat{x})$ and $\xi \in \partial f(\hat{x})(t)$, we have

$$\langle \xi, \delta\hat{d}\rangle = \langle \xi, \hat{x} + \delta\hat{d} - \hat{x}\rangle \le f(\hat{x} + \delta\hat{d})(t) - f(\hat{x})(t) = f(\hat{x} + \delta\hat{d})(t) \le 0.$$

Thus, $\langle \xi, \hat{d}\rangle$ for all $\xi \in \bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)$, and hence $\hat{d} \in \left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\odot}$. This means that

$$\{d \in \mathcal{H} \mid \psi'(\hat{x}; d) < 0\} \subseteq \left(\bigcup_{t\in T(\hat{x})} \partial f(\hat{x})(t)\right)^{\odot}.$$

This together with the continuity of $\psi'(\hat{x};\cdot)$, implies that

$$\{d\in\mathcal{H}\mid\psi'(\hat{x};d)\leq 0\}=\overline{\{d\in\mathcal{H}\mid\psi'(\hat{x};d)<0\}}\subseteq\overline{\left(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\right)^{\odot}}=\left(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\right)^{\odot}.$$

□

The following theorem presents some important results for convex-like $C(T)$-valued functions with compact T.

Theorem 3.13. Suppose that $f:\mathcal{H}\to C(T)$ is a convex-like $C(T)$-valued function and T is a compact space. If the SRP is satisfied, then

(i) the ITP holds at all $\hat{x}\in S$.

(ii) *cone* $\left(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\right)$ is closed for all $\hat{x}\in S$.

Proof .

(i) It is [4, Pshenichnyi-Levin-Valadier theorem, p. 267] or [14, Theorem 2.4.18] (The compactness assumption of T is essential).

(ii) Let x_0 be the Slater point of f, i.e., $f(x_0)(t)<0$ for all $t\in T$. We claim that

$$0_{\mathcal{H}}\notin conv\left(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\right). \tag{3.5}$$

Assume on the contrary that (3.5) doed not hold. Owing to (2.1), there exist some finite set $T_*\subseteq T(\hat{x})$, non-negative scalars $\alpha_1,\ldots,\alpha_{|T_*|}$, and $\xi_t\in\partial f(\hat{x})(t)$ as $t\in\{1,\ldots,|T_*|\}$, such that

$$0_{\mathcal{H}}=\sum_{t=1}^{|T_*|}\alpha_t\xi_t,\qquad\text{and}\qquad\sum_{t=1}^{|T_*|}\alpha_t=1.$$

Thus, the definition of subdifferential implies that

$$0=\langle 0_{\mathcal{H}},\hat{x}-x_0\rangle=\left\langle\sum_{t=1}^{|T_*|}\alpha_t\xi_t\ ,\ \hat{x}-x_0\right\rangle=\sum_{t=1}^{|T_*|}\alpha_t\langle\xi_t,\hat{x}-x_0\rangle\leq\sum_{t=1}^{|T_*|}\alpha_t\Big(\overbrace{f(\hat{x})(t)}^{=0}-\overbrace{f(x_0)(t)}^{<0}\Big)<0.$$

This contradiction shows that (3.5) holds. On the other hand, according to [14, p. 97], $\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)$ is a compact set. Owing to the Theorem 2.1, the proof is complete.

□

Thee following corollary collects Theorems 3.7, 3.9, 3.11, 3.12, and 3.13 in one diagram.

Corollary 3.14. Suppose that $f:\mathcal{H}\to C(T)$ is a convex-like $C(T)$-valued function and $\hat{x}\in S$. The implications of the following diagram hold true at $\hat{x}$, where the label $(*)$ besides an arrow stands for "the implication holds under the assumption that T is compact":

$$\begin{array}{ccccccc}
 & & & & [\text{KTRP}\wedge\ \text{ITP}] & & \\
 & & & & \uparrow & \searrow & \\
\text{SSRP} & \longrightarrow & \text{SRP} & \xrightarrow{(*)} & [\text{LSRP}\wedge\ \text{ITP}] & \longrightarrow & \text{ARP}\\
 & & \downarrow & \swarrow & & & \\
\text{PMFRP} & \longrightarrow & \text{LSRP} & & & &
\end{array}$$

Example 3.15. Considering Example 3.10, we have $cone\Big(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\Big)=[0,+\infty)$. So, the converse of the following implications are not true, even when $cone\Big(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\Big)$ is closed:

$$[\text{LSRP}\wedge\ \text{ITP}]\Longrightarrow\text{ARP},\qquad\text{and}\qquad[\text{KTRP}\wedge\ \text{ITP}]\Longrightarrow\text{ARP}.$$

This example, also, shows that the compactness condition of T is necessary in the following implication

$$\text{SRP}\overset{(*)}{\Longrightarrow}[\text{LSRP}\wedge\ \text{ITP}].$$

The following example shows that the closedness condition of $cone\Big(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\Big)$ can not be ignored in Theorem 3.3.

Example 3.16. Let $\mathcal{H}=\mathbb{R}^2$, $T:=\mathbb{N}\cup\{0\}$, $\hat{x}=(0,0)$, and

$$f(x)(t)=\sup\big\{\langle x,y\rangle\mid y\in B_t\big\},$$

where the compact convex set $B_t\subseteq\mathbb{R}^2$ is defined as

$$B_t:=\Big\{(y_1,y_2)\in[0,+\infty)\times[0,+\infty)\mid y_1^2+y_2^2-2(1+t)y_2\leq 0\Big\}.$$

We can see that

- $S=\Gamma(S,\hat{x})=(-\infty,0]\times(-\infty,0]$ and hence $N(S,\hat{x})=[0,+\infty)\times[0,+\infty)$.
- $T(\hat{x})=T$ and $\partial f(\hat{x})(t)=B_t$ as $t\in T$. Hence,

$$cone\Big(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\Big)=\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)=\big\{x\in\mathbb{R}^2\mid x_1\geq 0,\ x_2>0\big\}\cup\big\{(0,0)\big\}.$$

  So

$$\left(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\right)^{\odot}=(-\infty,0]\times(-\infty,0],\qquad\text{and}\qquad\left(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\right)^{\ominus}=\emptyset.$$

- $\psi(x)=\psi'(x;d)=0$ for all $x\in S$ and $d\in S$. So, $\partial\psi(\hat{x})=S$.
- $\bigcup_{t\in T_\varepsilon(\hat{x})}\partial f(\hat{x})(t)=\Big(\mathbb{R}\times(0,+\infty)\Big)\cup\big\{(0,0)\big\}$ for all $\varepsilon>0$.

Therefore, KTRP and ARP hold at $\hat{x}$ while the remaining regularity properties in Corollary 3.14 fail (as well as ITP). Also, $cone\Big(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\Big)$ is not closed and

$$N(S,\hat{x})\nsubseteq cone\Big(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\Big).$$

Thus, (3.2) does not hold at $\hat{x}$. Observe that

$$N(S,\hat{x})=\overline{cone}\Big(\bigcup_{t\in T(\hat{x})}\partial f(\hat{x})(t)\Big).$$

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