Int. J. Nonlinear Anal. Appl. 16 (2025) 7, 135–150
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.28395.3878

Modification and error analysis of an adaptive collocation method for solving delay differential equations

Ali Davari

Department of Mathematics, Khansar Campus, University of Isfahan, Iran

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, we introduce an effective multistep collocation method for solving delay differential equations (DDEs) with constant delays. We determine the convergence properties of the proposed method for delay differential equations with solutions in appropriate Sobolev spaces and show that the proposed scheme enjoys spectral accuracy. Numerical results show that the proposed method can be implemented efficiently and accurately for various DDE model problems.

Keywords: Delay differential equations, Collocation method, Spectral accuracy, Convergence analysis
2020 MSC: 34K40, 65L04, 65L06, 65L20, 65L60, 65L70, 41A05

1 Introduction

Delay differential equations (DDEs) are those differential equations in which the derivatives of functions are dependent on the values of the functions and possibly their derivatives at previous times. Various realistic and practical phenomena occurring can be modeled in applied mathematics by DDEs. In the modeling of many problems, for instance, in population dynamics and infectious diseases commonly spread, DDEs are useful tools [9, 13, 25, 11, 23]. In addition, the delays are useful instruments regarding expected times, incubation periods, and transport delays. In this manuscript, we consider the following model DDEs:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}U(t) = f\left(U(t), U(t-\tau), \dfrac{\mathrm{d}}{\mathrm{d}t}U(t-\tau), t\right), \quad 0 \leqslant t \leqslant T, \\ U(t) = \phi(t), \quad -\tau \leqslant t \leqslant 0, \quad \dfrac{\mathrm{d}}{\mathrm{d}t}U(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}\phi(t), \quad -\tau \leqslant t < 0, \end{cases} \tag{1.1}$$

in which f and ϕ are functions with some certain properties, τ is a positive constant delay or lag, and T is a positive constant. For the discussion about the existence and uniqueness of the solutions of the model (1.1), we refer the reader to [12, 16, 20]. If the model (1.1) contains the derivative delay term, we call it explicit neutral delay differential equation (NDDE); otherwise, it will be called retarded delay differential equation (RDDE).

In the study of population dynamics, the model problem (1.1) can be utilized, for example, from the balance laws of age-structured population dynamics, assuming that birth and death rates, as functions of age, are piecewise

*Corresponding author
Email address: a_davari@khc.ui.ac.ir (Ali Davari)

Received: June 2022 *Accepted:* September 2022

constant. Besides, this modeling process is applicable in the study of population dynamics of isolated populations [4], the interplay of predators and prey [30], and tumor modeling [5]. For easy, but without any loss of generality concerning the issues we consider, we take here the same time delay in U and $\frac{\mathrm{d}}{\mathrm{d}t}U$. It should be noticed that the set of breaking points of the model problem (1.1) is equal to $\{k\tau\}_{k=0}^{\infty}$; if we have a single delay. However, the evaluation of breaking points is more complicated when we have two or more different delays. Nevertheless, the numerical schemes and the error analysis that appear in the rest of the paper are almost the same, just the numbers of calculation steps are increased. Moreover, our obtained results in this manuscript for the DDE (1.1), one can generalize to systems of first-order ODEs. It is well-known that DDEs have some solutions that their behavior is quite different from solutions of ODEs. For instance, the characteristic equation corresponding to DDE can have infinitely many roots, and also that sometimes the solution oscillates rapidly. In the model (1.1) for the RDDE case, if there exists a discontinuity in the first derivative at the initial point, then there is a discontinuity in the second derivative at time τ, and for the corresponding higher derivatives this process is propagated in the interval of interest. In the other words, in the case of RDDEs, the solution becomes smoother as the integration proceeds, which is not necessarily the case with NDDEs.

There have been many numerical methods for solving RDDEs and some numerical methods for NDDEs mainly based on the Runge-Kutta scheme. This kind of numerical schemes have usually designed based on Taylor's expansions or quadrature formulas [9, 18, 15, 14, 37, 2, 27, 17, 21, 22, 8]. As we know, the spectral method employs global orthogonal polynomials as trial functions and provides exceedingly accurate numerical results for smooth solutions [10, 35]. However, so far, there are very few numerical methods with the spectral accuracy for DDEs and especially for NDDEs [19, 28, 29, 1, 36, 7]. The key problem is how to design proper algorithms and analyse numerical errors precisely.

In [28] a multistep Legendre-Gauss-Radau (LGR) collocation method using Lagrange interpolation for the numerical solution of RDDEs with constant delay is presented. In the present work, we introduce a modification of its. In the next section, some preliminaries required for our subsequent development are given. Then, based on the "method of steps", the modified multistep LGR collocation method for solving DDEs is explained. In Section 3, the detailed error analysis of the proposed method is derived for functions that lie in appropriate Sobolev spaces. Section 4 is devoted to some numerical results and justifies our theoretical analysis. We will show that the multistep LGR collocation method interacts well with the method of steps. The final section is for some concluding discussions.

2 Multistep LGR collocation method for retarded and neutral delay systems

In this section, we derive a novel and stable LGR collocation method for delay systems with continuous initial functions.

2.1 Preliminaries

Let $\alpha, \beta > -1$. The shifted Jacobi polynomial of degree n in the interval $I = [a, b]$ is defined by

$$J_{I,n}^{(\alpha,\beta)}(t) = J_n^{(\alpha,\beta)}\left(\frac{2}{b-a}t - \frac{b+a}{b-a}\right), \quad n = 0, 1, 2, \dots$$

where $J_n^{(\alpha,\beta)}(t)$ is the standard Jacobi polynomial of degree n [10]. We have that

$$J_{I,n}^{(\alpha,\beta)}(a) = (-1)^n \frac{\Gamma(n+\beta+1)}{n!\,\Gamma(\beta+1)}, \qquad J_{I,n}^{(\alpha,\beta)}(b) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}, \tag{2.1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}J_{I,n}^{(\alpha,\beta)}(t) = \frac{n+\alpha+\beta+1}{b-a}J_{I,n-1}^{(\alpha+1,\beta+1)}(t). \tag{2.2}$$

In particular, the shifted Legendre polynomial $L_{I,n}(t) = J_{I,n}^{(0,0)}(t)$. Thus, by (2.2),

$$\frac{\mathrm{d}}{\mathrm{d}t}L_{I,n}(t) = \frac{n+1}{b-a}J_{I,n-1}^{(1,1)}(t). \tag{2.3}$$

The set of $\{L_{I,n}(t)\}$ is a complete $L^2(I)$-orthogonal system, namely,

$$\int_I L_{I,m}(t)L_{I,n}(t)\,\mathrm{d}t = \frac{b-a}{2n+1}\delta_{m,n},$$

where $\delta_{m,n}$ is the Kronecker function. Thus for any $u \in L^2(I)$,

$$u(t) = \sum_{n=0}^{\infty} u_{I,n} L_{I,n}(t), \qquad u_{I,n} = \frac{2n+1}{b-a} \int_I u(t) L_{I,n}(t)\,\mathrm{d}t. \tag{2.4}$$

We denote by t_j, $0 \leqslant j \leqslant N$, the nodes of the standard LGR interpolation on the interval $[-1, 1)$. In particular, $t_0 = -1$ and $t_N < 1$. The corresponding Christoffel numbers are w_j, $0 \leqslant j \leqslant N$. Then the nodes of the shifted LGR interpolation on the interval $[a, b)$ are the distinct zeros of $L_{I,N}(t) + L_{I,N+1}(t)$, denoted by $t_{I,j}$, $0 \leqslant j \leqslant N$. In particular, $t_{I,0} = a$. Clearly, the nodes $t_{I,j}$ can be obtained by shifting the nodes t_j and the corresponding Christoffel numbers are $w_{I,j} = \frac{b-a}{2} w_j$, $0 \leqslant j \leqslant N$.

Let $\mathcal{P}_N$ be the set of polynomials of degree at most N. Thanks to the property of the standard LGR quadrature, it follows that for any $\psi \in \mathcal{P}_{2N}$ on I,

$$\begin{aligned} \int_I \psi(t)\,\mathrm{d}t &= \frac{b-a}{2} \int_{-1}^{1} \psi\left(\frac{b-a}{2} t + \frac{b+a}{2}\right) \mathrm{d}t \\ &= \frac{b-a}{2} \sum_{j=0}^{N} w_j \psi\left(\frac{b-a}{2} t_j + \frac{b+a}{2}\right) = \sum_{j=0}^{N} w_{I,j} \psi(t_{I,j}). \end{aligned} \tag{2.5}$$

Let $\langle u, v\rangle_I$ and $\|u\|_I$ be the inner product and the norm of space $L^2(I)$, respectively. We also define the following discrete inner product and norm,

$$\langle u, v\rangle_{I,N} = \sum_{j=0}^{N} w_{I,j} u(t_{I,j}) v(t_{I,j}), \qquad \|u\|_{I,N} = \sqrt{\langle u, u\rangle_{I,N}}. \tag{2.6}$$

Due to (2.5), for any $\psi\varphi \in \mathcal{P}_{2N}$ and $\varphi \in \mathcal{P}_N$,

$$\langle \varphi, \psi\rangle_I = \langle \varphi, \psi\rangle_{I,N}, \qquad \|\varphi\|_I = \|\varphi\|_{I,N}. \tag{2.7}$$

For any $U \in \mathcal{C}(I)$, the shifted LGR interpolation $\mathcal{I}_N U(t) \in \mathcal{P}_N$ is determined uniquely by

$$\mathcal{I}_N U(t_{I,j}) = U(t_{I,j}), \quad 0 \leqslant j \leqslant N. \tag{2.8}$$

Because of (2.7), for any $\varphi \in \mathcal{P}_N$,

$$\langle \mathcal{I}_N U, \varphi\rangle_I = \langle \mathcal{I}_N U, \varphi\rangle_{I,N} = \langle U, \varphi\rangle_{I,N}. \tag{2.9}$$

This shows that the interpolant $\mathcal{I}_N U$ is the orthogonal projection of u upon $\mathcal{P}_N$ on I with respect to the discrete inner product (2.6). The interpolation $\mathcal{I}_N U(t)$ in the interval I can be expanded as

$$\mathcal{I}_N U(t) = \sum_{n=0}^{N} \widetilde{u}_{I,n} L_{I,n}(t), \tag{2.10}$$

and with the aid of (2.4) and (2.9) we obtain

$$\widetilde{u}_{I,n} = \frac{2n+1}{b-a} \langle \mathcal{I}_N U, L_{I,n}\rangle_I = \frac{2n+1}{b-a} \langle U, L_{I,n}\rangle_{I,N}, \quad 0 \leqslant n \leqslant N. \tag{2.11}$$

2.2 The solution method

Let us start this section with the properties concerning the solution of DDEs. Assume that the model (1.1) has a unique solution. Then, this solution behavior is described based on the construction associated with the initial function $\phi(t)$. Let $\phi(t) \in \mathcal{C}^l[-\tau, 0]$ with $l \geqslant 1$. Then the solution $U(t)$ will be continuous for $t \in [0, T]$. A continuous solution with a discontinuous derivative could have occurred when $\phi(t)$ is continuous but has a jump at some points in $(-\tau, 0]$ for the first derivative. For problems with $\phi(t) \in \mathcal{C}^1[-\tau, 0]$, if the so called "sewing condition",

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(0) = f(\phi(0), \phi(-\tau), \frac{\mathrm{d}}{\mathrm{d}t}\phi(-\tau), 0), \tag{2.12}$$

is fulfilled, then the solution of both RDDEs and NDDEs is a class $\mathcal{C}^1$ function for $t \in [-\tau, T]$. Otherwise, the solution is just a class $\mathcal{C}$ function for $t \in [-\tau, T]$. In this case, the solution of RDDEs will not have a two-sided derivative at $t = 0$, whereas the solution of NDDEs will not have two-sided derivatives at all the breaking points $t = k\tau$, $k \in \mathbb{Z}^+$. In this case, we interpret the derivative at breaking points as the right-hand derivative. On the other hand, if $\phi(t)$ is piecewise continuous then the solution of RDDEs would still be continuous whilst the solution of NDDEs would be piecewise continuous as well [3].

Now, assume that $\phi(t) \in \mathcal{C}^l[-\tau, 0]$ with $l \geqslant 0$, and $\frac{\mathrm{d}^{l+1}}{\mathrm{d}t^{l+1}}\phi(t)$ is piecewise continuous. By this we mean that $\frac{\mathrm{d}^{l+1}}{\mathrm{d}t^{l+1}}\phi(t)$ is continuous on any compact subinterval of $[-\tau, 0]$ except at a finite number of points $\{-\sigma_r\}_{r=1}^{R} \subset (-\tau, 0]$ at each of which $\frac{\mathrm{d}^{l+1}}{\mathrm{d}t^{l+1}}\phi(t)$ has jump discontinuity with continuity from the right. In this case, the existence and uniqueness of the solutions of the model problem (1.1) are assured under suitable conditions on f [12, 16].

Without loss of generality we assume that $T = K\tau$ for a positive integer K, and we denote by $U_k(t)$, $k = 1, \ldots, K$ the local solution of the problem in subinterval $I_k = [(k-1)\tau, k\tau)$ (i.e., between two breaking points of the system). In this case, although the global solution $U(t)$ of both RDDEs and NDDEs is continuous, the local solution $U_k(t)$ would not be enough smooth unless l is large or $\phi(t) \in \mathcal{C}^\infty[-\tau, 0]$. Hence, in order to obtain better approximations and to keep the so called "spectral accuracy", we divide each subinterval I_k into $R + 1$ additional subintervals $[(k-1)\tau, k\tau - \sigma_1)$, $[k\tau - \sigma_1, k\tau - \sigma_2)$, …, $[k\tau - \sigma_R, k\tau)$ which are not necessarily equidistant. This defines a nonuniform mesh for the domain $[-\tau, T]$ with the mesh points

$$-\tau < -\sigma_1 < \cdots < -\sigma_R < 0 < \tau - \sigma_1 < \cdots < \tau - \sigma_R < \tau < \cdots < T,$$

denoted by $\mathfrak{D} = \{\zeta_i\}_{i=-R-1}^{K(R+1)}$. Evidently this set of mesh points includes all the breaking points of the system. Now let $I^{(i)} = [\zeta_{i-1}, \zeta_i)$ and $d_i = \zeta_i - \zeta_{i-1}$. We denote by $\chi_i(t)$ the smooth local solution of the problem on subinterval $I^{(i)}$. By virtue of the method of steps, the DDE (1.1) can be replaced with the following sequence of initial value problems:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\chi_i(t) = f\left(\chi_i(t), \chi_{i-R-1}(t-\tau), \frac{\mathrm{d}}{\mathrm{d}t}\chi_{i-R-1}(t-\tau), t\right), \quad t \in I^{(i)}, \\ \chi_i(\zeta_{i-1}) = \chi_{i-1}(\zeta_{i-1}), \quad i = 1, 2, \ldots, K(R+1), \end{cases} \tag{2.13}$$

where for $r = 1, \ldots, R+1$ and $t \in I^{(r-R-1)}$ we set $\chi_{r-R-1}(t) = \phi(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\chi_{r-R-1}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(t)$, which are well-defined since $\phi(t)$ and its first derivative are continuous on each subinterval $I^{(r-R-1)}$. Let $t_{I^{(i)},j}$ be the shifted LGR quadrature points on subinterval $I^{(i)}$. In the i^{th} step, the LGR collocation method for solving (2.13) is to seek $u_i^N(t) \in \mathcal{P}_N(I^{(i)})$, such that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u_i^N(t_{I^{(i)},j}) = f\left(u_i^N(t_{I^{(i)},j}), u_{i-R-1}^N(t_{I^{(i)},j} - \tau), \frac{\mathrm{d}}{\mathrm{d}t}u_{i-R-1}^N(t_{I^{(i)},j} - \tau), t_{I^{(i)},j}\right), \quad 1 \leqslant j \leqslant N, \\ u_i^N(t_{I^{(i)},0}) = u_{i-1}^N(t_{I^{(i)},0}). \end{cases} \tag{2.14}$$

Noteworthy, the possible jump discontinuities in the first derivative of $U(t)$ at breaking points is not an issue for approximating the solution of NDDEs, because the LGR scheme avoids collocation at breaking points.

We next describe the numerical implementation for (2.14). In [28], we expanded the unknown functions by the Lagrange interpolation, but as is well-known, the Lagrange interpolation is not stable for large N. In this work we derive a new implementation, in which one interpolates $u_i^N(t)$ by the shifted Legendre orthogonal polynomials. To do this, let

$$u_i^N(t) = \sum_{n=0}^{N} \widetilde{u}_{I^{(i)},n}^N L_{I^{(i)},n}(t), \quad t \in I^{(i)}. \tag{2.15}$$

Since $u_i^N(t)L_{I^{(i)},n}(t) \in \mathcal{P}_{2N}$, by integrating it over the interval $I^{(i)}$ and using (2.4) and (2.7) it can be verified that

$$\begin{aligned} \widetilde{u}_{I^{(i)},n}^N &= \frac{2n+1}{d_i}\langle u_i^N, L_{I^{(i)},n}\rangle_{I^{(i)}} = \frac{2n+1}{d_i}\langle u_i^N, L_{I^{(i)},n}\rangle_{I^{(i)},N} \\ &= \frac{2n+1}{d_i}\sum_{j=0}^{N} u_i^N(t_{I^{(i)},j})L_{I^{(i)},n}(t_{I^{(i)},j})w_{I^{(i)},j}, \quad 0 \leqslant n \leqslant N. \end{aligned} \tag{2.16}$$

Then, by virtue of (2.3), we deduce that

$$\frac{\mathrm{d}}{\mathrm{dt}}u_i^N(t) = \frac{1}{d_i}\sum_{n=1}^{N}(n+1)\widetilde{u}_{I^{(i)},n}^N J_{I^{(i)},n-1}^{(1,1)}(t), \quad t \in I^{(i)}. \tag{2.17}$$

Furthermore, using (2.1), a direct calculation shows $L_{I^{(i)},n}(t_{I^{(i)},0}) = (-1)^n$. Thereby, we have from (2.14) and (2.15) with $t = t_{I^{(i)},0} = \zeta_{i-1}$ that

$$\sum_{n=0}^{N}(-1)^n \widetilde{u}_{I^{(i)},n}^N = u_i^N(t_{I^{(i)},0}). \tag{2.18}$$

Consequently, we use (2.15)–(2.18) to obtain from (2.14) that

$$\begin{cases} \frac{1}{d_i}\sum_{n=1}^{N}(n+1)\widetilde{u}_{I^{(i)},n}^N J_{I^{(i)},n-1}^{(1,1)}(t_{I^{(i)},j}) = f(u_i^N(t_{I^{(i)},j}), u_{i-R-1}^N(t_{I^{(i-R-1)},j}), \frac{\mathrm{d}}{\mathrm{dt}}u_{i-R-1}^N(t_{I^{(i-R-1)},j}), t_{I^{(i)},j}), \\ \sum_{n=0}^{N}(-1)^n \widetilde{u}_{I^{(i)},n}^N = u_i^N(t_{I^{(i)},0}), \quad 1 \leqslant j \leqslant N, \quad 1 \leqslant i \leqslant K(R+1). \end{cases} \tag{2.19}$$

Note that the values of $u_{i-R-1}^N(t_{I^{(i-R-1)},j})$ and $\frac{\mathrm{d}}{\mathrm{dt}}u_{i-R-1}^N(t_{I^{(i-R-1)},j})$ are known and obtained in preceding steps. When the function f is nonlinear, we first use certain iteration process to solve (2.19) and obtain $u_i^N(t_{I^{(i)},j})$, $0 \leqslant j \leqslant N$. Next, we use (2.15) and (2.16) to obtain the approximation of $\chi_i(t)$ and we get $u_i^N(\zeta_i)$ by (2.15) with $L_{I^{(i)},n}(\zeta_i) = 1$. Finally, we use $u_i^N(t)$, $\frac{\mathrm{d}}{\mathrm{dt}}u_i^N(t)$ and $u_i^N(\zeta_i)$ as the history and the initial value to be used in the next step.

Remark 2.1. *When $\phi(t)$ is sufficiently smooth, the above multistep LGR collocation method can be performed on subintervals $I_k = [(k-1)\tau, k\tau)$, $k = 1, 2, \ldots, K$ to reduce the number of steps. However, when τ is large, we may need to resolve the resulted discrete system with very large mode N that is not convenient. On the other hand, for ensuring the convergence of scheme (2.14), the length of each subinterval is limited sometimes (see Section 3). Therefore, utilizing additional subintervals would be beneficial for keeping the spectral accuracy.*

3 Error analysis of the multistep LGR collocation method

For simplicity of statement, we analyse the convergence of scheme (2.14) with the assumption that the solutions between breaking points, i.e. $u_k(t)$ for $k = 1, 2, \ldots, K$ are smooth. The obtained results in this section can be extended in a straightforward manner to problems considered in Section 2.2 by replacing subintervals I_k with subintervals $I^{(i)}$, $i = 1, 2, \ldots, K(R+1)$.

We shall prove the spectral accuracy of numerical solutions $u_k^N(t)$. On each step we shall compare $u_k^N(t)$ with the interpolation approximation $\mathcal{I}_N U_k(t)$. Note that, in the first step of scheme (2.14), the exact values of the initial condition and the delay terms (i.e. $\phi(0)$, $\phi(t-\tau)$ and $\frac{\mathrm{d}}{\mathrm{dt}}\phi(t-\tau)$) are available, whereas in the subsequent steps they are approximated using the results of preceding steps. Therefore, in order to establish our convergence results, we proceed step by step on successive subintervals I_k. Furthermore, convergence results for RDDEs are a direct consequence of this section.

In the forthcoming discussions, we assume that there exists a Lipschitz constant $\gamma \geqslant 0$ such that

$$|f(y_1, y_2, y_3, t) - f(z_1, z_2, z_3, t)| \leqslant \gamma\left(|y_1 - z_1| + |y_2 - z_2| + |y_3 - z_3|\right). \tag{3.1}$$

We also make use of the following lemma and theorem in our subsequent development. In what follows, $H^r(I_k)$ denotes the Sobolev space of integer order r on I_k, and $\|\cdot\|_{I_k}$ denotes the norm of space $L^2(I_k)$.

Lemma 3.1. Let $u \in H^1(I_k)$, then for any $\varepsilon > 0$

$$\|u\|_{L^\infty(I_k)}^2 \leqslant \varepsilon^{-1}\|u\|_{I_k}^2 + \varepsilon\left\|\frac{\mathrm{d}}{\mathrm{d}t}u\right\|_{I_k}^2 + |u(t_{I_k,0})|^2. \tag{3.2}$$

Proof . If $u \in H^1(I_k)$, then it is continuous and piecewise continually differentiable. Thus using Hölder's inequality one has

$$|u(t)|^2 - |u(t_{I_k,0})|^2 = \int_{t_{I_k,0}}^{t} \frac{\mathrm{d}}{\mathrm{d}x}(u^2(x))\,\mathrm{d}x \leqslant 2\,\|u\|_{I_k} \left\|\frac{\mathrm{d}}{\mathrm{d}t}u\right\|_{I_k} \leqslant \varepsilon^{-1}\,\|u\|_{I_k}^2 + \varepsilon \left\|\frac{\mathrm{d}}{\mathrm{d}t}u\right\|_{I_k}^2,$$

as desired. □

Theorem 3.1. For any $U \in H^r(I_k)$ with integers $r \geqslant 1$, we have

$$\|U - \mathcal{I}_N U\|_{I_k}^2 \leqslant cN^{-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l} \left\|U^{(l)}\right\|_{I_k}^2, \tag{3.3}$$

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}(U - \mathcal{I}_N U)\right\|_{I_k}^2 \leqslant cN^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l-2} \left\|U^{(l)}\right\|_{I_k}^2, \tag{3.4}$$

where $U^{(l)}$ is the l^{th} distributional derivative of U and c is a generic positive constant dependent only on r.

Proof . Consult (5.4.33) and (5.4.34) of [3] and Theorem 1 of [28]. □

Thus, by virtue of (3.3) and setting $U \to \frac{\mathrm{d}}{\mathrm{d}t}U$ and $r \to r-1$,

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}U - \mathcal{I}_N \frac{\mathrm{d}}{\mathrm{d}t}U\right\|_{I_k}^2 \leqslant cN^{2-2r} \sum_{l=\min\{r-1,N\}}^{r-1} \tau^{2l} \left\|U^{(l+1)}\right\|_{I_k}^2. \tag{3.5}$$

3.1 Error analysis of step one

Theorem 3.2. If the function f in scheme (2.14) satisfies the Lipschits condition (3.1) and β_1 be a positive constant such that $\sqrt{8}\gamma\tau \leqslant \beta_1 < 1$, then for any $U_1 \in H^r(I_1)$ and integer $r \geq 2$,

$$\left\|U_1 - u_1^N\right\|_{I_1}^2 \leqslant c_{\beta_1} N^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l} \|U_1\|_{I_1}^2, \tag{3.6}$$

$$\left|U_1(\tau) - u_1^N(\tau)\right|^2 \leqslant c_{\beta_1} N^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l-1} \|U_1\|_{I_1}^2, \tag{3.7}$$

where c_{β_1} is a positive constant depending only on β_1 and r.

Proof . Consider (2.13) and (2.14) with k instead of i. Let $t \in I_1$ and

$$\xi_{1,1}^N(t) = \mathcal{I}_N \frac{\mathrm{d}}{\mathrm{d}t}U_1(t) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_N U_1(t), \tag{3.8}$$

$$\xi_{1,2}^N(t) = f(u_1^N(t), \phi(t-\tau), \frac{\mathrm{d}}{\mathrm{d}t}\phi(t-\tau), t) - f(\mathcal{I}_N U_1(t), \phi(t-\tau), \frac{\mathrm{d}}{\mathrm{d}t}\phi(t-\tau), t), \tag{3.9}$$

$$E_1^N(t) = u_1^N(t) - \mathcal{I}_N U_1(t). \tag{3.10}$$

It is clear that $\xi_{1,1}^N, E_1^N \in \mathcal{P}_N(I_1)$. Using (2.13) we have

$$\mathcal{I}_N \frac{\mathrm{d}}{\mathrm{d}t}U_1(t_{I_1,j}) = \frac{\mathrm{d}}{\mathrm{d}t}U_1(t_{I_1,j}) = f(U_1(t_{I_1,j}), \phi(t_{I_1,j}-\tau), \frac{\mathrm{d}}{\mathrm{d}t}\phi(t_{I_1,j}-\tau), t_{I_1,j}).$$

Then, by adding the term $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_N U_1(t_{I_1,j})$ to both sides, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_N U_1(t_{I_1,j}) = f(U_1(t_{I_1,j}), \phi(t_{I_1,j}-\tau), \frac{\mathrm{d}}{\mathrm{d}t}\phi(t_{I_1,j}-\tau), t_{I_1,j}) - \xi_{1,1}^N(t_{I_1,j}). \tag{3.11}$$

Subtracting (3.11) from (2.14) with $k = 1$, results

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}E_1^N(t_{I_1,j}) = \xi_{1,1}^N(t_{I_1,j}) + \xi_{1,2}^N(t_{I_1,j}), \quad 1 \leqslant j \leqslant N, \\ E_1^N(0) = 0. \end{cases} \tag{3.12}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}E_1^N(t) \in \mathcal{P}_{N-1}(I_1)$, we use (2.7) and (3.12) to obtain that

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t}E_1^N \right\|_{I_1}^2 = \left\| \frac{\mathrm{d}}{\mathrm{d}t}E_1^N \right\|_{I_1,N}^2 = \left\| \xi_{1,1}^N + \xi_{1,2}^N \right\|_{I_1,N}^2 \leqslant 2\left\| \xi_{1,1}^N \right\|_{I_1,N}^2 + 2\left\| \xi_{1,2}^N \right\|_{I_1,N}^2. \tag{3.13}$$

Therefore, Lemma 3.1 with $\varepsilon = \varepsilon_1 > 0$ and (3.13) yield

$$\left| E_1^N(\tau) \right|^2 \leqslant \varepsilon_1 \left\| \frac{\mathrm{d}}{\mathrm{d}t}E_1^N \right\|_{I_1}^2 + \varepsilon_1^{-1} \left\| E_1^N \right\|_{I_1}^2 \leqslant 2\varepsilon_1 (\left\| \xi_{1,1}^N \right\|_{I_1,N}^2 + \left\| \xi_{1,2}^N \right\|_{I_1,N}^2) + \varepsilon_1^{-1} \left\| E_1^N \right\|_{I_1}^2. \tag{3.14}$$

We next estimate $\left\| \xi_{1,1}^N \right\|_{I_1,N}$. Since $\xi_{1,1}^N \in \mathcal{P}_N$, if $U_1 \in H^r(I_1)$ with $r \geq 2$, then Theorem 3.1 together with (3.5), gives

$$\begin{aligned} \left\| \xi_{1,1}^N \right\|_{I_1,N}^2 = \left\| \xi_{1,1}^N \right\|_{I_1}^2 &\leqslant 2\left\| \frac{\mathrm{d}}{\mathrm{d}t}(U_1 - \mathcal{I}_N U_1) \right\|_{I_1}^2 + 2\left\| \frac{\mathrm{d}}{\mathrm{d}t}U_1 - \mathcal{I}_N \frac{\mathrm{d}}{\mathrm{d}t}U_1 \right\|_{I_1}^2 \\ &\leqslant cN^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l-2} \left\| U_1^{(l)} \right\|_{I_1}^2. \end{aligned} \tag{3.15}$$

In order to estimate $||\xi_{1,2}^N||_{I_1,N}$, we assume that f satisfies the Lipschitz condition (3.1). Note that, in step one we have $y_2 = z_2 = \phi(t)$ and $y_3 = z_3 = \frac{\mathrm{d}}{\mathrm{d}t}\phi(t)$. Since $E_1^N(t) \in \mathcal{P}_N(I_1)$, we use (2.7), (3.1) and (3.9) to obtain

$$\left\| \xi_{1,2}^N \right\|_{I_1,N}^2 \leqslant \gamma^2 \left\| E_1^N \right\|_{I_1,N}^2 = \gamma^2 \left\| E_1^N \right\|_{I_1}^2. \tag{3.16}$$

Next, substituting from (3.16) into (3.14) results

$$\left| E_1^N(\tau) \right|^2 \leqslant 2\varepsilon_1 \left\| \xi_{1,1}^N \right\|_{I_1}^2 + (2\varepsilon_1\gamma^2 + \varepsilon_1^{-1}) \left\| E_1^N \right\|_{I_1}^2. \tag{3.17}$$

Again using Lemma 3.1 with $\varepsilon = \varepsilon_2 > 0$, we have

$$\left| E_1^N(t) \right|^2 \leqslant \varepsilon_2 \left\| \frac{\mathrm{d}}{\mathrm{d}t}E_1^N \right\|_{I_1}^2 + \varepsilon_2^{-1} \left\| E_1^N(t) \right\|_{I_1}^2.$$

Integrating the above inequality over I_1 with respect to t, and using (3.13), (3.16) yields

$$\begin{aligned} \left\| E_1^N \right\|_{I_1}^2 &\leqslant \varepsilon_2\tau \left\| \frac{\mathrm{d}}{\mathrm{d}t}E_1^N \right\|_{I_1}^2 + \varepsilon_2^{-1}\tau \left\| E_1^N \right\|_{I_1}^2 \\ &\leqslant 2\varepsilon_2\tau \left\| \xi_{1,1}^N \right\|_{I_1}^2 + 2\varepsilon_2\tau\gamma^2 \left\| E_1^N \right\|_{I_1}^2 + \varepsilon_2^{-1}\tau \left\| E_1^N \right\|_{I_1}^2. \end{aligned} \tag{3.18}$$

Consequently, we have from (3.15) that

$$\begin{aligned} \left\| E_1^N \right\|_{I_1}^2 &\leqslant \frac{2\varepsilon_2\tau}{1 - 2\varepsilon_2\tau\gamma^2 - \varepsilon_2^{-1}\tau} \left\| \xi_{1,1}^N \right\|_{I_1}^2 \\ &\leqslant \frac{2\varepsilon_2\tau}{1 - 2\varepsilon_2\tau\gamma^2 - \varepsilon_2^{-1}\tau} cN^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l-2} \left\| U_1^{(l)} \right\|_{I_1}^2. \end{aligned} \tag{3.19}$$

Obviously, the constant coefficient in (3.19) is positive provided that $8\tau^2\gamma^2 < 1$. To meet this requirement, we take $\varepsilon_2 = 2\tau$ to obtain

$$\left\|E_1^N\right\|_{I_1}^2 \leqslant c_{\beta_1} N^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l} \left\|U_1^{(l)}\right\|_{I_1}^2, \tag{3.20}$$

where the positive constant β_1 is such that $\sqrt{8}\tau\gamma \leqslant \beta_1 < 1$. Moreover, due to (3.13), (3.15), (3.16) and (3.20) and since $\gamma^2 < \frac{1}{8\tau^2}$, we have

$$\begin{aligned} \left\|\frac{\mathrm{d}}{\mathrm{d}t}E_1^N\right\|_{I_1}^2 &\leqslant 2\left\|\xi_{1,1}^N\right\|_{I_1}^2 + 2\gamma^2\left\|E_1^N\right\|_{I_1}^2 \\ &\leqslant c_{\beta_1} N^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l-2} \left\|U_1^{(l)}\right\|_{I_1}^2. \end{aligned} \tag{3.21}$$

Now, substituting (3.15) and (3.20) into (3.17) gives

$$\left|E_1^N(\tau)\right|^2 \leqslant 2\varepsilon_1 c N^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l-2} \left\|U_1^{(l)}\right\|_{I_1}^2 + \left(2\varepsilon_1\gamma^2 + \varepsilon_1^{-1}\right) c_{\beta_1} N^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l} \left\|U_1^{(l)}\right\|_{I_1}^2. \tag{3.22}$$

Taking $\varepsilon_1 = 2\tau$, we have $2\varepsilon_1\gamma^2 + \varepsilon_1^{-1} \leqslant \tau^{-1}$. Therefore,

$$\left|E_1^N(\tau)\right|^2 \leqslant c_{\beta_1} N^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l-1} \left\|U_1^{(l)}\right\|_{I_1}^2. \tag{3.23}$$

Consequently, we use Theorem 3.1 and (3.20) to derive that

$$\begin{aligned} \left\|U_1 - u_1^N\right\|_{I_1}^2 &\leqslant 2\left\|U_1 - \mathcal{I}_N U_1\right\|_{I_1}^2 + 2\left\|E_1^N\right\|_{I_1}^2 \\ &\leqslant c_{\beta_1} N^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l} \left\|U_1^{(l)}\right\|_{I_1}^2. \end{aligned}$$

Furthermore, Lemma 3.1, along with Theorem 3.1, leads to

$$\begin{aligned} \left|U_1(\tau) - \mathcal{I}_N U_1(\tau)\right|^2 &\leqslant \tau^{-1}\left\|U_1 - \mathcal{I}_N U_1\right\|_{I_1}^2 + \tau\left\|\frac{\mathrm{d}}{\mathrm{d}t}(U_1 - \mathcal{I}_N U_1)\right\|_{I_1}^2 \\ &\leqslant c N^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l-1} \left\|U_1^{(l)}\right\|_{I_1}^2. \end{aligned} \tag{3.24}$$

So, we use (3.23) and (3.24) to obtain

$$\begin{aligned} \left|U_1(\tau) - u_1^N(\tau)\right|^2 &\leqslant 2\left|U_1(\tau) - \mathcal{I}_N U_1(\tau)\right|^2 + 2\left|E_1^N(\tau)\right|^2 \\ &\leqslant c_{\beta_1} N^{3-2r} \sum_{l=\min\{r,N+1\}}^{r} \tau^{2l-1} \left\|U_1^{(l)}\right\|_{I_1}^2, \end{aligned}$$

which completes the proof. □

Remark 3.1. *The same upper bound as (3.7) can also be derived for* $\left\|U_1(t) - u_1^N(t)\right\|_{L^\infty(I_1)}$. *In addition, we see from the error bounds in Theorem 3.2 that the first step of the scheme (2.14) possesses the spectral accuracy.*

3.2 Error analysis of subsequent steps

As stated earlier, in steps two onwards, the exact values of $x_k^N((k-1)\tau)$, $x_{k-1}^N(t-\tau)$ and $\frac{\mathrm{d}}{\mathrm{d}t}x_{k-1}^N(t-\tau)$, $t \in I_k$ are not available and we approximate them using the results obtained in step $k-1$. This affects on the accuracy of the solution of the k^{th} step, $2 \leqslant k \leqslant K$, which is analysed now.

Theorem 3.3. If f in scheme (2.14) satisfies the Lipschitz condition (3.1) and β_k with $k \geq 2$ be a positive constant such that $\sqrt{24}\gamma\tau \leqslant \beta_k < 1$, then for any $X_k \in H^r(I_k)$ and integer $r \geqslant 2$,

$$\left\|X_k - x_k^N\right\|_{I_k}^2 \leqslant d_{\beta_k} N^{3-2r} \sum_{l=min\{r,N+1\}}^{r} \sum_{\kappa=1}^{k} \tau^{2l-2} \left\|X_\kappa^{(l)}\right\|_{I_k}^2, \tag{3.25}$$

$$\left|X_k(k\tau) - x_k^N(k\tau)\right|^2 \leqslant d_{\beta_k} N^{3-2r} \sum_{l=min\{r,N+1\}}^{r} \sum_{\kappa=1}^{k} \tau^{2l-3} \left\|X_\kappa^{(l)}\right\|_{I_k}^2, \tag{3.26}$$

where d_{β_k} is a positive constant depending only on β_k and r.

Proof . Consider (2.13), (2.14) with $i = k \geqslant 2$. Let $t \in I_k$ and

$$\xi_{k,1}^N(t) = \mathcal{I}_N \frac{\mathrm{d}}{\mathrm{d}t} X_k(t) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_N X_k(t), \tag{3.27}$$

$$\xi_{k,2}^N(t) = f(x_k^N(t), x_{k-1}^N(t-\tau), \frac{\mathrm{d}}{\mathrm{d}t}x_{k-1}^N(t-\tau), t) - f(\mathcal{I}_N X_k(t), \mathcal{I}_N X_{k-1}^N(t-\tau), \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_N X_{k-1}^N(t-\tau), t), \tag{3.28}$$

$$E_k^N(t) = x_k^N(t) - \mathcal{I}_N X_k(t). \tag{3.29}$$

Then, similar to the derivation of (3.12), since $t_{I_k,0} = (k-1)\tau$ we obtain

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}E_k^N(t_{I_k,j}) = \xi_{k,1}^N(t_{I_k,j}) + \xi_{k,2}^N(t_{I_k,j}), \quad 1 \leqslant j \leqslant N \\ E_k^N(t_{I_k,0}) = x_{k-1}^N(t_{I_k,0}) - X_{k-1}(t_{I_k,0}). \end{cases} \tag{3.30}$$

Further, since $\frac{\mathrm{d}}{\mathrm{d}t}E_k^N(t) \in \mathcal{P}_{N-1}(I_k)$, we have

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}E_k^N\right\|_{I_k}^2 = \left\|\frac{\mathrm{d}}{\mathrm{d}t}E_k^N\right\|_{I_k,N}^2 = \left\|\xi_{k,1}^N + \xi_{k,2}^N\right\|_{I_k,N}^2 \leqslant 2\left\|\xi_{k,1}^N\right\|_{I_k,N}^2 + 2\left\|\xi_{k,2}^N\right\|_{I_k,N}^2. \tag{3.31}$$

Considering that in the k^{th} step $E_k^N(t_{I_k,0}) \neq 0$, here, Lemma 3.1 reads,

$$\left|E_k^N(k\tau)\right|^2 \leqslant \varepsilon_1 \left\|\frac{\mathrm{d}}{\mathrm{d}t}E_k^N\right\|_{I_k}^2 + \varepsilon_1^{-1}\left\|E_k^N\right\|_{I_k}^2 + \left|E_k^N(t_{I_k,0})\right|^2. \tag{3.32}$$

Therefore, using (3.30)-(3.32) we obtain

$$\left|E_k^N(k\tau)\right|^2 \leqslant 2\varepsilon_1\left\|\xi_{k,1}^N\right\|_{I_k,N}^2 + 2\varepsilon_1\left\|\xi_{k,2}^N\right\|_{I_k,N}^2 + \varepsilon_1^{-1}\left\|E_k^N\right\|_{I_k}^2 + \left|X_{k-1}(t_{I_k,0}) - x_{k-1}^N(t_{I_k,0})\right|^2. \tag{3.33}$$

Now, similar to the derivation of the error bound (3.15), if $X_k \in H^r(I_k)$ with $r \geq 2$, using (2.7) we have

$$\left\|\xi_{k,1}^N\right\|_{I_k,N}^2 = \left\|\xi_{k,1}^N\right\|_{I_k}^2 \leqslant cN^{3-2r} \sum_{l=min\{r,N+1\}}^{r} \tau^{2l-2}\left\|X_k^{(l)}\right\|_{I_k}^2. \tag{3.34}$$

Moreover, utilizing the Lipschitz condition (3.1), yields

$$\left\|\xi_{k,2}^N\right\|_{I_k,N}^2 \leqslant 3\gamma^2\left(\left\|E_k^N\right\|_{I_k,N}^2 + \left\|E_{k-1}^N\right\|_{I_{k-1},N}^2 + \left\|\frac{\mathrm{d}}{\mathrm{d}t}E_{k-1}^N\right\|_{I_{k-1},N}^2\right). \tag{3.35}$$

Since $E_{k-1}^N, E_k^N \in \mathcal{P}_N$ and $\left|\xi_{k,1}^N(t_{I_k,0})\right| < \infty$, we have from (2.7), (3.33) and (3.35) that

$$\begin{aligned}\left|E_k^N(k\tau)\right|^2 &\leqslant (6\varepsilon_1\gamma^2+\varepsilon_1^{-1})\left\|E_k^N\right\|_{I_k}^2 + 2\varepsilon_1\left\|\xi_{k,1}^N\right\|_{I_k}^2\\ &\quad +6\varepsilon_1\gamma^2\left(\left\|E_{k-1}^N\right\|_{I_{k-1}}^2+\left\|\frac{\mathrm{d}}{\mathrm{d}t}E_{k-1}^N\right\|_{I_{k-1}}^2\right)+\left|X_{k-1}(t_{I_k,0})-x_{k-1}^N(t_{I_k,0})\right|^2. \qquad (3.36)\end{aligned}$$

On the other hand, like (3.18), (3.32) with $\varepsilon_1=\varepsilon_2=2\tau$, we have

$$\left\|E_k^N\right\|_{I_k}^2 \leqslant 2\tau\left|X_{k-1}(t_{I_k,0})-x_{k-1}^N(t_{I_k,0})\right|^2+4\tau^2\left\|\frac{\mathrm{d}}{\mathrm{d}t}E_k^N\right\|_{I_k}^2.$$

The above with (3.31) and (3.35) yields

$$\begin{aligned}(1-24\gamma^2\tau^2)\left\|E_k^N\right\|_{I_k}^2 &\leqslant 8\tau^2\left\|\xi_{k,1}^N\right\|_{I_k}^2+24\gamma^2\tau^2\left(\left\|E_{k-1}^N\right\|_{I_{k-1}}^2+\left\|\frac{\mathrm{d}}{\mathrm{d}t}E_{k-1}^N\right\|_{I_{k-1}}^2\right)\\ &\quad +2\tau\left|X_{k-1}(t_{I_k,0})-x_{k-1}^N(t_{I_k,0})\right|^2.\end{aligned}$$

Consequently, if $\sqrt{24}\gamma\tau \leqslant \beta_k < 1$, then

$$\left\|E_k^N\right\|_{I_k}^2 \leqslant c_{\beta_k}\left(\tau^2\left\|\xi_{k,1}^N\right\|_{I_k}^2+\left\|E_{k-1}^N\right\|_{I_{k-1}}^2+\left\|\frac{\mathrm{d}}{\mathrm{d}t}E_{k-1}^N\right\|_{I_{k-1}}^2+\tau\left|X_{k-1}(t_{I_k,0})-x_{k-1}^N(t_{I_k,0})\right|^2\right).$$

Now, taking $k=2$ and substituting (3.7), (3.20), (3.21) and (3.34) into the above inequality, results

$$\begin{aligned}\left\|E_2^N\right\|_{I_2}^2 &\leqslant c_{\beta_2}N^{3-2r}\sum_{l=min\{r,N+1\}}^{r}\left(\tau^{2l}\left\|X_2^{(l)}\right\|_{I_2}^2+(\tau^{2l}+\tau^{2l-2})\left\|X_1^{(l)}\right\|_{I_1}^2\right)\\ &\leqslant d_{\beta_2}N^{3-2r}\sum_{l=min\{r,N+1\}}^{r}\tau^{2l-2}\left(\left\|X_2^{(l)}\right\|_{I_2}^2+\left\|X_1^{(l)}\right\|_{I_1}^2\right). \qquad (3.37)\end{aligned}$$

Moreover, by (3.37) and (3.36) with $k=2$ and $\varepsilon_1=2\tau$, and since $6\varepsilon_1\gamma^2+\varepsilon_1^{-1}\leqslant\tau^{-1}$,

$$\begin{aligned}\left|E_2^N(2\tau)\right|^2 &\leqslant c_{\beta_2}N^{3-2r}\sum_{l=min\{r,N+1\}}^{r}\left(\tau^{2l-1}\left\|X_2^{(l)}\right\|_{I_2}^2+(\tau^{2l-1}+\tau^{2l-3})\left\|X_1^{(l)}\right\|_{I_1}^2\right)\\ &\leqslant d_{\beta_2}N^{3-2r}\sum_{l=min\{r,N+1\}}^{r}\tau^{2l-3}\left(\left\|X_2^{(l)}\right\|_{I_2}^2+\left\|X_1^{(l)}\right\|_{I_1}^2\right). \qquad (3.38)\end{aligned}$$

Therefore, using Theorem 3.1 and (3.37), we conclude that

$$\begin{aligned}\left\|X_2-x_2^N\right\|_{I_2}^2 &\leqslant 2\left\|X_2-\mathcal{I}_NX_2\right\|_{I_2}^2+2\left\|E_2^N\right\|_{I_2}^2\\ &\leqslant d_{\beta_2}N^{3-2r}\sum_{l=min\{r,N+1\}}^{r}\tau^{2l-2}\left(\left\|X_2^{(l)}\right\|_{I_2}^2+\left\|X_1^{(l)}\right\|_{I_1}^2\right). \qquad (3.39)\end{aligned}$$

Further, by virtue of (3.24),

$$\left|X_2(2\tau)-\mathcal{I}_NX_2(2\tau)\right|^2\leqslant cN^{3-2r}\sum_{l=\min\{r,N+1\}}^{r}\tau^{2l-1}\left\|X_2^{(l)}\right\|_{I_2}^2.$$

This, together with (3.38), leads to

$$\begin{aligned}\left|X_2(2\tau)-x_2^N(2\tau)\right|^2 &\leqslant 2\left|E_2^N(2\tau)\right|^2+2\left|X_2(2\tau)-\mathcal{I}_N X_2(2\tau)\right|^2\\ &\leqslant d_{\beta_2}N^{3-2r}\sum_{l=min\{r,N+1\}}^{r}\tau^{2l-3}\left(\left\|X_2^{(l)}\right\|_{I_2}^2+\left\|X_1^{(l)}\right\|_{I_1}^2\right).\end{aligned}\tag{3.40}$$

Repeating the above process, we conclude for $k\geqslant 2$ that,

$$\left\|X_k-x_k^N\right\|_{I_k}^2\leqslant d_{\beta_k}N^{3-2r}\sum_{l=min\{r,N+1\}}^{r}\sum_{\kappa=1}^{k}\tau^{2l-2}\left\|X_\kappa^{(l)}\right\|_{I_k}^2,$$

and

$$\left|X_k(k\tau)-x_k^N(k\tau)\right|^2\leqslant d_{\beta_k}N^{3-2r}\sum_{l=min\{r,N+1\}}^{r}\sum_{\kappa=1}^{k}\tau^{2l-3}\left\|X_\kappa^{(l)}\right\|_{I_k}^2,$$

and the proof is completed. □

Remark 3.2. *Theorem 3.3 indicates the spectral accuracy of scheme (2.14) for $k\geqslant 2$. In addition, if $2\leqslant r\leqslant N+1$ and $X^{(r)}\in L^\infty(0,k\tau)$, then for $k\geqslant 1$ we have*

$$\left\|X-x^N\right\|_{L^2(0,k\tau)}\leqslant d_\beta^{\frac12}kN^{\frac32-r}\tau^{r-\frac12}\left\|X^{(r)}\right\|_{L^\infty(0,k\tau)},\tag{3.41}$$

$$\left|U(k\tau)-x^N(k\tau)\right|\leqslant d_\beta^{\frac12}k^{\frac12}N^{\frac32-r}\tau^{r-1}\left\|X^{(r)}\right\|_{L^\infty(0,k\tau)},\tag{3.42}$$

which demonstrate that the L^2 and the pointwise absolute errors accumulate linearly in terms of t.

4 Numerical results

In this section, we present some numerical results to illustrate the efficiency and accuracy of our multistep algorithm.

4.1 RDDE with discontinuous initial function

This example shows that the method of Subsection 2.2 may also work for RDDEs with discontinuous initial function. We consider the nonlinear RDDE,

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t)=U(t-\pi)U(t),\quad 0\leqslant t\leqslant 2\pi,\tag{4.1}$$

with discontinuous initial function

$$\phi(t)=\begin{cases}0 & t<-\frac{\pi}{2},\\ -2 & -\frac{\pi}{2}\leqslant t<0.\\ -1 & t=0.\end{cases}\tag{4.2}$$

The exact solution to this problem is

$$U(t)=\begin{cases}-1 & 0\leqslant t<\frac{\pi}{2},\\ -e^{\pi-2t} & \frac{\pi}{2}\leqslant t<\pi,\\ -e^{-t} & \pi\leqslant t<\frac{3\pi}{2},\\ -e^{-\frac{3\pi}{2}+\frac12(e^{3\pi-2t}-1)} & \frac{3\pi}{2}\leqslant t<2\pi,\end{cases}$$

which is continuous but has an n^{th} order discontinuity at $t=\{\frac{(2n-1)\pi}{2},n\pi\}$. Consequently, the scheme of Subsection 2.2 can be employed and there is no need to utilize the perturbation scheme of Subsection 4.2. In Fig. 1, we plot the maximum absolute errors for various values of R and N. As expected, in contrast to the case $R=0$, the numerical errors for $R=1$ and $R=3$ decay exponentially as N increases. This is due to discontinuity of the initial function at the points $t=-\frac{\pi}{2}$ and $t=0$.

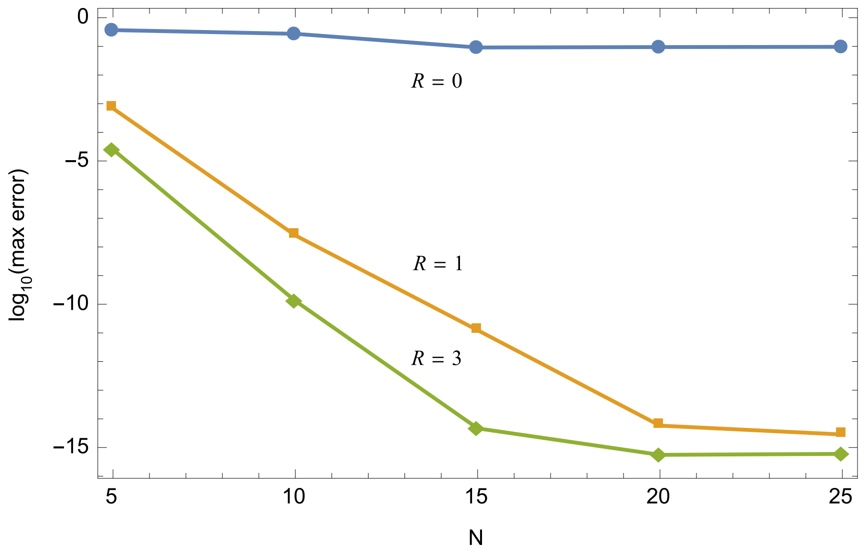

Figure 1: The maximum absolute errors of multistep LGR collocation method for Example 5.2.

4.2 A model of food-limited population

Consider the following nonlinear NDDE which models a food-limited population [26],

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = rU(t)\left(1 - U(t-1) - c\frac{\mathrm{d}}{\mathrm{d}t}U(t-1)\right), \quad t \geqslant 0, \tag{4.3}$$

with smooth initial functions

$$\phi(t) = t+2, \quad -1 \leqslant t \leqslant 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = 1, \quad -1 \leqslant t < 0. \tag{4.4}$$

Although the initial function is smooth, the sewing condition (2.12) is not fulfilled; hence, this equation has a first-order discontinuity at $t = n$ for integers $n \geqslant 0$. For $r = \frac{\pi}{\sqrt{3}} + \frac{1}{20}$ and $c = \frac{\sqrt{3}}{2\pi} - \frac{1}{25}$ a reference solution at $t = 40$ is $U(40) = 0.8044138361971349$. We utilize the scheme of Subsection 2.2 to approximate the solution of this NDDE. Fig. 2 shows the time history and the oscillatory behavior of $U(t)$. In Fig. 3, we plot the numerical errors at $t = 40$ for various values of N and R. Again, they indicate that the numerical errors decay exponentially as N and R increase. In this example, the solution between each two consecutive breaking points is smooth; hence, the case $R = 0$ naturally provides an exponential convergency. Nonetheless, the convergence rate for $R = 1$ is higher. Table 1, compares the errors of our method with the methods DDVERK, DRKLAG and ARCHI [18]. We can observe that our method provides more accurate numerical results.

To show the efficiency and convergence of the multistep LGR collocations method for large domain calculations, we also solve this problem for $0 \leqslant t \leqslant 1000$ with $R = 0$ (1000 steps) and various values of N. In Table 2, we give the approximate values of $U(1000)$. The convergence of the results is apparent.

4.3 Nonlinear stiff NDDE

Consider the following nonlinear stiff NDDE adopted from [34],

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}X_1(t) = -2X_1(t) + X_2(t) + 0.1\sin(X_1(t)) + 0.05\sin(X_2(t)) + 0.05\sin(X_1(t-\frac{\pi}{2})) \\ \qquad +\frac{1}{2}\sin(X_2(t-\frac{\pi}{2})) + 10^{-4}\frac{\mathrm{d}}{\mathrm{d}t}X_1(t-\frac{\pi}{2}) + 0.5\times 10^{-4}\frac{\mathrm{d}}{\mathrm{d}t}X_2(t-\frac{\pi}{2}) + J_1(t), \\ \frac{\mathrm{d}}{\mathrm{d}t}X_2(t) = X_1(t) - 9999X_2(t) + 0.05\sin(X_1(t)) + 0.15\sin(X_2(t)) - 0.05\sin(X_1(t-\frac{\pi}{2})) \\ \qquad +0.1\sin(X_2(t-\frac{\pi}{2})) + 0.5\times 10^{-4}\frac{\mathrm{d}}{\mathrm{d}t}X_1(t-\frac{\pi}{2}) + 10^{-4}\frac{\mathrm{d}}{\mathrm{d}t}X_2(t-\frac{\pi}{2}) + J_2(t), \end{cases} \tag{4.5}$$

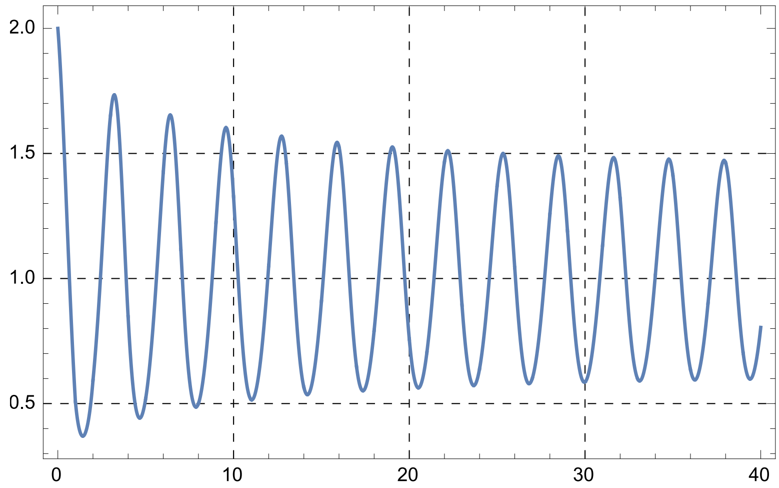

Figure 2: Solution $U(t)$ for NDDE in Example 5.3 with smooth initial function.

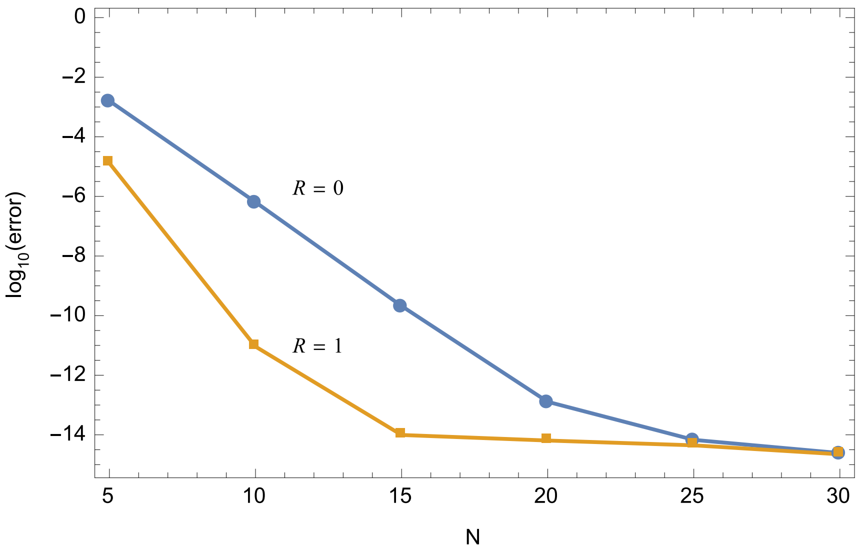

Figure 3: The numerical errors of multistep LGR collocation method at $t = 40$ for Example 5.3.

Table 1: Comparison of the numerical errors at $t = 40$ for Example 5.3.

Method	Steps	Error
DRKLAG ($TOL = 10^{-6}$, without root finding option)	241	$5.66\mathrm{e}+00$
DRKLAG ($TOL = 10^{-2}$, with root finding option)	80	$3.29\mathrm{e}-01$
DDVERK	620	$4.78\mathrm{e}+00$
ARCHI	1590	$3.41\mathrm{e}-01$
Present method ($N = 20$)	40 ($R = 0$)	$1.28\mathrm{e}-13$
	80 ($R = 1$)	$6.44\mathrm{e}-15$

Table 2: Numerical results at $t = 1000$ using the multistep LGR collocation method with $R = 0$ for Example 5.3.

N	5	10	15	20	25
$U(1000)$	0.8341844212	0.8015301619	0.8015311515	0.8015311564	0.8015311565

Table 3: Comparison of global errors for Example 5.4.

h	Mesh points	Method I [34]	N	Mesh points	Present method (R=0)
$\pi/400$	4000	$1.63\text{e}-04$	5	120	$2.12\text{e}-02$
$\pi/1600$	16000	$1.02\text{e}-05$	10	220	$1.41\text{e}-06$
$\pi/6400$	64000	$6.37\text{e}-07$	15	320	$5.35\text{e}-10$

for $0 \leqslant t \leqslant 10\pi$ with smooth initial functions $\phi_1(t) = \sin(3t)$ and $\phi_2(t) = \cos(\frac{t}{2})$ for $-\frac{\pi}{2} \leqslant t \leqslant 0$. $J_1(t)$ and $J_2(t)$ are assigned functions such that problem (4.5) has exact solutions $X_1(t) = \sin(3t)$ and $X_2(t) = \cos(\frac{t}{2})$.

In Table 3, comparison of global errors between the present method and Method I (an implicit-explicit one-leg method) proposed in [34] is shown. We see that for $R = 0$ and $N = 15$ with 320 mesh points, our method is much more accurate than the method in [34] by 64000 mesh points.

5 Concluding remarks

In this paper, we proposed a multistep LGR collocation method for solving delay differential equations with constant delays. This method enables us to evaluate the numerical solutions with moderate N, step by step. Benefiting from the rapid convergence of the LGR interpolation, this method possesses the spectral accuracy.Conversely, the existing numerical methods for delay differential equations do not usually have such fascinating merits. In particular, for any fixed mode N, the numerical solutions have a higher convergence rate than the usual implicit Runge-Kutta methods. In addition, in the existing Runge-Kutta methods, the Lagrange interpolation is usually used which is unstable for large N; whereas, we used the Gauss-type interpolation that makes our method much more stable for large N. Moreover, this method often works well even for a large time lag τ and so saves work, and is specially appropriate for long-time calculations. The error analysis carried out in this paper indicates that the multistep LGR collocation method, when implemented, provides a reliable and efficient approach for solving DDEs with constant delays.

References

[1] I. Ali, H. Brunner, and T. Tang, *A spectral method for pantograph–type delay differential equations and its convergence analysis*, J. Comput. Math. **27** (2009), 254–265.

[2] U.M. Ascher and L.R. Petzold, *The numerical solution of delay-differential algebraic equations of retarded and neutral type*, SIAM J. Numer. Anal. **32** (1995), 1635–1657.

[3] C.T.H. Baker and C.A.H. Paul, *Discontinuous solutions of neutral delay differential equations*, Appl. Numer. Math. **56** (2006), 284–304.

[4] M.V. Barbarossa, K.P. Hadeler, and C. Kuttler, *State–dependent neutral delay equations from population dynamics*, J. Math. Bio. **69** (2014), no. 4, 1027–1056.

[5] M.V. Barbarossa, C. Kuttler, and J. Zinsl, *Delay equations modeling the effects of phase–specific drugs and immunotherapy on proliferating tumor cells*, Math. Biosc. Eng. **9** (2012), no. 2, 241–257.

[6] A. Bellen and N. Guglielmi, *Solving neutral delay differential equations with state-dependent delays*, J. Comput. Appl. Math. **229** (2009), 350–362.

[7] H. Brunner, H. Xie, and R. Zhang, *Analysis of collocation solutions for a class of functional equations with vanishing delays*, IMA J. Numer. Anal. **31** (2011), no. 2, 698–718.

[8] A. Bellen, *Preservation of superconvergence in the numerical integration of delay differential equations with proportional delay*, IMA J. Numer. Anal. 22 (2002), no. 4, 529-536.

[9] A. Bellen and M. Zennaro, *Numerical Methods for Delay Differential Equations*, Oxford University Press, Oxford, 2003.

[10] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang, *Spectral Methods: Fundamentals in Single Domains*, Springer–Verlag, Berlin, 2006.

[11] M.J. Crawley, *Natural Enemies: The Population Biology of Predators, Parasites and Disease*, Blackwell Scientific Publications, Oxford, 1992.

[12] R.D. Driver, *Existence and continuous dependence of solutions of a neutral functional-differential equation*, Rational, Mech. Anal. **19** (1965), no. 2, 149–186.

[13] L.E. El'sgol'ts and S.B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press, New York, 1973.

[14] W.H. Enright, *Software for ordinary and delay differential equations: Accurate discrete approximate solutions are not enough*, Appl. Numer. Math. **56** (2006), no. 3-4, 459–471.

[15] W.H. Enright and H. Hayashi, *Convergence analysis of the solution of retarded and neutral delay differential equations by continuous numerical methods*, SIAM J. Numer. Anal. **35** (1998), no. 2, 572–585.

[16] L.J. Grimm, *Existence and continuous dependence for a class of nonlinear neutral differential equations*, Proc. Amer. Math. Soc. **29** (1971), no. 3, 467–473.

[17] N. Guglielmi and E. Hairer, *Implementing Radau IIA methods for stiff delay differential equations*, Computing **67** (2001), 1–12.

[18] H. Hayashi, *Numerical solution of retarded and neutral delay differential equations using continuous Runge-Kutta methods*, Ph.D. thesis, Department of Computer Science, University of Toronto, ON, Canada, 1996.

[19] K. Ito, H.T. Tran, and A. Manitius, *A fully–discrete spectral method for delay-differential equations*, SIAM J. Numer. Anal. **28** (1991), 1121–1140.

[20] Z. Jackiewicz, *Existence and uniqueness of solutions of neutral delay-differential equations with state dependent delays*, Funk. Ekvac. **30** (1987), 9–17.

[21] Z. Jackiewicz and E. Lo, *Numerical solution of neutral functional differential equations by Adams methods in divided difference form*, J. Comput. Appl. Math. **189** (2006), 592–605.

[22] J.C. Jimenez, L.M. Pedroso, F. Carbonell, and V. Hernandez, *Local linearization method for numerical integration of delay differential equations*, SIAM J. Numer. Anal. 44 (2006), no. 6, 2584–2609.

[23] H. Kocak and A. Yildirim, *Series solution for a delay differential equation arising in electrodynamics*, Int. J. Numer. Method. Biomed. Engin. **25** (2009), no. 11, 1084–1096.

[24] V.B. Kolmanovski and A. Myshkis, *Introduction to the Theory and Applications of Functional-Differential Equations*, Kluwer Academic, Dordrecht, 1999.

[25] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Mathematics in Science and Engineering, vol. 191, Academic Press, Boston, 1993.

[26] Y. Kuang and A. Feldstein, *Boundedness of solutions of a nonlinear nonautonomous neutral delay equation*, J. Math. Anal. Appl. 156 (1991), 293–304.

[27] Y. Liu, *Numerical solution of implicit neutral functional-differential equations*, SIAM J. Numer. Anal. **36** (1999), 516–528.

[28] M. Maleki, I. Hashim, and S. Abbasbandy, *Solution of time–varying delay systems using an adaptive collocation method*, Appl. Math. Comput. **219** (2012), 1434–1448.

[29] M. Maleki and I. Hashim, *Adaptive pseudospectral methods for solving constrained linear and nonlinear time-delay optimal control problems*, J. Franklin. Inst. **351** (2014), 811–839.

[30] M. Mohr, M.V. Barbarossa, and C. Kuttler, *Predator–prey interactions, age structures and delay equations*, Math. Model. Nat. Phenom. **9** (2014), no. 1, 92–107.

[31] K.W. Neves and S. Thompson, *Software for the numerical solution of system of functional differential equations with state-dependent delays*, Appl. Numer. Math. **9** (1992), 385–401.

[32] H.J. Oberle and H.J. Pesch, *Numerical treatment of delay differential Equations by Hermite interpolation*, Numer. Math. **37** (1981), 235–255.

[33] C.A.H. Paul, *Developing a delay differential equation solver*, Appl. Numer. Math. **9** (1992), 403–414.

[34] Z. Tan and C. Zhang, *Implicit–explicit one–leg methods for nonlinear stiff neutral equations*, Appl. Math. Comput. **335** (2018), 196–210.

[35] Z. Wang and B. Guo, *Legendre-Gauss–Radau collocation method for solving initial value problems of first order ordinary differential equations*, J. Sci. Comput. **52** (2012), no. 1, 226-225.

[36] Z.Q. Wang and L.L. Wang, *A Legendre-Gauss collocation method for nonlinear delay differential equations*, Discrete Contin. Dyn. Syst. Ser. B **13** (2010), 685–708.

[37] H. ZivariPiran and H. Enright, *An efficient unified approach for the numerical solution of delay differential equations*, Numer. Algor. **53** (2010), no. 2-3, 397–417.