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New progress on refinements Young type inequalities with Kantorovich constant

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Abstract

In this article, we introduce refinements of Young-type inequalities for scalars. We then extend these inequalities to encompass versions based on Hilbert-Schmidt and trace norms. The results presented in this paper represent refinements of the findings originally established by Nasiri and Shakoori[6].

Keywords: Young Type Inequalities, Hilbert-Schmidt Norm (Frobinius), trace norm, Positive Semidefinite Matrices, Singular Values, Refinement
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1 Introduction

Yang's inequality states that for any $a, b \geq 0$ and $0 \leq \nu \leq 1$, we have:

$$a^{\nu} b^{1-\nu} \leq \nu a + (1-\nu) b, \tag{1.1}$$

with equality if and only if $a = b$. When $\nu = 1/2$, Yang's inequality transforms into the arithmetic-geometric mean inequality:

$$\sqrt{ab} \leq \frac{a+b}{2}. \tag{1.2}$$

Zhang and Wu [7] refined this inequality, introducing the Kantorovich constant, as follows:

$$\nu a + (1-\nu) b \geq r_0 \left(\sqrt{a} - \sqrt{b}\right)^2 + K(\sqrt{h}, 2)^{r_1} a^{\nu} b^{1-\nu}, \tag{1.3}$$

where a and b are non-negative, $0 \leq \nu \leq 1$, and $h = \frac{a}{b}$. The Kantorovich constant is defined as:

$$K(h, 2) = \frac{(h+2)^2}{4h}, \quad h > 0,$$

with $r_0 = \min \nu, 1-\nu$ and $r_1 = \min 2r_0, 1-2r_0$. It is important to note that the Kantorovich constant satisfies the following conditions:

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(i) $K(1, 2) = 1$.
(ii) $K(h, 2) = K(1/h, 2)$.
(iii) $K(h, 2)$ is monotonically increasing on $[1, \infty)$ and monotonically decreasing on $(0, 1]$.

Using inequality (1.3), Nasiri and Shakoori in their article [6] derived refinements of Young's inequality as follows: Let $a, b \geq 0$ and $0 \leq \nu \leq 1$.

(i) If $0 \leq \nu \leq 1/2$, then:

$$\nu^2a^2 + (1-\nu)^2b^2 \geq \nu^2(a-b)^2 + r_0 b\left(\sqrt{b} - \sqrt{\nu a}\right)^2 + K(\sqrt{\nu h}, 2)^{r_1}\left[(\nu a)^{\nu} b^{1-\nu}\right]^2, \tag{1.4}$$

where $r_0 = \min 2\nu, 1-2\nu$, $h = \frac{a}{b}$, and $r_1 = \min 2r_0, 1-2r_0$.

(ii) If $1/2 \leq \nu \leq 1$, then:

$$\nu^2a^2 + (1-\nu)^2b^2 \geq (1-\nu)^2(a-b)^2 + r_0 a\left(\sqrt{a} - \sqrt{(1-\nu)b}\right)^2 + K\left(\sqrt{\frac{h}{1-\nu}}, 2\right)^{r_1}\left[a^{\nu}(1-\nu)b^{1-\nu}\right]^2, \tag{1.5}$$

where $r_0 = \min 2\nu - 1, 2-2\nu$, and $r_1 = \min 2r_0, 1-2r_0$.

In the realm of inequalities, a fascinating area of study involves the generalization of numerical inequalities into matrix form or even for bounded linear operators on Hilbert spaces. The results obtained from such generalizations often rely on unitary norms, particularly the Hilbert-Schmidt norm and the trace norm.

Let M_n represent the space of $n \times n$ complex matrices, and let $\|.\|$ denote any unitarily invariant norm on M_n. This implies that $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices U and $V \in M_n$.

For a matrix $A = [a_{ij}] \in M_n$, the Hilbert-Schmidt norm (or Frobenius norm) and the trace norm of A are respectively defined as follows:

$$\|A\|_2 = \left(\sum_{j=1}^{n} s_j^2(A)\right)^{1/2}, \qquad \|A\|_1 = \sum_{j=1}^{n} s_j(A).$$

Here, the singular values of A are denoted as $s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A)$. These singular values represent the eigenvalues of the positive matrix $\|A\| = (A^*A)^{1/2}$, arranged in decreasing order and repeated according to their multiplicity.

It is crucial to emphasize that the Hilbert-Schmidt norm possesses the valuable property of unitary invariance, making it exceptionally versatile in a wide range of mathematical applications. For a more in-depth understanding of Young-type inequalities and their matrix versions, readers are encouraged to explore the works of Bhatia [1], Hu [2], Wu and Zhao [4] and Zuo et al [5] as well as other relevant references. In a specific case, utilizing the inequalities (1.4) and (1.5), Nasiri and Shakoori [6] introduced the matrix version with the Hilbert-Schmidt norm, while Zhang and Wu [7] obtained refinements of inequality (1.4).

In this paper, we first derive new refinements of the results of Nasiri and Shakoori in [6]. We then extend the matrix form of these inequalities with the Hilbert-Schmidt norm and trace norm.

2 A Refinements of the Classical Young Inequality

We are about to present our key result, centered on refining the classical Young inequality (1.1). To begin, we introduce the following theorem applicable to scalar values. These refinements extend the inequalities (2.1) and (2.2) that were obtained by Nasiri and Shakoori in [6].

Theorem 2.1. Let $a, b \geq 0$ and $0 < \nu \leq 1$.

(i) If $0 < \nu < 1/4$, then

$$\begin{aligned}\nu^2a^2 + (1-\nu)^2b^2 \geq{}& \nu^2(a-b)^2 + 2\nu b\left(\sqrt{b} - \sqrt{\nu a}\right)^2 + 4\nu b\left((\nu ab)^{\frac{1}{4}} - (\nu a)^{\frac{1}{2}}\right)^2 + r_0 b\left(b^{\frac{1}{2}} - (\nu b^3 a)^{\frac{1}{8}}\right)^2 \\ &+ K\left(h^{\frac{1}{8}}, 2\right)^{r_1}(\nu a)^{2\nu}b^{2-2\nu},\end{aligned} \tag{2.1}$$

where $r_0 = \min\{1-8\nu, 8\nu\}$, $r_1 = \min\{2r_0, 1-2r_0\}$, and $h = \frac{\nu a}{b}$.

(ii) If $1/4 \leq \nu < 1/2$, then

$$\nu^2a^2 + (1-\nu)^2b^2 \geq \nu^2(a-b)^2 + \nu b\left(\sqrt{b} - \sqrt{\nu a}\right)^2 + (1-4\nu)b\left((\nu ab)^{\frac{1}{8}} - b^{\frac{1}{2}}\right)^2 + r_0 b\left((\nu ab)^{\frac{1}{4}} - (\nu b^3 a)^{\frac{1}{2}}\right)^2 + K\left(h^{\frac{1}{8}}, 2\right)^{r_1}\left(a^{\nu}((1-\nu)b)^{1-\nu}\right)^2, \tag{2.2}$$

where $r_0 = \min\{2-8\nu, 8\nu-1\}$, $r_1 = \min\{2r_0, 1-2r_0\}$, and $h = \frac{\nu a}{b}$.

(iii) If $1/2 \leq \nu < 3/4$, then

$$\nu^2a^2 + (1-\nu)^2b^2 \geq (1-\nu)^2(a-b)^2 + (2-2\nu)a\left(\sqrt{(1-\nu)b} - \sqrt{a}\right)^2 + (4\nu-3)a\left(((1-\nu)ab)^{\frac{1}{8}} - a^{\frac{1}{2}}\right)^2 + r_0 a\left(((1-\nu)ab)^{\frac{1}{4}} - ((1-\nu)ba^3)^{\frac{1}{8}}\right)^2 + K\left(h^{\frac{1}{8}}, 2\right)^{r_1}\left(a^{\nu}((1-\nu)b)^{1-\nu}\right)^2, \tag{2.3}$$

where $r_0 = \min\{7-8\nu, 8\nu-6\}$, $r_1 = \min\{2r_0, 1-2r_0\}$, and $h = \frac{a}{(1-\nu)b}$.

(iv) If $3/4 < \nu < 1$, then

$$\nu^2a^2 + (1-\nu)^2b^2 \geq (1-\nu)^2(a-b)^2 + (2-2\nu)a\left(\sqrt{(1-\nu)b} - \sqrt{a}\right)^2 + (4-4\nu)a\left(((1-\nu)ab)^{\frac{1}{8}} - a^{\frac{1}{2}}\right)^2 + ra\left(a^{\frac{1}{2}} - ((1-\nu)ba^3)^{\frac{1}{8}}\right)^2 + K\left(h^{\frac{1}{8}}, 2\right)^{r_1}\left(a^{\nu}((1-\nu)b)^{1-\nu}\right)^2, \tag{2.4}$$

where $r_0 = \min\{8-8\nu, 8\nu-7\}$, $r_1 = \min\{2r_0, 1-2r0\}$, and $h = \frac{a}{(1-\nu)b}$.

Proof .

(i) Let $0 < \nu < 1/4$. In this case, concerning inequality (2.1) from [7, Theorem 2.1], we have

$$\begin{aligned}\nu^2a^2 + (1-\nu)^2b^2 - \nu^2(a-b)^2 =& b\left[(1-2\nu)b + 2\nu(\nu a)\right] \\ \geq& b\Bigg[2\nu\left(\sqrt{b} - \sqrt{\nu a}\right)^2 + 4\nu\left((\nu ab)^{\frac{1}{4}} - b^{\frac{1}{2}}\right)^2 \\ &+ r_0\left(b^{\frac{1}{2}} - (\nu b^3 a)^{\frac{1}{8}}\right) + K\left((\nu h)^{\frac{1}{2}}, 2\right)^{r_1} b^{1-2\nu}(\nu a)^{2\nu}\Bigg] \\ =& 2\nu b\left(b^{\frac{1}{2}} - (\nu a)^{\frac{1}{2}}\right)^2 + 4\nu b\left((\nu ab)^{\frac{1}{4}} - b^{\frac{1}{2}}\right)^2 \\ &+ r_0 b\left(b^{\frac{1}{2}} - (\nu b^3 a)^{\frac{1}{8}}\right)^2 + K(h^{\frac{1}{8}}, 2)^{r_1} b^{2-2\nu}(\nu a)^{2\nu},\end{aligned} \tag{2.5}$$

where $r_0 = \min\{1-8\nu, 8\nu\}$, $r_1 = \min\{2r_0, 1-2r_0\}$, and $h = \frac{\nu a}{b}$. Therefore, theorem (i) is proved.

(ii) If $1/4 \leq \nu < 1/2$, then

$$\begin{aligned}\nu^2a^2 + (1-\nu)^2b^2 - \nu^2(a-b)^2 =& b\left[(1-2\nu)b + 2\nu(\nu a)\right] \\ \geq& b\Bigg[2\nu\left(\sqrt{b} - \sqrt{\nu a}\right)^2 + (1-4\nu)\left((\nu ab)^{\frac{1}{8}} - b^{\frac{1}{2}}\right)^2 \\ &+ r_0\left((\nu ab)^{\frac{1}{4}} - (\nu b^3 a)^{\frac{1}{8}}\right) + K\left(h^{\frac{1}{8}}, 2\right)^{r_1} b^{1-2\nu}(\nu a)^{2\nu}\Bigg] \\ =& 2\nu b\left(b^{\frac{1}{2}} - (\nu a)^{\frac{1}{2}}\right)^2 + (1-4\nu)b\left((\nu ab)^{\frac{1}{8}} - b^{\frac{1}{2}}\right)^2 \\ &+ r_0 b\left((\nu ab)^{\frac{1}{4}} - (\nu b^3 a)^{\frac{1}{8}}\right)^2 + K(h^{\frac{1}{8}}, 2)^{r_1} b^{2-2\nu}(\nu a)^{2\nu},\end{aligned} \tag{2.6}$$

where $r_0 = \min\{2-8\nu, 8\nu-1\}$, $r_1 = \min\{2r_0, 1-2r_0\}$, and $h = \frac{\nu a}{b}$. Therefore, theorem (ii) is proved.

(iii) For $1/2 \leq \nu < 3/4$. In this case, due to inequality (2.3) from [[7], Theorem 2.1], we have

$$\begin{aligned}\nu^2a^2+(1-\nu)^2b^2-(1-\nu)^2(a-b)^2 =&a\left[(2\nu-1)a+2(1-\nu)((1-\nu)b)\right]\\ \geq& a[(2-2\nu)\left(a^{\frac{1}{2}}-((1-\nu)b)^{\frac{1}{2}}\right)^2+(4\nu-2)\left(((1-\nu)ab)^{\frac{1}{4}}-a^{\frac{1}{2}}\right)^2\\ &+r_0\left(((1-\nu)ab)^{\frac{1}{4}}-((1-\nu)ba^3)^{\frac{1}{8}}\right)^2+K(h^{\frac{1}{8}},2)^{r_1}a^{2\nu-1}((1-\nu)b)^{2(1-\nu)}]\\ =&a(2-2\nu)\left(a^{\frac{1}{2}}-((1-\nu)b)^{\frac{1}{2}}\right)^2+(4\nu-2)a\left(((1-\nu)ab)^{\frac{1}{4}}-a^{\frac{1}{2}}\right)^2\\ &+r_0a\left(((1-\nu)ab)^{\frac{1}{4}}-((1-\nu)ba^3)^{\frac{1}{8}}\right)^2+K(h^{\frac{1}{8}},2)^{r_1}a^{2\nu-2}((1-\nu)b)^{2(1-\nu)}\end{aligned} \tag{2.7}$$

where $r_0 = \min\{7-8\nu, 8\nu-6\}$, $r_1 = \min\{2r_0, 1-2r_0\}$, and $h = \frac{a}{(1-\nu)b}$. Therefore, theorem (iii) is proved.

(iv) For $3/4 \leq \nu < 1$, by inequality (2.4) from [7, Theorem 2.1], we have

$$\begin{aligned}\nu^2a^2+(1-\nu)^2b^2-(1-\nu)^2(a-b)^2 =&a\left[(2\nu-1)a+2(1-\nu)((1-\nu)b)\right]\\ \geq& a\Bigg[(2-2\nu)\left(a^{\frac{1}{2}}-((1-\nu)b)^{\frac{1}{2}}\right)^2+(4-4\nu)\left(((1-\nu)ab)^{\frac{1}{4}}-a^{\frac{1}{2}}\right)^2\\ &+r_0\left((a)^{\frac{1}{2}}-((1-\nu)ba^3)^{\frac{1}{8}}\right)^2+K(h^{\frac{1}{8}},2)^{r_1}a^{2\nu-1}((1-\nu)b)^{2(1-\nu)}\\ =&a(2-2\nu)\left(a^{\frac{1}{2}}-((1-\nu)b)^{\frac{1}{2}}\right)^2+(4-4\nu)a\left(((1-\nu)ab)^{\frac{1}{4}}-a^{\frac{1}{2}}\right)^2\\ &+r_0a\left((a)^{\frac{1}{2}}-((1-\nu)ba^3)^{\frac{1}{8}}\right)^2+K(h^{\frac{1}{8}},2)^{r_1}a^{2\nu-1}((1-\nu)b)^{2(1-\nu)},\end{aligned} \tag{2.8}$$

where $r_0 = \min\{8-8\nu, 8\nu-7\}$ and $h = \frac{a}{(1-\nu)b}$. Thus, the proof of the theorem is complete.

□

3 Refinements of the Matrix Version of Young's Inequality

In this section, we will employ the inequalities obtained from (2.1)-(2.4) to introduce a series of matrix inequalities. These inequalities will be presented in terms of the Hilbert-Schmidt norm (also known as the Frobenius norm) and the trace norm.

Theorem 3.1. Let $A, B, X \in M_n$ such that A and B are semidefinite positive matrices and $0 < \nu < 1$.

(i) If $0 < \nu < 1/4$ then

$$\begin{aligned}\|\nu AX+(1-\nu)XB\|_2^2 \geq&\nu^2\|AX-XB\|_2^2+2\nu\left[\nu\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+\|XB\|_2^2-2\nu\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right]\\ &+2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+4\nu[\sqrt{\nu}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2+\|XB\|_2^2-2\nu^{\frac{1}{4}}\left\|A^{\frac{1}{8}}XB^{\frac{7}{8}}\right\|_2^2\\ &+r_0\left[\|XB\|_2^2+\nu^{\frac{1}{4}}\left\|A^{\frac{1}{8}}XB^{\frac{7}{8}}\right\|_2^2-2\nu^{\frac{1}{8}}\left\|A^{\frac{1}{16}}XB^{\frac{15}{16}}\right\|_2^2\right]+K^{r_1}\nu^{2\nu}\left\|A^{\nu}XB^{1-\nu}\right\|_2^2,\end{aligned} \tag{3.1}$$

where $r_0 = \min\{1-8\nu, 8\nu\}$, $r_1 = \min\{2r_0, 1-2r_0\}$ and $K = \min\left\{K\left((\frac{\nu\lambda_i}{\mu_j})^{\frac{1}{8}},2\right),\ 1\leq i,j\leq n\right\}$.

(ii) If $1/4 \leq \nu < 1/2$ then

$$\begin{aligned}\|\nu AX+(1-\nu)XB\|_2^2 \geq&\nu^2\|AX-XB\|_2^2+2\nu\left[\nu\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+\|XB\|_2^2-2\sqrt{\nu}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right]\\ &+2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+(1-4\nu)[\sqrt{\nu}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2+\|XB\|_2^2-2\nu^{\frac{1}{4}}\left\|A^{\frac{1}{8}}XB^{\frac{7}{8}}\right\|_2^2\\ &+r_0\left[\nu^{\frac{1}{2}}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2+\nu^{\frac{1}{4}}\left\|A^{\frac{1}{8}}XB^{\frac{7}{8}}\right\|_2^2\right]+2\nu^{\frac{3}{8}}K^{r_1}\nu^{2\nu}\left\|A^{\nu}XB^{1-\nu}\right\|_2^2,\end{aligned} \tag{3.2}$$

where $r_0=\min\{2-8\nu,8\nu-1\}$, $r_1=\min\{2r_0,1-2r_0\}$ and $K=\min\left\{K\left((\frac{\nu\lambda_i}{\mu_j})^{\frac{1}{8}},2\right),\ 1\le i,j\le n\right\}$.

(iii) If $1/2\le\nu<3/4$ then

$$\begin{aligned}\|\nu AX+(1-\nu)XB\|_2^2\ge&(1-\nu)^2\|AX-XB\|_2^2\\&+(2-2\nu)\left[(1-\nu)\nu\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+\|AX\|_2^2-2\sqrt{1-\nu}\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\right]\\&+2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+(4\nu-3)[\sqrt{1-\nu}\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\\&+\|AX\|_2^2-2(1-\nu)^{\frac{1}{4}}\left\|A^{\frac{7}{8}}XB^{\frac{1}{8}}\right\|_2^2+r_0\left[(1-\nu)^{\frac{1}{2}}\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2+(1-\nu)^{\frac{1}{4}}\left\|A^{\frac{7}{8}}XB^{\frac{1}{8}}\right\|_2^2\right]\\&-2(1-\nu)^{\frac{5}{8}}\left\|A^{\frac{5}{16}}XB^{\frac{11}{16}}\right\|_2^2+K^{r_1}(1-\nu)^{2-2\nu}\left\|A^{\nu}XB^{1-\nu}\right\|_2^2,\end{aligned}\tag{3.3}$$

where $r_0=\min\{7-8\nu,8\nu-6\}$, $r_1=\min\{2r_0,1-2r_0\}$ and $K=\min\left\{K\left((\frac{\lambda_i}{(1-\nu)\mu_j})^{\frac{1}{8}},2\right),\ 1\le i,j\le n\right\}$.

(iv) If $3/4\le\nu<1$ then

$$\begin{aligned}\|\nu AX+(1-\nu)XB\|_2^2\ge&(1-\nu)^2\|AX-XB\|_2^2\\&+(2-2\nu)\left[(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+\|AX\|_2^2-2\sqrt{1-\nu}\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\right]\\&+2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2\\&+r_0\left[(1-\nu)\left\|A^{\frac{7}{8}}XB^{\frac{1}{8}}\right\|_2^2+\|AX\|_2^2+2(1-\nu)^{\frac{1}{8}}\left\|A^{\frac{15}{16}}XB^{\frac{1}{16}}\right\|_2^2\right]\\&+K^{r_1}(1-\nu)^{2(-\nu)}\left\|A^{\nu}XB^{1-\nu}\right\|_2^2,\end{aligned}\tag{3.4}$$

where $r_0=\min\{8-8\nu,8\nu-7\}$, $r_1=\min\{2r_0,1-2r_0\}$ and $K=\min\left\{K\left((\frac{\lambda_i}{(1-\nu)\mu_j})^{\frac{1}{8}},2\right),\ 1\le i,j\le n\right\}$.

Proof . Since A and B are semi-definite matrices, according to the spectral theorem, there exist unitary matrices U and V and diagonal matrices $D_1=\operatorname{diag}(\lambda_i)$ and $D_2=\operatorname{diag}(\mu_i)$ with $\lambda_i,\mu_i\ge 0$ such that:

$$A=UD_1U^* \quad\text{and}\quad B=VD_2V^*.$$

To perform the calculations, we define $Y=U^*XV=[y_{ij}]$. In this case, we have:

$$\nu AX+(1-\nu)XB=U\left[(\nu\lambda_i+(1-\nu)\mu_j)\,y_{ij}\right]V^*,$$

$$A^{\frac{1}{2}}XB^{\frac{1}{2}}=U\left[(\lambda_i)^{\frac{1}{2}}(\mu_j)^{\frac{1}{2}}\right]V^*,\qquad A^{\nu}XB^{1-\nu}=U\left[(\lambda_i)^{\nu}(\mu_j)^{1-\nu}\right]V^*.$$

If $0<\nu\le 1/4$, by (2.1) and the unitarily invariant property of $\|.\|_2$ (Hilbert-Schmidt norm), we have:

$$\begin{aligned}\|\nu AX+(1-\nu)XB\|_2^2 \ \ge\ & K^{r_1}\nu^{2\nu}\sum_{i,j=1}^{n}\left((\lambda_i)^{\nu}(\mu_j)^{1-\nu}\right)^2+\nu^2\sum_{i,j=1}^{n}(\lambda_i-\mu_j)^2|y_{ij}|^2\\&+2\nu\left[\sum_{i,j=1}^{n}\left((\lambda_i)^{\frac{1}{2}}(\mu_j)^{\frac{1}{2}}\right)^2|y_{ij}|^2+\sum_{i,j=1}^{n}(\mu_j)^2|y_{ij}|^2-2\sqrt{\nu}\sum_{i,j=1}^{n}\left((\lambda_i)^{\frac{1}{4}}(\mu_j)^{\frac{3}{4}}\right)^2|y_{ij}|^2\right]\\&+4\nu[\nu\sum_{i,j=1}^{n}\left((\lambda_i)^{\frac{1}{2}}(\mu_j)^{\frac{1}{2}}\right)^2|y_{ij}|^2+\sqrt{\nu}\sum_{i,j=1}^{n}\left((\lambda_i)^{\frac{1}{4}}(\mu_j)^{\frac{3}{4}}\right)^2|y_{ij}|^2+\nu\sum_{i,j=1}^{n}\left(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}}\right)^2|y_{ij}|^2\\&{}_2\nu^{\frac{1}{4}}\sum_{i,j=1}^{n}\left((\lambda_i)^{\frac{3}{8}}(\mu_j)^{\frac{5}{8}}\right)^2|y_{ij}|^2]+\left[\sum_{i,j=1}^{n}\mu_j^2|y_{ij}|^2+\sqrt{\nu}\sum_{i,j=1}^{n}\left((\lambda_i)^{\frac{1}{8}}(\mu_j)^{\frac{7}{8}}\right)^2\right]\\=\ &\nu^2\|AX-XB\|_2^2+2\nu\left[\nu\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+\|XB\|_2^2-2\nu\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right]\\&+2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+4\nu[\sqrt{\nu}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2+\|XB\|_2^2-2\nu^{\frac{1}{4}}\left\|A^{\frac{1}{8}}XB^{\frac{7}{8}}\right\|_2^2\\&+r_0\left[\|XB\|_2^2+\nu^{\frac{1}{4}}\left\|A^{\frac{1}{8}}XB^{\frac{7}{8}}\right\|_2^2-2\nu^{\frac{1}{8}}\left\|A^{\frac{1}{16}}XB^{\frac{15}{16}}\right\|_2^2\right]+K^{r_1}\nu^{2\nu}\left\|A^{\nu}XB^{1-\nu}\right\|_2^2,\end{aligned}\tag{3.5}$$

where $r_0 = \min\{1-8\nu, 8\nu\}$, $r_1 = \min\{2r_0, 1-2r_0\}$, and $K = \min\left\{K\left((\frac{\nu\lambda_i}{\mu_j})^{\frac{1}{8}}, 2\right), 1 \leq i,j \leq n\right\}$. Thus, (3.1) is proved. The proof of inequalities (3.2)-(3.4) is similar to that of inequality (3.1). □

Remark 3.2. It is clear that inequalities (3.1)-(3.4) are refinements of the inequalities (3.3) and (3.4) obtained by Nasiri and Shakoori in [6].

In the end, we obtain refinements of the trace versions of Young-type inequalities. To do this, we rely on the Cauchy-Schwarz inequality and the following lemma:

Lemma 3.3. Let $A, B \in M_n(\mathbb{C})$, then [1]

$$\sum_{j=1}^{n} s_j(AB) \leq \sum_{j=1}^{n} s_j(A)s_j(B).$$

For more details, please refer to the book by Bhatia in [1].

Theorem 3.4. Let $A, B \in M_n(\mathbb{C})$ such that A and B are semi-definite positive and $0 < \nu < 1$. Then, we have the following inequalities:

$$\begin{aligned}\operatorname{tr}\left(\nu^2A^2+(1-\nu)^2B^2\right) \geq{}& K^{r_1}(\nu)^{2\nu}\left[\|A^\nu B^{1-\nu}\|_2^2\right] + \nu^2\left[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1\right]\\ &+2\nu\left[\nu\|AB\|_1+\|B\|_2^2-2\sqrt{\nu}\left(\sqrt{\|A\|_1}\sqrt{\|B^3\|_1}\right)\right]\\ &+4\nu\left[\sqrt{\nu}\left\|A^{\frac{1}{4}}B^{\frac{3}{4}}\right\|_2^2+\|B\|_2^2-4\sqrt{\nu}\sqrt{\|A^{\frac{1}{2}}\|_1}\sqrt{\|B^{\frac{7}{4}}\|_1}\right]\\ &+r_0\left[\|B\|_2^2+\nu^{\frac{1}{4}}\left\|A^{\frac{1}{8}}B^{\frac{7}{8}}\right\|_2^2-2\nu^{\frac{1}{8}}\left\|A^{\frac{1}{16}}B^{\frac{15}{16}}\right\|_2^2\right],\end{aligned} \tag{3.6a}$$

$$\begin{aligned}\operatorname{tr}\left(\nu^2A^2+(1-\nu)^2B^2\right) \geq{}& K^{r_1}(\nu)^{2\nu}\left[\|A^\nu B^{1-\nu}\|_2^2\right] + \nu^2\left[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1\right]\\ &+2\nu\left[\nu\|AB\|_1+\|B\|_2^2-2\sqrt{\nu}\left(\sqrt{\|A\|_1}\sqrt{\|B^3\|_1}\right)\right]\\ &+(1-4\nu)\left[\sqrt{\nu}\left\|A^{\frac{1}{4}}B^{\frac{3}{4}}\right\|_2^2+\|B\|_2^2-2\nu^{\frac{1}{4}}\sqrt{\|A^{\frac{1}{2}}\|_1}\sqrt{\|B^{\frac{7}{2}}\|_1}\right]\\ &+r_0\left[\left\|A^{\frac{1}{4}}B^{\frac{3}{4}}\right\|_2^2+\nu^{\frac{1}{4}}\left\|A^{\frac{1}{8}}B^{\frac{3}{8}}\right\|_2^2-2\nu^{\frac{3}{8}}\sqrt{\left\|A^{\frac{3}{4}}\right\|_1}\sqrt{\left\|B^{\frac{13}{4}}\right\|_1}\right],\end{aligned} \tag{3.6b}$$

$$\begin{aligned}\operatorname{tr}\left(\nu^2A^2+(1-\nu)^2B^2\right) \geq{}& K^{r_1}(1-\nu)^{2-2\nu}\left[\|A^\nu B^{1-\nu}\|_2^2\right] + (1-\nu)^2\left[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1\right]\\ &+2\nu\left[\nu\|AB\|_1+\|B\|_2^2-2\sqrt{\nu}\left(\sqrt{\|A\|_1}\sqrt{\|B^3\|_1}\right)\right]\\ &+(2-2\nu)\left[(1-\nu)\left\|A^{\frac{1}{2}}B^{\frac{3}{2}}\right\|_2^2+\|A\|_2^2-2\sqrt{1-\nu}\sqrt{\|A^3\|_1}\sqrt{\|B\|_1}\right]\\ &+(4\nu-3)\left[\left\|A^{\frac{3}{4}}B^{\frac{1}{4}}\right\|_2^2+\|A\|_2^2-2(1-\nu)^{\frac{1}{4}}\sqrt{\left\|A^{\frac{7}{2}}\right\|_1}\sqrt{\left\|B^{\frac{3}{2}}\right\|_1}\right]\\ &+r_0\left[\sqrt{1-\nu}\left\|A^{\frac{3}{4}}B^{\frac{1}{4}}\right\|_2^2+(1-\nu)^{\frac{1}{4}}\left\|A^{\frac{7}{8}}B^{\frac{1}{8}}\right\|_2^2-2(1-\nu)^{\frac{3}{8}}\sqrt{\left\|A^{\frac{13}{4}}\right\|_1}\sqrt{\left\|B^{\frac{3}{4}}\right\|_1}\right],\end{aligned} \tag{3.6c}$$

$$\begin{aligned}\operatorname{tr}\left(\nu^2A^2+(1-\nu)^2B^2\right) \geq{}& K^{r_1}(1-\nu)^{2-2\nu}\left[\|A^\nu B^{1-\nu}\|_2^2\right] + (1-\nu)^2\left[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1\right]\\ &+(4-4\nu)\left[\sqrt{1-\nu}\left\|A^{\frac{3}{4}}B^{\frac{1}{4}}\right\|_2^2+\|A\|_2^2-2\sqrt{1-\nu}\left(\sqrt{\|B^{\frac{1}{2}}\|_1}\sqrt{\|A^{\frac{7}{2}}\|_1}\right)\right]\\ &+r_0\left[\sqrt{1-\nu}\left\|A^{\frac{3}{4}}B^{\frac{1}{4}}\right\|_2^2+\|A\|_2^2+(1-\nu)^{\frac{1}{4}}\left\|A^{\frac{7}{8}}B^{\frac{1}{8}}\right\|_2^2-2(1-\nu)^{\frac{3}{8}}\left\|A^{\frac{15}{16}}B^{\frac{1}{16}}\right\|_2^2\right].\end{aligned} \tag{3.6d}$$

Here, r_0 is defined as $\min\{1-8\nu, 8\nu\}$, r_1 as $\min\{2r_0, 1-2r_0\}$, and K is defined as $\min\left\{K\left((\frac{\nu s_j(A)}{s_j(B)})^{\frac{1}{8}}, 2\right),\ 1 \leq i,j \leq n\right\}$.

Proof . We shall prove the first inequality, and leave the second to the reader because the proof is very similar. If $0 < \nu < 1/4$ then using Lemma 3.3 and the inequality (2.1), we have:

$$\begin{aligned}
\operatorname{tr}\left(\nu^2A^2+(1-\nu)^2B^2\right) &= \nu^2\operatorname{tr}(A^2)+(1-\nu)^2\operatorname{tr}(B^2)\\
&= \sum_{j=1}^{n}\left(\nu^2 s_j^2(A)+(1-\nu)^2 s_j^2(B)\right)\\
&\geq K^{r_1}\nu^{2\nu}\sum_{j=1}^{n}\left[s_j(A^\nu)s_j(B^{1-\nu})\right]^2+\nu^2\left[\sum_{j=1}^{n}s_j^2(A)+\sum_{j=1}^{n}s_j^2(B)-2\sum_{j=1}^{n}s_j(A)s_j(B)\right]\\
&\quad +2\nu\left[\nu\sum_{j=1}^{n}s_j(A)s_j(B)+\sum_{j=1}^{n}s_j^2(B)-2\sqrt{\nu}\left(\sum_{j=1}^{n}s_j^{\frac14}(A)s_j^{\frac34}(B)\right)^2\right]\\
&\quad +4\nu\left[\sqrt{\nu}\sum_{j=1}^{n}s_j^{\frac12}(A)s_j^{\frac32}(B)+\sum_{j=1}^{n}s_j^2(B)-2\nu^{\frac14}\left(\sum_{j=1}^{n}s_j^{\frac18}(A)s_j^{\frac78}(B)\right)^2\right]\\
&\quad +r_0\left[\sum_{j=1}^{n}s_j^2(B)+\nu^{\frac14}\sum_{j=1}^{n}s_j^{\frac14}(A)s_j^{\frac74}(B)\right]-2\nu^{\frac18}\sum_{j=1}^{n}s_j^{\frac18}(A)s_j^{\frac{15}{8}}(B)\\
&\geq K^{r_1}(\nu)^{2\nu}\left[\|A^\nu B^{1-\nu}\|_2^2\right]+\nu^2\left[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1\right]\\
&\quad +2\nu\left[\nu\|AB\|_1+\|B\|_2^2-2\sqrt{\nu}\left(\sqrt{\|A\|_1}\sqrt{\|B^3\|_1}\right)\right]\\
&\quad +4\nu\left[\sqrt{\nu}\left\|A^{\frac14}B^{\frac34}\right\|_2^2+\|B\|_2^2-4\sqrt{\nu}\sqrt{\|A^{\frac12}\|_1}\sqrt{\|B^{\frac74}\|_1}\right]\\
&\quad +r_0\left[\|B\|_2^2+\nu^{\frac14}\left\|A^{\frac18}B^{\frac78}\right\|_2^2-2\nu^{\frac18}\left\|A^{\frac1{16}}B^{\frac{15}{16}}\right\|_2^2\right],
\end{aligned}\tag{3.7}$$

where $r_0=\min\{1-8\nu,8\nu\}$, $r_1=\min\{2r_0,1-2r_0\}$, and $K=\min\left\{K\left((\frac{\nu s_j(A)}{s_j(B)})^{\frac18},2\right),\ 1\leq i,j\leq n\right\}$. This estimate completes the proof of (3.6a). On the other hand, we have:

$$\begin{aligned}
\operatorname{tr}\left(\nu^2A^2+(1-\nu)^2B^2\right) &= \nu^2\operatorname{tr}(A^2)+(1-\nu)^2\operatorname{tr}(B^2)\\
&= \nu^2\|A\|_2^2+(1-\nu)^2\|B\|_2^2.
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\nu^2\|A\|_2^2+(1-\nu)^2\|B\|_2^2 &\geq K^{r_1}(\nu)^{2\nu}\left[\|A^\nu B^{1-\nu}\|_2^2\right]+\nu^2\left[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1\right]\\
&\quad +2\nu\left[\nu\|AB\|_1+\|B\|_2^2-2\sqrt{\nu}\left(\sqrt{\|A\|_1}\sqrt{\|B^3\|_1}\right)\right]\\
&\quad +4\nu\left[\sqrt{\nu}\left\|A^{\frac14}B^{\frac34}\right\|_2^2+\|B\|_2^2-4\sqrt{\nu}\sqrt{\|A^{\frac12}\|_1}\sqrt{\|B^{\frac74}\|_1}\right]\\
&\quad +r_0\left[\|B\|_2^2+\nu^{\frac14}\left\|A^{\frac18}B^{\frac78}\right\|_2^2-2\nu^{\frac18}\left\|A^{\frac1{16}}B^{\frac{15}{16}}\right\|_2^2\right].
\end{aligned}$$

This completes the proof of (3.6a). The proofs of (3.6b) and (3.6d) are similar. □

Obviously, inequalities (3.6a)-(3.6d) are refinement of the well-known results in [7, Theorem 3.2].

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