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Uniqueness of meromorphic (entire) functions regarding general difference-differential polynomials

Harina P. Waghamore*, Megha M. Manakame

Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bangalore, 560 056, India

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Abstract

In this article, we study the uniqueness of higher order difference-differential polynomial $\Delta_c^u f(z)$ and meromorphic(entire) functions with weights $l \geq 2$, $l = 1$ and $l = 0$. We obtained the results which generalize and extend due to [2, 13].

Keywords: Meromorphic functions, difference operator, weighted sharing, difference-differential, uniqueness
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1 Introduction, Definitions and Main Results

Nevanlinna theory is a branch of complex analysis that deals with the value distribution of meromorphic functions. Rolf Nevanlinna developed it in the early 20th century and provides powerful tools to study the behaviour of meromorphic functions, particularly about their zeros, poles, and growth rates. Nevanlinna's theory provides a deep and rigorous framework for understanding the complex behaviour of meromorphic functions, making it an essential study area in complex analysis.

For the elementary definitions and standard notations of the Nevanlinna value distribution theory such as $T(r, f)$, $N(r, f)$, $N\left(r, \frac{1}{f}\right)$, $m(r, f)$ *etc* see Hayman [5]. The uniqueness theory of meromorphic functions focuses on the criteria that allow for the existence of essentially only one function that meets these conditions. It demonstrated that any non-constant meromorphic function may be uniquely defined by five values, i.e., if two non-constant meromorphic functions f and g take the same five values at the same locations, then $f \equiv g$.

Let f and g be two non-constant meromorphic functions defined in the open complex plane and $S(r, f)$ denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to +\infty$ possibly outside a set I with finite linear measure. A meromorphic function $a(z)$ is called a small function concerning $f(z)$ if $T(r, a) = S(r, f)$. For $a \in \mathbb{C} \cup \{\infty\}$, if $f(z) - a$ and $g(z) - a$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicity). If $f(z) - a$ and $g(z) - a$ assume the same zeros ignoring the multiplicities, then we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicity).

Definition 1.1. [5] The order $\rho(f)$ and hyper-order $\rho_2(f)$ of a meromorphic function $f(z)$ are defined as,

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{log\, T(r, f)}{log\, r}$$

*Corresponding author
Email addresses: harinapw@gmail.com (Harina P. Waghamore), meghamm@bub.ernet.in (Megha M. Manakame)

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and

$$\rho_2(f) = \overline{\lim_{r\to\infty}} \frac{log\,log\,T(r,f)}{log\,r}.$$

Definition 1.2. [8] For a positive integer p, we denote by $N_p\left(r, \frac{1}{f-a}\right)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \le p$ and p times if $m > p$.

Definition 1.3. [8] Let f and g share the value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.
Clearly, $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$, where $\overline{N}_L(r, a; f)$ denotes the counting function of those 1-points of f and g, when two meromorphic functions f and g share the value 1 IM and z_0 is a 1-point of f of order p, and a 1-point of g of order q, such that $q < p$.

Definition 1.4. [7] For a complex number $a \in \mathbb{C}\cup\{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point with multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. For a complex number $a \in \mathbb{C}\cup\{\infty\}$, such that $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with weight k.

The definition implies that if f, g share value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\le k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\le k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f, g share (a, k) to mean that f, g share the value a with wieght k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.5. [12] For a meromorphic function f, let us denote its difference operator $\Delta_c^u f$ by,

$$\Delta_c^u f(z) = \Delta_c^{u-1}\left(\Delta_c f(z)\right) = \sum_{r=0}^{u}(-1)^r\binom{u}{r}f(z+(u-r)c),$$

where $c \in \mathbb{C}$, $u(\ge 2) \in \mathbb{N}$, $0 \le r \le u$.

We can see that $\Delta_c f(z) = f(z+c) - f(z)$. recently, there has been a surge in interest in difference analogues of Nevanlinna's theory, with numerous works concentrating on the uniqueness and value distribution of difference polynomials of whole or meromorphic functions. In 2010, Qi et al. [11] proved the following uniqueness theorem regarding shift operator.

Theorem A.[11] Let f and g be transcendental entire functions of finite order, let c be a non-zero complex constant and let $n \ge 6$ be an integer. If $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share z CM, then $f(z) \equiv tg(z)$ for a constant t satisfying $t^{n+1} = 1$.

In 2015, Y. Liu et al. [10] obtained the following results.

Theorem B. [10] Let $c \in \mathbb{C}\backslash\{0\}$, $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order and $n \ge 14$, $k \ge 3$ be two positive integers. If $E_k(1, f^n f(z+c)) = E_k(1, g^n g(z+c))$, then $f(z) = t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Theorem C. [10] Let $c \in \mathbb{C}\backslash\{0\}$, $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order and $n \ge 16$ be an positive integer. If $E_2(1, f^n f(z+c)) = E_2(1, g^n g(z+c))$, then $f(z) = t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Theorem D. [10] Let $c \in \mathbb{C}\backslash\{0\}$, $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order and $n \ge 22$ be an positive integer. If $E_1(1, f^n f(z+c)) = E_1(1, g^n g(z+c))$, then $f(z) = t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

In 2018, Banerjee and Majumder [2] considered transcedental entire functions of finite order and difference - differential polynomial of the form $[f^n \Delta_c f(z)]^{(k)}$ obtained following results.

Theorem E.[2] Let $f(z)$ be a transcendental entire function of finite order such that $\Delta_c f(z) \not\equiv 0$ and $\alpha(z)$ be a small function with respect to $f(z)$. If $n \ge k+2$, then the difference-differential polynomial $[f^n \Delta_c f(z)]^{(k)} - \alpha(z)$ has infinitely many zeros.

Theorem F.[2] Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order and n, k be two positive integers. Suppose that c is a non-zero complex constant such that $\Delta_c f(z) \not\equiv 0$ and $\Delta_c g(z) \not\equiv 0$. Let $[f^n \Delta_c f(z)]^{(k)}$ and

$[g^n\Delta_c g(z)]^{(k)}$ share $(1, k_1)$ and one of the following conditions holds:
(i) $k_1 \geq 2$ and $n > 2k + 5$;
(ii) $k_1 = 1$ and $n > \frac{5k}{2} + 6$;
(iii) $k_1 = 0$ and $n > 5k + 11$.

Then one of the following conclusions holds:

(1) $f^n\Delta_c f(z) \equiv g^n\Delta_c g(z)$;

(2) $f(z) = c_1e^{az}$ and $g(z) = c_2e^{-az}$, where a, c_1 and c_2 are non-zero constants such that $(-1)^k(c_1c_2)^{n+1}[(n + 1)a]^{2k}(2 - e^{ac} - e^{-ac}) = 1$.

In 2022, Waghamore and Naveenkumar S. H.[13] considered the value distribution of difference - differential polynomials and obtained following results.

Theorem G.[13] Let f and g be two transcendental meromorphic functions of finite order, and let $c \in \mathbb{C}$. If $E_l(1, [f^n\Delta_c f]^{(k)})=E_l(1, [g^n\Delta_c g]^{(k)})$ and l, m, n are integers satisfying one of the following conditions:
(1) $l \geq 2$ and $n > 5k + 19$;
(2) $l = 1$ and $n > 6k + 21$;
(3) $l = 0$ and $n > 11k + 31$.

Then one of the following conclusions holds:
(i) $f(z) \equiv tg(z)$ for a constant t with $t^{n+1} = 1$;
(ii) $f(z) = c_1e^{az}$ and $g(z) = c_2e^{-az}$, where a, c_1 and c_2 are non-zero constants such that

$$(-1)^k(c_1c_2)^{n+1}[(n + 1)a]^{2k}(2 - e^{ac} - e^{-ac}) = 1.$$

Let $P(z) = a_mz^m + a_{m-1}z^{m-1} + ... + a_0$ is a non-zero polynomial of degree m and $\Gamma_0 = m_1 + m_2$, where m_1 is the number of the simple zeros of $P(z)$ and m_2 is the number of multiple zeros of $P(z)$. Here, we used the idea of weighted sharing values to extend the above results

Now, it will be interesting to study what happens to Theorem G when we consider a more generalized form of the difference operator $\Delta_c^u f$ instead of $\Delta_c f(z)$, by considering the difference-differential polynomial of the form $f^nP(f)\Delta_c^u f$ and obtained the following results.

Theorem 1.6. Let $f(z)$ and $g(z)$ be any two transcendental meromorphic functions of finite order. Let l, m, n and u are positive integers and $c \in \mathbb{C}$. If $E_l(1, [f^nP(f)\Delta_c^u f]^{(k)}) = E_l(1, [g^nP(g)\Delta_c^u g]^{(k)})$ then it satisfying one of the following conditions:

(I) $l \geq 2$ and $n > 7u + ku - m + 3k + 2\Gamma_0 + 10$;

(II) $l = 1$ and $n > 8u + \frac{3ku}{2} + 4k + \frac{5\Gamma_0-2m}{2} + \frac{23}{2}$;

(III) $l = 0$ and $n > 13u + 4ku + 9k + 5\Gamma_0 - m + 19$.

Then one of the following conclusions holds:

(i) $[f^n(z)P(f)\Delta_c^u f(z)]^{(k)} \cdot [g^n(z)P(g)\Delta_c^u g(z)]^{(k)} \equiv a^2(z)$;

(ii) $f \equiv tg$ for a constant t with $t^d = 1$, where $d = GCD\{k \in (n+m+1, ..., n+m+1-i, ..., n+1) : a_{k-n-1} \neq 0\}$;

(iii) f and g satisfy algebraic equation $R(f, g) \equiv 0$, where

$$R(\omega_1, \omega_2) = \omega_1^n(z)P(\omega_1)\Delta_c^u\omega_1(z) - \omega_2^n(z)P(\omega_2)\Delta_c^u\omega_2(z).$$

Corollary 1.7. Let $f(z)$ and $g(z)$ be any two transcendental meromorphic functions of finite order. Let l, m, n and u are positive integers and $c \in \mathbb{C}$. If $E_l(1, [f^n\Delta_c^u f]^{(k)}) = E_l(1, [g^n\Delta_c^u g]^{(k)})$ then it satisfying one of the following conditions.
(i) $l \geq 2$ and $n > 7u + ku + 3k + 10$;
(ii) $l = 1$ and $n > 8u + \frac{3ku}{2} + 4k + \frac{23}{2}$;
(iii) $l = 0$ and $n > 13u + 4ku + 9k + 19$.

Then one of the conclusions of Theorem 1.6 holds.

Theorem 1.8. Let $f(z)$ and $g(z)$ be any two transcendental entire functions of finite order. Let l, m, n and u are positive integers and $c \in \mathbb{C}$. If $E_l(1, [f^nP(f)\Delta_c^u f]^{(k)}) = E_l(1, [g^nP(g)\Delta_c^u g]^{(k)})$ then it satisfying one of the following conditions:

(i) $l \geq 2$ and $n > 2k + 4 + 2\Gamma_0 - m$;
(ii) $l = 1$ and $n > \frac{1}{2}[5k + 9 + 5\Gamma_0 - 2m]$;
(iii) $l = 0$ and $n > 5k + 7 + 5\Gamma_0 - m$.

Then one of the conclusions of Theorem 1.6 holds.

2 Preliminaries

In this section we provide all the necessary Lemmas required to prove our theorems. Let us define,

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1. [3] Let $f(z)$ be a non-constant meromorphic function of finite order, Then

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.2. [6] Let $f(z)$ be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then

$$N(r, \infty; f(z+c)) \leq N(r, \infty; f) + S(r, f),$$
$$N(r, 0; f(z+c)) \leq N(r, 0; f) + S(r, f).$$

Lemma 2.3. [3] Let $f(z)$ be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$, we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Lemma 2.4. [7] Let f and g be two non-constant meromorphic functions. If $E_2(1; F) = E_2(1; G)$ then one of the following cases holds:

(i) $T(r) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + S(r)$,

(ii) $F = G$,

(iii) $FG = 1$, where $T(r) = \max\{T(r, F), T(r, G)\}$ and $S(r) = o\{T(r)\}$.

Lemma 2.5. [1] Let F and G be two non-constant meromorphic functions. If $E_1(1; F) = E_1(1; G)$ and $H \not\equiv 0$, then

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2}\overline{N}(r, 0; F) + \frac{1}{2}\overline{N}(r, \infty; F) + S(r, F) + S(r, G).$$

The same inequality holds for $T(r, G)$.

Lemma 2.6. [1] Let F and G be two non-constant meromorphic functions sharing 1 IM and $H \not\equiv 0$. Then

$$\begin{aligned} T(r, F) \leq & N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) \\ & + 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, F) + S(r, G) \end{aligned}$$

The same inequality holds for $T(r, G)$.

Lemma 2.7. [14] Let $f(z)$ be a non-constant meromorphic function and let $a_0(z)$, $a_1(z)$,...,$a_n(z)(\not\equiv 0)$ be small functions with respect to f. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.8. [9] Let $f(z)$ be a non-constant meromorphic function and p, k be positive integers. Then,

$$T(r, f^{(k)}) \leq T(r, f) + k\overline{N}(r, f) + S(r, f),$$
$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f),$$
$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.9. [15] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Then

$$N\left(r, \frac{f}{g}\right) - N\left(r, \frac{g}{f}\right) = N(r, f) + N\left(r, \frac{1}{g}\right) - N(r, g) - N\left(r, \frac{1}{f}\right).$$

Lemma 2.10. Let $f(z)$ be a transcendental meromorphic function of finite order, $c \in \mathbb{C}\backslash\{0\}$ be finite complex constants and $n \in \mathbb{N}$ and let $F_1(z) = f^n(z)P(f)\Delta_c^u f(z)$, where n, m and u are positive integers. Then

$$(n+m-u)T(r,f) + S(r,f) \leq T(r, F_1).$$

Proof . From Lemmas 2.1, 2.2, 2.3 and 2.7, we obtain

$$\begin{aligned}
(n+m+1)T(r,f) + S(r,f) &= T(r, f^{n+1}P(f)) + S(r,f) \\
&\leq T\left(r, \frac{f(z)\cdot F_1}{\Delta_c^u f}\right) + S(r,f) \\
&\leq T(r,F_1) + T\left(r, \frac{\Delta_c^u f}{f(z)}\right) + S(r,f) \\
&\leq T(r,F_1) + m\left(r, \frac{\Delta_c^u f}{f(z)}\right) + N\left(r, \frac{\Delta_c^u f}{f(z)}\right) + S(r,f) \\
&\leq T(r,F_1) + N\left(r, \frac{\sum_{r=0}^{u}(-1)^r\binom{u}{r} f(z+(u-r)c)}{f(z)}\right) + S(r,f) \\
&\leq T(r,F_1) + N\left(r, \frac{\sum_{r=0}^{u-1}(-1)^r\binom{u}{r} f(z+(u-r)c)}{f(z)}\right) + N\left(r, (-1)^r\frac{f(z)}{f(z)}\right) + S(r,f) \\
&\leq T(r,F_1) + N\left(r, \frac{1}{f(z)}\right) + \sum_{r=0}^{u-1} N(r, f(z+(u-r)c)) + S(r,f) \\
&\leq T(r,F_1) + T(r,f) + uN(r,f) + S(r,f) \\
&\leq T(r,F_1) + (u+1)T(r,f) + S(r,f).
\end{aligned}$$

Therefore, we have $(n+m-u)T(r,f) + S(r,f) \leq T(r,F_1)$. This completes the proof of Lemma 2.10. □

Lemma 2.11. [4] Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C}\backslash\{0\}$ be finite complex constants and $n \in \mathbb{N}$. Let $\Phi(z) = f^n(z)P(f)\Delta_c^u f(z)$, where $\Delta_c^u f(z) \not\equiv 0$. Then, we have

$$(n+m)T(r,f) \leq T(r,\Phi) - N\left(r, \frac{1}{\Delta_c^u f(z)}\right) + S(r,f).$$

Proof . Using the same arguments as in [4, Lemma 2.7], we can quickly obtain Lemma 2.11 □

3 Proof of Theorems

3.1 Proof of Theorem 1.6.

Let $F(z) = [f^n(z)P(f)\Delta_c^u f(z)]^{(k)}$, $G(z) = [g^n(z)P(g)\Delta_c^u g(z)]^{(k)}$, $F_1(z) = [f^n(z)P(f)\Delta_c^u f(z)]$ and $G_1(z) = [g^n(z)P(g)\Delta_c^u g(z)]$, where F and G are transcendental meromorphic functions satisfying $E_l(1;F) = E_l(1;G)$. By using Lemma 2.8

$$\begin{aligned}
N_2(r,0;F) \leq& N_2(r,0;(F_1)^{(k)}) + S(r,f), \\
\leq& T(r,(F_1)^{(k)}) - T(r,F_1) + N_{k+2}(r,0;F_1) + S(r,f), \\
\leq& T(r,F) - (n+m-u)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f).
\end{aligned}$$

From this we get,

$$(n+m-u)T(r,f) \leq T(r,F) - N_2(r,0;F) + N_{k+2}(r,0;F_1) + S(r,f). \tag{3.1}$$

Again, from Lemma 2.8 we have

$$\begin{aligned} N_2(r,0;F) \leq & N_2(r,0;(F_1)^{(k)}) + S(r,f) \\ \leq & k\overline{N}(r,\infty;F_1) + N_{k+2}(r,0;F_1) + S(r,f). \end{aligned} \tag{3.2}$$

We now discuss the following three cases separately.

Case 1. Let $l \geq 2$. Suppose that, if possible, that (i) of Lemma 2.4 holds. Then using (3.2), we obtain (3.1)

$$\begin{aligned} (n+m-u)T(r,f) \leq & N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + N_{k+2}(r,0;F_1) + S(r,f) + S(r,g) \\ \leq & N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + k\overline{N}(r,\infty;G_1) + 2\overline{N}(r,\infty;F_1) \\ & + 2\overline{N}(r,\infty;G_1) + S(r,f) + S(r,g). \end{aligned}$$

Then,

$$\begin{aligned} (n+m-u)T(r,f) \leq & (2u+2+k+\Gamma_0+u+3)T(r,f) + (k+\Gamma_0+u+3+ku+k+2u+2)T(r,g) \\ & + S(r,f) + S(r,g) \end{aligned} \tag{3.3}$$

Similarly,

$$\begin{aligned} (n+m-u)T(r,g) \leq & (2u+2+k+\Gamma_0+u+3)T(r,g) + (k+\Gamma_0+u+3+ku+k+2u+2)T(r,f) \\ & + S(r,f) + S(r,g). \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4) we obtain,

$$(n+m-u)[T(r,f)+T(r,g)] \leq [6u+ku+2\Gamma_0+3k+10][T(r,f)+T(r,g)] + S(r,f) + S(r,g),$$

contradicting the fact that $n > 7u + ku - m + 3k + 2\Gamma_0 + 10$. Therefore, by Lemma 2.4, we have either $FG = 1$ or $F = G$. We assume that $F \equiv G$, then

$$[f^n P(f)\Delta_c^u f]^{(k)} \equiv [g^n P(g)\Delta_c^u g]^{(k)}.$$

Integrating for k times, we get

$$[f^n P(f)\Delta_c^u f] \equiv [g^n P(g)\Delta_c^u g] + p(z),$$

where $p(z)$ is a polynomial of degree atmost $k-1$. If $p(z) \not\equiv 0$, above equation can be written as

$$\frac{f^n P(f)\Delta_c^u f}{p(z)} = \frac{g^n P(g)\Delta_c^u g}{p(z)} + 1.$$

By the Nevanlinna's Second Fundamental Theorem and Lemma 2.10, we have

$$\begin{aligned} (n+m-u)T(r,f) \leq & T\left(r, \frac{f^n P(f)\Delta_c^u f}{p(z)}\right) \\ \leq & \overline{N}\left(r, \frac{f^n P(f)\Delta_c^u f}{p(z)}\right) + \overline{N}\left(r, \frac{p(z)}{f^n P(f)\Delta_c^u f}\right) + \overline{N}\left(r, \frac{p(z)}{g^n P(g)\Delta_c^u g}\right) + S(r,f), \end{aligned}$$

which implies,

$$(n+m-u)T(r,f) \leq (2u+\Gamma_0+3)T(r,f) + (\Gamma_0+u+2)T(r,g) + S(r,f).$$

Similarly, we have

$$(n+m-u)T(r,g) \leq (2u+\Gamma_0+3)T(r,g) + (\Gamma_0+u+2)T(r,f) + S(r,g).$$

Now, by combining the above two inequalities, we have

$$(n+m-4u-2\Gamma_0-5) \leq S(r,f) + S(r,g),$$

which is a contradiction to $n > 7u + ku - m + 3k + 2\Gamma_0 + 10$. Thus, we have $p(z) \equiv 0$ and hence

$$f^n P(f)\Delta_c^u f \equiv g^n P(g)\Delta_c^u g. \tag{3.5}$$

Let $h = \frac{f}{g}$, we consider the following two subcases.

Subcase 1.1. Suppose h is non-constant then f and g will be a solution of the algebraic equation $R(f,g) \equiv 0$, with $R(w_1, w_2) = w_1^n P(w_1)\Delta_c^u w_1 - w_2^n P(w_2)\Delta_c^u w_2$. This is the conclusion (iii) of Theorem 1.6.

Subcase 1.2. If h is constant, then substituting $f = gh$ in (3.5), we get

$$\begin{aligned} &f^n P(f)[f(z+uc) - uf(z+(u-1)c) + \frac{u(u-1)}{2} f(z+(u-2)c) + \cdots + (-1)^{u-1} uf(z+c) + (-1)^u f(z)] \\ =&g^n P(g)[g(z+uc) - ug(z+(u-1)c) + \frac{u(u-1)}{2} g(z+(u-2)c) + \cdots + (-1)^{u-1} ug(z+c) + (-1)^u g(z)]. \end{aligned}$$

This implies,

$$\begin{aligned} &[g^{n+m}(h^{n+m+1} - 1) + a_{m-1} g^{n+m-1}(h^{n+m} - 1) + \cdots + a_1 g^{n+1}(h^{n+2} - 1) \\ &+ g^n(h^{n+1} - 1)]g(z+uc) + [g^{n+m}(h^{n+m+1} - 1) + a_{m-1} g^{n+m-1}(h^{n+m} - 1) \\ &+ \cdots + a_1 g^{n+1}(h^{n+2} - 1) + g^n(h^{n+1} - 1)](-1)ug(z+(u-1)c)+ \\ &\vdots \\ &+ [g^{n+m}(h^{n+m+1} - 1) + a_{m-1} g^{n+m-1}(h^{n+m} - 1) + \cdots + a_1 g^{n+1}(h^{n+2} - 1) \\ &+ g^n(h^{n+1} - 1)](-1)^u g(z) = 0. \end{aligned}$$

Therefore, we get

$$[g^{n+m}(h^{n+m+1} - 1) + a_{m-1} g^{n+m-1}(h^{n+m} - 1) + \cdots + g^n(h^{n+1} - 1)]\Delta_c^u g = 0,$$

which implies $h^d = 1$, where $d = GCD\{k \in (n+m+1, ..., n+m+1-i, ..., n+1) : a_{k-n-1} \neq 0\}$. Thus $f \equiv tg$, where t is a constant with $t^d = 1$. This is the conclusion *(ii)* of Theorem 1.6.

Case 2. Let $l = 1$ and $H \not\equiv 0$. Using Lemma 2.5 and (3.2), we obtain from (3.1)

$$\begin{aligned} (n+m-u)T(r,f) \leq& N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) \\ &+ \frac{1}{2}\overline{N}(r,\infty;F) + N_{k+2}(r,0;F_1) + S(r,f) + S(r,g) \\ \leq& N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + k\overline{N}(r,\infty;G_1) + 2\overline{N}(r,\infty;F_1) \\ &+ 2\overline{N}(r,\infty;G_1) + \frac{1}{2}\left[N_{k+1}(r,0;F_1) + k\overline{N}(r,F_1)\right] \\ &+ \frac{1}{2}\overline{N}(r,F_1) + S(r,f) + S(r,g). \end{aligned}$$

Then

$$(n+m-u)T(r,f) \leq \frac{1}{2}\left[8u + 4k + 3\Gamma_0 + ku + 13\right]T(r,f) + \left[2k + ku + 3u + \Gamma_0 + 5\right]T(r,g) + S(r,f) + S(r,g). \tag{3.6}$$

Similarly,

$$(n+m-u)T(r,g) \leq \frac{1}{2}\left[8u + 4k + 3\Gamma_0 + ku + 13\right]T(r,g) + \left[2k + ku + 3u + \Gamma_0 + 5\right]T(r,f) + S(r,f) + S(r,g). \tag{3.7}$$

Now, by combining the above two inequalities (3.6) and (3.7), we have

$$(n+m-u)[T(r,f) + T(r,g)] \leq \frac{1}{2}\left[8k + 3ku + 14u + 5\Gamma_0 + 23\right]\left[T(r,f) + T(r,g)\right] + S(r,f) + S(r,g),$$

contradicting the fact that $n > 8u + \frac{3ku}{2} + 4k + \frac{5\Gamma_0 - 2m}{2} + \frac{23}{2}$. We now assume that $H \equiv 0$. Then

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get,

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.8}$$

where $A(\neq 0)$ and B are constants. From (3.8) it is obvious that F, G share the value 1 CM and so they share (1, 2). Hence we have $n > 7u + ku - m + 3k + 2\Gamma_0 + 10$. Now we discuss the following three subcases.

Subcase 2.1. Let $B \neq 0$ and $A = B$ then from (3.8), we get

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.9}$$

If $B = -1$, then from (3.9), we obtain $FG = 1$ i.e

$$[f^n P(f)\Delta_c^u f]^{(k)} \cdot [g^n P(g)\Delta_c^u g]^{(k)} = a^2(z),$$

which is one of the conclusion of Theorem 1.6. If $B \neq -1$, from (3.9), we have

$$\frac{1}{F} = \left(\frac{BG}{(1+B)G - 1}\right)$$

and so $\overline{N}\left(r, \frac{1}{1+B}; G\right) = \overline{N}(r, 0; F)$. Now from the Second Fundamental Theorem and Lemma 2.8, we have

$$\begin{aligned}
T(r, G) &\leq \overline{N}(r, 0; G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r, \infty; G) + S(r, G) \\
&\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + S(r, G) \\
&\leq N_{k+1}(r, 0; F_1) + T(r, G) + N_{k+1}(r, 0; G_1) + k\overline{N}(r, F_1) - T(r, G_1) + \overline{N}(r, G_1) + S(r, G).
\end{aligned}$$

This gives,

$$(n + m - u)T(r, g) \leq (2k + ku + u + 2 + \Gamma_0)T(r, f) + (k + 2u + \Gamma_0 + 3)T(r, g) + S(r, g). \tag{3.10}$$

Similarly,

$$(n + m - u)T(r, f) \leq (2k + ku + u + 2 + \Gamma_0)T(r, g) + (k + 2u + \Gamma_0 + 3)T(r, f) + S(r, f). \tag{3.11}$$

By combining (3.10) and (3.11), we obtain

$$(n + m - 4u - 3k - 5 - 2\Gamma_0 - ku) \leq S(r, f) + S(r, g). \tag{3.12}$$

This is contradiction to $n > 7u + ku - m + 3k + 2\Gamma_0 + 10$.

Subcase 2.2. Let $B \neq 0$ and $A \neq B$. Then from (3.8) we obtain that

$$F = \left(\frac{(B+1)G - (B - A + 1)}{BG + (A - B)}\right)$$

and therefore

$$\overline{N}\left(r, \frac{B - A + 1}{B + 1}; G\right) = \overline{N}(r, 0; F).$$

Proceeding similarly as in Subcase 2.1, we can get a contradiction.

Subcase 2.3. Let $B = 0$ and $A \neq 0$. Then from (3.8) we get $F = \left(\frac{G+A-1}{A}\right)$ and $G = AF - (A - 1)$. If $A \neq 1$, we have $\overline{N}\left(r, \frac{A-1}{A}; F\right)=\overline{N}(r, 0; G)$ and $\overline{N}(r, 1 - A; G) = \overline{N}(r, 0; F)$. Proceeding similarly as in Subcase 2.1, we can get a contradiction.Thus $A = 1$ and then $F = G$. Now the result follows from proof of Case 1.

Case 3. Let $l = 0$ and $H \not\equiv 0$. Using Lemma 2.6, then using (3.2), we obtain (3.1)

$$\begin{aligned}(n+m-u)T(r,f) \leq & N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) \\ & + N_{k+2}(r,0;F_1) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,f) + S(r,g) \\ \leq & N_{k+2}\left(r,\frac{1}{F_1}\right) + 2\overline{N}(r,F) + N_{k+2}\left(r,\frac{1}{G_1}\right) + k\overline{N}(r,G_1) + 2\overline{N}(r,G) \\ & + 2\left[\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F)\right] + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + S(r,F) + S(r,G).\end{aligned}$$

Thus,

$$\begin{aligned}(n+m-u)T(r,f) \leq & [7u+5k+3\Gamma_0+2ku+11]T(r,f) + [4k+2ku+5u+2\Gamma_0+8]T(r,g) \\ & + S(r,f) + S(r,g).\end{aligned} \tag{3.13}$$

Similarly,

$$\begin{aligned}(n+m-u)T(r,g) \leq & [7u+5k+3\Gamma_0+2ku+11]T(r,g) + [4k+2ku+5u+2\Gamma_0+8]T(r,f) \\ & + S(r,f) + S(r,g).\end{aligned} \tag{3.14}$$

Now, by combining the above two inequalities (3.13) and (3.14), we have

$$(n+m-u)[T(r,f)+T(r,g)] \leq [9k+4ku+12u+5\Gamma_0+19][T(r,f)+T(r,g)] + S(r,f) + S(r,g),$$

contradicting the fact that $n > 13u + 4ku + 9k + 5\Gamma_0 - m + 19$. Proceeding in a similar manner for $H \equiv 0$, as in Case 2, the result follows. This completes the proof of Theorem 1.6.

3.2 Proof of Theorem 1.8.

Theorem 1.8 can be proved in a similar manner as Theorem 1.6, by using Lemma 2.11 and taking $N(r,f) = \overline{N}(r,f) = S(r,f)$. This completes the proof of Theorem 1.8.

We can pose the following open questions for further research.

1. Can the condition for n in Theorem 1.6 be still reduced?

2. What happens if we replace $F = [f^n(z)P(f)\Delta_c^u f(z)]^{(k)}$ in Theorem 1.6, by $F = [f^n(z)P_m(f(qz+c))H(f)]$ where $P(z)$ is a polynomial of finite degree 'm' and $H[f] = \prod_{j=1}^{k} f^{(j)}(z)$.

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