Progress in Physics of Applied Materials 5 (2025), 61 – 66

Semnan University

Progress in Physics of Applied Materials

journal homepage: https://ppam.semnan.ac.ir/

The Effect of Gap Fluctuations on Specific Heat of a Superconducting Nanograin

Alireza Donyavi [a], Neda Ebrahimian [a*]

[a] *Department of Physics, Faculty of Basic Sciences, Shahed University, Tehran 18155-159, Iran*

ARTICLE INFO

Article history:
Received: 10 December 2024
Revised: 2 January 2024
Accepted: 21 January 2024

Keywords:
Superconductivity;
Heat Capacity;
Nano Grain;
Order Parameter.

ABSTRACT

Recently, the finite size effects, via superconducting nanograins, have attracted much attention from physicists. The effect of the small size can enter via the interaction matrix element and the spectral energy. We suppose that the mean level spacing near the Fermi energy is smaller than the bulk gap, allowing the BCS formalism to remain a valid approximation. For a nanograin, the gap function, in general, depends on the size of the system, and the Fermi energy. By entering the effect of the small size on the gap equation for a rectangular nanograin, specific heat in terms of temperature and length of a superconducting nanograin is obtained. Our results reveal that the spectral energy of the nanograin does not affect the change in the behavior of specific heat. However, the effect of the energy gap of nanograin strongly affects the behavior of specific heat. One of the interesting results is that at some fixed temperatures, the behaviour of specific heat shows a peak in a special length. Also, we compare specific heat in 2- and 3-dimensional cases.

1. Introduction

Nanotechnology is highly interesting to researchers of various sciences and will continue to be in the next few decades. This field has also made notable advances in superconductivity. Recently, the finite size effects, via considering nanograins, have attracted much attention from physicists. As Anderson initially noted, BCS theory becomes inconsistent when the size of the superconductor is such that the mean level spacing, d, approaches the superconducting gap, Δ [1]. According to BCS theory, the dominant contribution to pairing correlations comes from levels within a range of order Δ around the Fermi surface. Still, there are no levels left within this range when d > Δ. When it became possible to reach this regime experimentally by doing transport measurements on superconducting grains, interest was spurred in a description of the pair-correlated state that is also valid for d > Δ [2-4].

Because of the importance of this issue, the thermodynamic properties of these grains have been extensively analyzed [5]. Also, Experimental investigations into the characteristics of these grains have been conducted in detail. The results have further stimulated the interest in superconducting nanograin [6-10].

These systems are altered with thickness down to the nanometre scale has also significant technological implications. Subsequent experimental investigations of superconducting confined systems showed either a decrease or an increase in critical temperatures with sample size, depending on the material.

Since 1960s, most experiments focusing on superconducting correlations in grains were performed with either grain powders or with granular films, where each metallic crystallized grain is surrounded by an insulating, amorphous barrier [11-26].

* Corresponding author. Tel.: +98-21-51212242
E-mail address: n.ebrahimian@shahed.ac.ir

Cite this article as:
Donyavi A., and Ebrahimian N., 2025. The Effect of Gap Fluctuations on Specific Heat of a Superconducting Nanograin. *Progress in Physics of Applied Materials*, *5*(1), pp.61-66.DOI: **10.22075/PPAM.2025.36219.1125**

The experimental results demonstrated an increase in critical temperatures for Indium (In), Tin (Sn), and Aluminum (Al) [15,21]. The studies indicated good agreement between theoretical predictions and experimental observations for weak and intermediate coupling superconductors. For Lead (Pb) and Niobium (Nb) films, which consist of crystalline grains, a decrease in critical temperature was observed as the thickness decreased, suggesting that the system behaves like a disordered network of weakly coupled grains [22-26]. Detecting the superconducting gap in a single physically isolated ultra-small Pb/Sn grain was achieved using a scanning tunneling microscope (STM) [27,28]. These studies examined the size evolution of superconductivity in isolated nanoparticles that were grown on a substrate. In the case of Sn particles, oscillations of the superconducting energy gap with varying particle sizes were observed, with enhancements of the gap reported to be as large as 60%. Conversely, Pb particles demonstrated a decrease in the gap as particle size decreased [27-29].

One of the most important thermodynamic properties of superconductors is specific heat [30-33]. In the present paper, we calculate the specific heat for rectangular-shaped grains based on gap fluctuations. Our work relies on numerical calculations, focusing on ballistic grains. The mean field potential is approximated as an infinite well representative of the grain's shape. Additionally, the mean level spacing near the Fermi energy is smaller than the bulk gap, allowing the BCS formalism to remain a valid approximation.

The paper is organized as follows. In Sec. 2, the theoretical Frame of the calculation of specific heat is given. In Sec. 3, the effect of the finite size of a superconductor is entered via the dependence of the energy gap to the length of the superconductor, and numerical calculations are brought. In Sec. 4, some remarks and conclusions are given.

2. Theoretical Frame

BCS Hamiltonian can be written as

$$H = \sum_{n\sigma} \varepsilon_n c_{n\sigma}^{\dagger} c_{n\sigma} - \sum_{n,n'} I_{n,n'} c_{n\uparrow}^{\dagger} c_{n\downarrow}^{\dagger} c_{n'\downarrow} c_{n'\uparrow} \tag{1}$$

where $c_{n\sigma}$ and $c_{n\sigma}^{\dagger}$ are annihilation and creation operators for the state n and spin σ. ε_n is an eigenvalue of electron in a grain. Also, interaction matrix element, $I_{n,n'}$, is given by [34]

$$I_{n,n'} = \lambda V \delta \int \psi_n^2(\vec{r})\ \psi_{n'}^2(\vec{r}) d\vec{r} \tag{2}$$

The matrix shows the interaction between electrons. λ is a constant. C$\psi_n$is the eigenstate of a free electron in a grain, δ is mean level spacing and V is the volume of the system. Gap energy is defined by

$$\Delta(\varepsilon_k) = \sum_{k'} I_{kk'} u_{k'} v_{k'} \tag{3}$$

where u_k and v_k are given by

$$u_k = \sqrt{\frac{1}{2}\left(1 + \frac{\varepsilon_k}{E_k}\right)}, \quad v_k = \sqrt{\frac{1}{2}\left(1 - \frac{\varepsilon_k}{E_k}\right)} \tag{4}$$

where E_k is the excitation energy and ε_k is the kinetic energy measured with respect to the chemical potential, $\varepsilon_k = \left(\frac{\hbar^2 k^2}{2m}\right) - \mu$, where m, μ and $\vec{k}$ are the mass of the electron, the chemical potential and the wave vector. Also, the excitation energy is given by $E_k = \sqrt{\varepsilon_k^2 + \Delta(\varepsilon_k)^2}$. It should be noted that u_k and v_k existing in Eq. (3) can be obtained by minimizing the following total energy of the system

$$E = \sum_{n\sigma} \varepsilon_n v_n^2 - \sum_{n,n'} I_{n.n'} v_n u_n v_{n'} u_{n'} \tag{5}$$

For finite temperatures, the first term and the last term should be multiplied by the factors $(1-2f_k)$ and $(1-2f_k)(1-2f_{k'})$, respectively. f_k is Fermi-Dirac distribution function and is given by [35,36]

$$f_\uparrow = f_\downarrow \equiv f_k = \frac{1}{1 + e^{\varepsilon/k_B T}} \tag{6}$$

k_B and T are Boltzmann constant and temperature, respectively. Then, u_k and v_k given by Eq. 4 lead to the following result for total energy

$$2\sum_k \varepsilon_k \left[v_k^2 + \left(u_k^2 - v_k^2\right) f_k \right] + \sum_{k,k'} I_{k,k'} v_k u_k v_{k'} u_{k'} \left(1-2f_k\right)\left(1-2f_{k'}\right) \tag{7}$$

Or by using Eq. (4), one has

$$2\sum_k \frac{1}{2E_k}[(\varepsilon_k + E_k)^2 f_k + (E_k - \varepsilon_k)^2(1 - f_k)] \tag{8}$$

At finite temperatures, the integral form of the energy gap, which is, in general, dependent on energy and temperature, in the grand canonical approximation (by introducing spectral energy, $N(\varepsilon') = \sum_{n'} \delta(\varepsilon - \varepsilon')$), is

$$\Delta(\varepsilon, T) = \frac{1}{2}\int_{-\varepsilon_D}^{\varepsilon_D} \frac{\Delta(\varepsilon', T) I(\varepsilon, \varepsilon')}{\sqrt{\varepsilon'^2 + \Delta^2(\varepsilon', T)}} N(\varepsilon') \times (1 - 2f(\varepsilon'))d\varepsilon' \tag{9}$$

It should be mentioned that we can modify the gap equation (Eq, 9) for finite temperatures by the factor $(1 - 2f(\varepsilon'))$.

The small size effect influences in two ways: First from $I(\varepsilon, \varepsilon')$, which is related to the energy, and second via spectral energy, $N(\varepsilon')$. When the volume of nanograin approaches infinity, the spectral density, $N(\varepsilon')$, can be taken to be energy independent and is equal to that of bulk superconductor. Also, matrix elements are energy-independent and the gap is equal to the bulk value. $N(\varepsilon)$ for the superconducting nanograin in the semiclassical approximation is given by

$$N(\varepsilon) \approx N(0)\left(1 + \bar{g}(0) + \tilde{g}(\varepsilon)\right) \tag{10}$$

where $N(0)$ is spectral energy at the Fermi surface. $N(\varepsilon)$ is accompanied by a monotonous $\bar{g}(0)$ (at the Fermi energy) and an oscillatory function, $\tilde{g}(\varepsilon)$. Spectral energy is related to the length of a superconductor. It should be mentioned that the dependence of the energy gap on the length of the superconductor comes in two ways, i.e., from spectral energy and energy gap. By considering our numerical calculations, we saw that the effect of spectral energy was not effective in changing our results except for a somewhat change in the value of specific heat. However, the gap energy of the superconducting nanograin has more effects on the specific heat concerning that of a bulk superconductor.

The entropy for a system of fermions is given by

$$S = -k_B \sum_{k,\alpha} \left[f_k \, ln \, f_k + (1 - f_k) \, ln(1 - f_k) \right] \quad (11)$$

It should be noted that f_k is not dependent on spins, therefore, the summation on spins is trivial and is given by 2. Then, one has

$$TS = -\sum_k \left[E_k (1 - 2f_k) - 2k_B T \, ln \left(2 \cosh \left(\frac{E_k}{2k_B T} \right) \right) \right] \quad (12)$$

By using $C = T \, dS/dT$, the specific heat is

$$C = 2\beta^2 k_B \sum_k f_k \, (1 - f_k) \left[{E_k}^2 + \beta E_k \frac{dE_k}{d\beta} \right] \quad (13)$$

where $\beta = 1/k_B T$. Then, after transforming the summation to the integral, one has

$$C = 2\beta^2 k_B N(0) \int_0^\infty d\varepsilon \, f(\varepsilon) \left(1 - f(\varepsilon) \right) \left[E^2 - T\Delta \frac{d\Delta}{dT} \right] \quad (14)$$

Now we proceed to enter the effect of the small size of the system. Temperature-dependence of the gap function in the BCS model is given by

$$\Delta_{BCS}(T) \approx 1.76 k_B T_c \, tanh \left(1.74 \sqrt{\frac{T_c}{T} - 1} \right) = \Delta(T = 0) \, tanh \left(1.74 \sqrt{\frac{T_c}{T} - 1} \right) \quad (15)$$

where T_c is the critical temperature.

At zero temperature, for a nanograin, the gap function, in general, depends on the single-particle energy, the size of the system, and the number of particles (or, equivalently, Fermi energy). For a rectangular box in two and three dimensions, the gap equation is energy-independent. In the chaotic case, however, one gets an integral equation due to the energy dependence of the interaction matrix elements. We consider only a rectangular box. Using the knowledge of spectral energy and interaction matrix elements as series in $\frac{1}{(k_F L)}$, where k_F and L are Fermi number wave and the length of the nanograin, respectively, and by using the solution of the gap equation in the semiclassical region for a 3D case, one has

$$\Delta = \Delta_{BCS} \left(1 + f^{(1)} + f^{\left(\frac{3}{2}\right)} + f^{(2)} \right) \quad (16)$$

where $f^{(n)} \propto \frac{1}{(k_F L)^n}$ and Δ_{BCS} is gap energy for bulk limit. For finite temperatures, we apply Eq. (16), by replacing $\Delta_{BCS} \to \Delta_{BCS}(T)$. This is correct since $\Delta(\varepsilon') \equiv \Delta$ and the gap is energy-independent for a rectangular nanograin [34]. Using Eqs. (14) and (16), we obtain the heat capacity of grain for numerical calculation

$$C_V = (1.764^2) \left(\frac{N(0)V}{2} \right) \left(\frac{\delta^3}{t^2} \right) \int_0^{\delta^{-1} \sinh(1/N(0)V)} dz \times \frac{1}{\cosh^2 \left[\left(\frac{0.88\delta}{t} \right) \left(\sqrt{1 + z^2} \right) \right]} \times \left(z^2 + 1 + \left(\frac{0.87t/\delta}{(t^2)\sqrt{(1/t) - 1}} \right) \right) \times \left(\frac{1}{\cosh^2 \left[1.74 \sqrt{(1/t) - 1} \right]} \right) \quad (17)$$

where we use the definitions $t = T/T_c$ and

$$\delta = tanh \left(1.74 \sqrt{\frac{T_c}{T} - 1} \right) \times \left(1 + \left(\frac{1}{k_F L} \right) + \left(\frac{1}{k_F L} \right)^2 + \left(\frac{1}{k_F L} \right)^{3/2} \right) \quad (18)$$

3. Numerical results

Before investigating the specific heat, we examine the behavior of gap function to observe how dimensionality and length influence the value of the gap, as shown in Fig. 1.

In Fig. 1, for the 3D case, the gap energy (measured by bulk gap at zero temperature and in arbitrary units) is plotted with respect to the temperature ratio (T/T_c) for both 2D and 3D cases. Fig. 1 shows that at a fixed length, the gap function decreases with increasing temperature, in both 2D and 3D cases. However, at a fixed temperature (or a fixed length), the gap in 2D case is higher than that in the 3D case. The inset of Fig. 1 displays the behavior of the gap function versus the length of a superconducting nanograin. At a fixed temperature, the gap function decreases as the length increases for both the 2D and 3D cases.

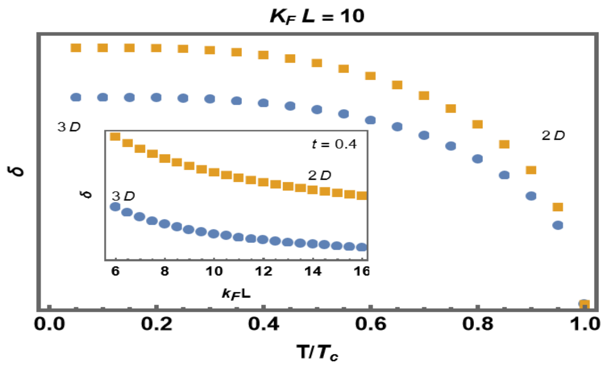

Fig. 1. Gap function (measured by bulk gap at zero temperature and in arbitrary units) in terms of the T/T_c for both 2D and 3D cases. Inset: Gap function (measured by bulk gap at zero temperature and in arbitrary units) in terms of the $k_F L$ for both 2D and 3D cases

In Fig. 2, the specific heat concerning the T/T_c at two fixed different $k_F L$ is plotted. Although, the specific heat increases with the increasing temperature, the rate of increase varies for nanograins of different lengths.

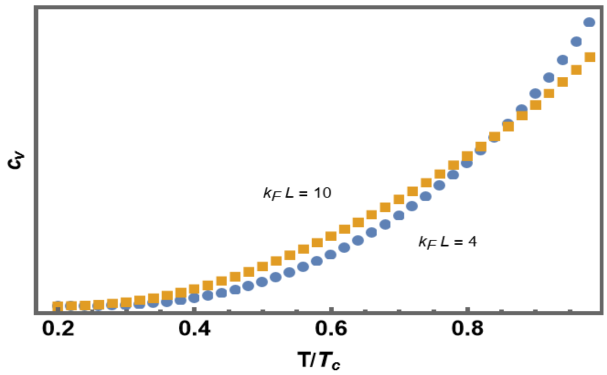

Fig. 2. Specific heat (in arbitrary units) in terms of T/T_c at two different $k_F L = 4$ and $k_F L = 10$

The more significant issue is the variations in specific heat capacity concerning the length of the superconductor, as illustrated in Figs. 3-5, which belonged at different temperatures, yield very noteworthy results. At T/T_c values around 0.8 up to 0.84, the specific heat displays a peak in the curve; however, at temperatures above 0.84, the specific heat shows a decreasing trend. Conversely, at temperatures below approximately 0.8, there is an increasing trend in specific heat as a function of length for the superconductor.

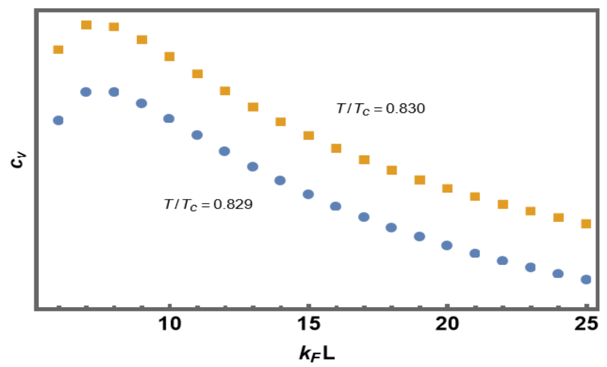

Fig. 3. Specific heat (in arbitrary units) in terms of $k_F L$ and at two different temperature ratios 0.829 and 0.83

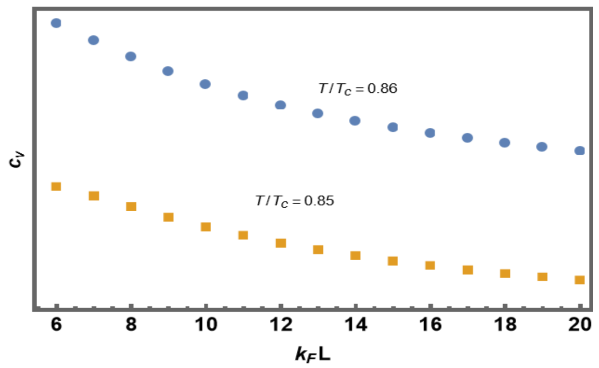

Fig. 4. Specific heat (in arbitrary units) in terms of $k_F L$ and at two different temperature ratios 0.85 and 0.86

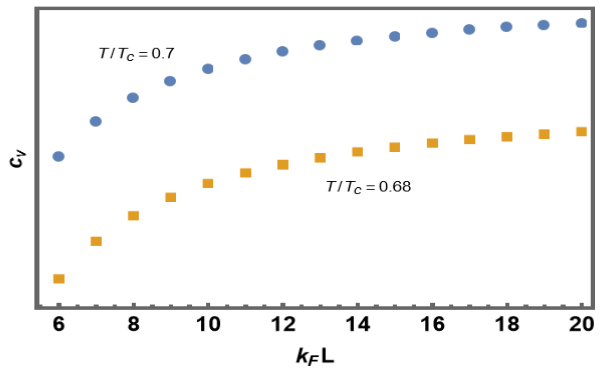

Fig. 5. Specific heat (in arbitrary units) in terms of $k_F L$ and at two different temperature ratios 0.68 and 0.70

Now we will compare the effects of small size on 3D and 2D cases. $\Delta = \left(\Delta_{BCS}(T)\right)\left(1 + f^{(1)} + f^{(1/2)}\right)$ is used for 2D cases. In Fig. 6, the specific heat concerning the temperature is plotted at a fixed $k_F L = 10$ for two cases, i.e., two- and three-dimensional cases. It is seen that the behavior of the specific heat is different in these cases.

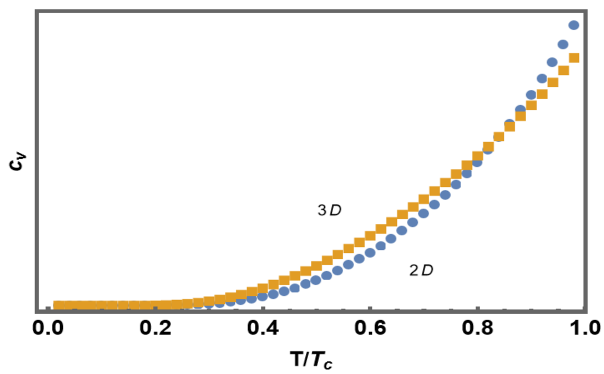

Fig. 6. Specific heat (in arbitrary units) in terms of the T/T_c at fixed $k_F L = 10$for 2D and 3D

In Fig. 7, the specific heat (in arbitrary units) in terms of $k_F L$ for 2-dimensional and 3-dimensional at a fixed $T/T_c = 0.86$ is plotted. For example, at $T/T_c = 0.86$, which is the typical temperature, at lower value of $k_F L = 13$, for the 3D case, specific heat is increasing with length, while, for the 2D case, that is decreasing. Of course, when the circumstance is changed, the results about increasing and decreasing are changed.

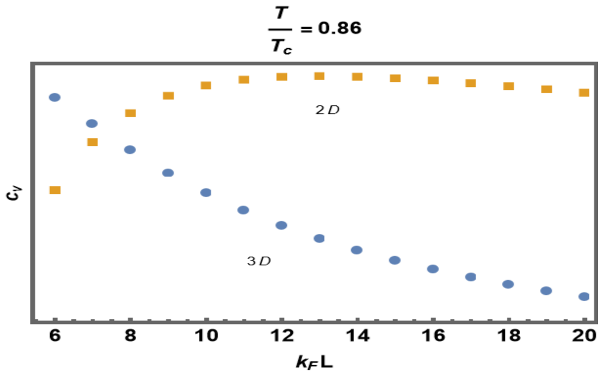

Fig. 7. Specific heat (in arbitrary units) in terms of $k_F L$ for 2-dimension and 3-dimension at a fixed $T/T_c = 0.86$

4. Conclusions

We have considered a superconducting nanograin and numerically obtained a thermodynamic property of the system, namely, its specific heat. When a bulk superconductor is replaced by a superconducting nanograin, the effect of small size is incorporated through the spectral energy and the energy gap. Our results reveal that the spectral energy of the nanograin does not change the behavior of specific heat, except for its value. However, the energy gap of the nanograin strongly affects the behavior of specific heat. Generally, the energy gap depends on the excitation energy, the size of the system, and the Fermi energy. In particular, the present investigation shows that the specific heat is related to the length of the superconducting nanograin.

An important issue is that, at different temperatures, the behavior of specific heat varies. For example, at $T/T_c = 0.68$ ($T/T_c = 0.86$), we do not observe any extremum in specific heat; it increases (decreases) with $k_F L$ (when the calculation continues, at least $k_F L = 2000$, the results do not change). However, in the temperature ratio interval of about $T/T_c = 0.79$ and $T/T_c = 0.84$, the specific heat shows a maximum value with respect to $k_F L$. Additionally, we focused on comparing the 2-dimensional and 3-dimensional cases. The dependence of the energy gap on the length differs, and the behavior of specific heat with respect to $k_F L$ is also notably different in 2- and 3-dimensions cases.

References

[1] Anderson, P.W., 1959. Theory of dirty superconductors. *Journal of Physics and Chemistry of Solids*, *11*(1-2), pp.26-30.

[2] Black, C.T., Ralph, D.C. and Tinkham, M., 1996. Spectroscopy of the superconducting gap in individual nanometer-scale aluminum particles. *Physical review letters*, *76*(4), p.688.

[3] Von Delft, J. and Ralph, D.C., 2001. Spectroscopy of discrete energy levels in ultrasmall metallic grains. *Physics Reports*, *345*(2-3), pp.61-173.

[4] Richardson, R., Sherman, N., 1964. Pairing models of Pb206, Pb204 and Pb202. Nucl. Phys., 52, p. 253.

[5] Brack, M. and Bhaduri, R.K., 1997. Semiclassical Physics Addison.

[6] Balian, R. and Bloch, C., 1971. Distribution of eigenfrequencies for the wave equation in a finite domain. II. Electromagnetic field. Riemannian spaces. *Annals of Physics*, *64*(1), pp.271-307.

[7] Parmenter, R.H., 1968. Size effect in a granular superconductor. *Physical Review*, *166*(2), p.392.

[8] Blatt, J.M. and Thompson, C.J., 1963. Shape resonances in superconducting thin films. *Physical Review Letters*, *10*(8), p.332.

[9] Mühlschlegel, B., Scalapino, D.J. and Denton, R., 1972. Thermodynamic properties of small superconducting particles. *Physical Review B*, *6*(5), p.1767.

[10] Bardeen, J., Cooper, L.N. and Schrieffer, J.R., 1957. Theory of superconductivity. *Physical review*, *108*(5), p.1175.

[11] Abeles, B., Cohen, R.W. and Cullen, G.W., 1966. Enhancement of superconductivity in metal films. *Physical Review Letters*, *17*(12), p.632.

[12] Giaever, I. and Zeller, H.R., 1968. Superconductivity of small tin particles measured by tunneling. *Physical Review Letters*, *20*(26), p.1504.

[13] Ohshima, K., Kuroishi, T. and Fujita, T., 1976. Superconducting transition temperature of aluminium fine particles. *Journal of the Physical Society of Japan*, *41*(4), pp.1234-1236.

[14] Tsuboi, T. and Suzuki, T., 1977. Specific heat of superconducting fine particles of tin. I. Fluctuations in zero magnetic field. *Journal of the Physical Society of Japan*, *42*(2), pp.437-444.

[15] Matsuo, S., Sugiura, H. and Noguchi, S., 1974. Superconducting transition temperature of aluminum, indium, and lead fine particles. *Journal of Low Temperature Physics*, *15*, pp.481-490.

[16] Abeles, B., Sheng, P., Coutts, M.D. and Arie, Y., 1975. Structural and electrical properties of granular metal films. *Advances in Physics*, *24*(3), pp.407-461.

[17] Shapira, Y. and Deutscher, G., 1983. Semiconductor-superconductor transition in granular Al-Ge. *Physical Review B*, *27*(7), p.4463.

[18] Li, W.H., Yang, C.C., Tsao, F.C. and Lee, K.C., 2003. Quantum size effects on the superconducting parameters of zero-dimensional Pb nanoparticles. *Physical Review B*, *68*(18), p.184507.

[19] Li, W.H., Yang, C.C., Tsao, F.C., Wu, S.Y., Huang, P.J., Chung, M.K. and Yao, Y.D., 2005. Enhancement of superconductivity by the small size effect in In nanoparticles. *Physical Review B—Condensed Matter and Materials Physics*, *72*(21), p.214516.

[20] Bose, S., Raychaudhuri, P., Banerjee, R., Vasa, P. and Ayyub, P., 2005. Mechanism of the size dependence of the superconducting transition of nanostructured Nb. *Physical review letters*, *95*(14), p.147003.

[21] Li, W.H., Wang, C.W., Li, C.Y., Hsu, C.K., Yang, C.C. and Wu, C.M., 2008. Coexistence of ferromagnetism and superconductivity in Sn nanoparticles. *Physical Review B—Condensed Matter and Materials Physics*, *77*(9), p.094508.

[22] Bose, S., Galande, C., Chockalingam, S.P., Banerjee, R., Raychaudhuri, P. and Ayyub, P., 2009. Competing effects of surface phonon softening and quantum size effects on thesuperconducting properties of nanostructured Pb. *Journal of Physics: Condensed Matter*, *21*(20), p.205702.

[23] Jin, Y., Song, X. and Zhang, D., 2009. Grain-size dependence of superconductivity in dc sputtered Nb films. *Science in China Series G: Physics, Mechanics and Astronomy*, *52*(9), pp.1289-1292.

[24] Delacour, C., Ortega, L., Faucher, M., Crozes, T., Fournier, T., Pannetier, B. and Bouchiat, V., 2011. Persistence of superconductivity in niobium ultrathin films grown on R-plane sapphire. *Physical Review B—Condensed Matter and Materials Physics*, *83*(14), p.144504.

[25] Reich, S., Leitus, G., Popovitz-Biro, R. and Schechter, M., 2003. Magnetization of small lead particles. *Physical review letters*, *91*(14), p.147001.

[26] Kresin, V.Z. and Tavger, B.A., 1966. Superconducting transition temperature of a thin film. *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki (USSR) For English translation see Sov. Phys.-JETP (Engl. Transl.)*, *50*.

[27] Brihuega, I., García-García, A.M., Ribeiro, P., Ugeda, M.M., Michaelis, C.H., Bose, S. and Kern, K., 2011. Experimental observation of thermal fluctuations in single superconducting Pb nanoparticles through tunneling measurements. Physical Review B—Condensed Matter and Materials Physics, 84(10), p.104525.

[28] Liu, J., Wu, X., Ming, F., Zhang, X., Wang, K., Wang, B. and Xiao, X., 2011. Size-dependent superconducting state of individual nanosized Pb islands grown on Si (111) by tunneling spectroscopy. *Journal of Physics: Condensed Matter*, *23*(26), p.265007.

[29] Romero-Bermúdez, A. and Garcia-Garcia, A.M., 2014. Size effects in superconducting thin films coupled to a substrate. *Physical Review B*, *89*(6), p.064508.

[30] Gordon, J.E., Tan, M.L., Fisher, R.A. and Phillips, N.E., 1989. Specific heat data of high-Tc superconductors: Lattice and electronic contributions. *Solid state communications*, *69*(6), pp.625-629.

[31] Chou, C., White, D. and Johnston, H.L., 1958. Heat capacity in the normal and superconducting states and critical field of niobium. *Physical Review*, *109*(3), p.788.

[32] Wen, H.H., 2020. Specific heat in superconductors. *Chinese Physics B*, *29*(1), p.017401.

[33] Jiang, C., Zaccone, A., Setty, C. and Baggioli, M., 2023. Glassy heat capacity from overdamped phasons and hypothetical phason-induced superconductivity in incommensurate structures. *Physical Review B*, *108*(5), p.054203.

[34] García-García, A.M., Urbina, J.D., Yuzbashyan, E.A., Richter, K. and Altshuler, B.L., 2011. BCS superconductivity in metallic nanograins: Finite-size corrections, low-energy excitations, and robustness of shell effects. *Physical Review B—Condensed Matter and Materials Physics*, *83*(1), p.014510.

[35] De Gennes, P.G., 2018. Superconductivity of metals and alloys. CRC press.

[36] Ketterson, J.B. and Song, S.N., 1999. Superconductivity. Cambridge university press.