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Ulam-Hyers-Rassias-stability of a Cauchy-Jensen additive mapping In fuzzy Banach spaces

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Abstract

In this paper, We prove the Ulam-Hyers-Rassias stability of a Cauchy-Jensen additive functional equation in fuzzy Banach spaces. The concept of Ulam-Hyers-Rassias stability originated from Th. M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297-300.

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1 Introduction

The stability problem of functional equations was originated from a question of Ulam [37] concerning the stability of group homomorphisms. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Th. M. Rassias [30] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1. $(Th.M.Rassias)$: Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \le p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The Ulam-Hyers-Rassias stability of the quadratic functional equation was proved by Skof [36]

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for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [8] proved the Ulam-Hyers-Rassias stability of the quadratic functional equation.
The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [1, 2, 3, 9]–[12], [16], [23]–[35]).

Katsaras [18] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view (see [13], [20], [24]).
In particular, Bag and Samanta [4], following Cheng and Mordeson [6], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [19]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [5].
Now we consider a mapping $f : X \to Y$ satisfying the following functional equation, which is introduced by the first author,

$$\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l(\neq i_j,\forall j\in\{1,\cdots,m\})\le n}} f\left(\frac{\sum_{j=1}^{m} x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) = \frac{(n-m+1)}{n}\begin{pmatrix} n\\ m\end{pmatrix}\sum_{i=1}^{n} f(x_i) \tag{1.1}$$

for all $x_1, \cdots, x_n \in X$, where $m, n \in \mathbb{N}$ are fixed integers with $n \geq 2$, $1 \leq m \leq n$. Specially, we observe that in case $m = 1$ the equation (1.1) yields Cauchy additive equation

$$f\left(\sum_{l=1}^{n} x_{k_l}\right) = \sum_{l=1}^{n} f(x_i).$$

We observe that in case $m = n$ the equation (1.1) yields Jensen additive equation

$$f\left(\frac{\sum_{j=1}^{n} x_j}{n}\right) = \frac{1}{n}\sum_{l=1}^{n} f(x_i).$$

Therefore, the equation (1.1) is a generalized form of the Cauchy-Jensen additive equation and thus every solution of the equation (1.1) may be analogously called general (m, n)-Cauchy- Jensen additive. For the case $m = 2$, we have established new theorems about the Ulam-Hyers-Rassias stability in quasi β-normed spaces [29]. Let X and Y be linear spaces. For each m with $1 \leq m \leq n$, a mapping $f : X \to Y$ satisfies the equation (1.1) for all $n \geq 2$ if and only if $f(x)-f(0) = A(x)$ is Cauchy additive, where $f(0) = 0$ if $m < n$. In particular, we have $f((n-m+1)x) = (n-m+1)f(x)$ and $f(mx) = mf(x)$, for all $x \in X$.

Definition 1.2. Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

- $(N1)$ $N(x, t) = 0$ for $t \leq 0$;
- $(N2)$ $x = 0$ if and only if $N(x, t) = 1$, for all $t > 0$;
- $(N3)$ $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;
- $(N4)$ $N(x + y, c + t) \geq min\{N(x, s), N(y, t)\}$;
- $(N5)$ $N(x, .)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
- $(N6)$ for $x \neq 0$, $N(x, .)$ is continuous on $\mathbb{R}$.

Example 1.3. Let $(X, \|.\|)$ be a normed linear space and $\alpha, \beta > 0$. Then

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t+\beta\|x\|} & t > 0, x \in X\\ 0 & t \leq 0, x \in X\end{cases}$$

is a fuzzy norm on X.

2 Preliminaries

Definition 2.1. Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N - \lim_{t\to\infty} x_n = x$.

Definition 2.2. Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [5]).

Definition 2.3. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$, for all $x, y \in X$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

Theorem 2.4. Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

3 Fuzzy stability of $(m, n)-$Cauchy-Jensen additive functional equation (1.1): A fixed point method

In this section, using the fixed point alternative approach we prove the Ulam-Hyers-Rassias stability of functional equation (1.1) in fuzzy Banach spaces. Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Theorem 3.1. Let $\varphi : X^n \to [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\varphi\left(\frac{x_1}{n-m+1}, \ldots, \frac{x_n}{n-m+1}\right) \leq \frac{L\varphi(x_1, x_2, \ldots, x_n)}{n-m+1}$$

for all $x_1, \cdots, x_n \in X$. Let $f : X \to Y$ with $f(0) = 0$ is a mapping satisfying

$$N\left(\sum_{\substack{1 \leq i_1 < \cdots < i_m \leq n \\ 1 \leq k_l (\neq i_j, \forall j \in \{1, \cdots, m\}) \leq n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) - \frac{(n-m+1)}{n}\begin{pmatrix} n \\ m \end{pmatrix} \sum_{i=1}^n f(x_i), t\right) \geq \frac{t}{t + \varphi(x_1, \ldots, x_n)} \tag{3.1}$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Then there exists a unique $(m, n)-$Cauchy-Jensen additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix}(1-L)t}{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix}(1-L)t + L\varphi(x, \cdots, x)}, \tag{3.2}$$

for all $x \in X$ and all $t > 0$.

Proof . Replacing $(x_1, \cdots, x_n)$ by $(x, \cdots, x)$ in (3.1), we have

$$N\left(\begin{pmatrix} n \\ m \end{pmatrix} f((n-m+1)x) - \begin{pmatrix} n \\ m \end{pmatrix}(n-m+1)f(x), t\right) \geq \frac{t}{t+\varphi(x,\cdots,x)} \tag{3.3}$$

for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y \ ; \ g(0) = 0\}$ and the generalized metric d in S defined by

$$d(f,g) = \inf\left\{\mu \in \mathbb{R}^+ : N(g(x)-h(x), \mu t) \geq \frac{t}{t+\varphi(x,\cdots,x)}, \forall x \in X, \ t > 0\right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S,d) is complete (see [22]). Now, we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := (n-m+1)g\left(\frac{x}{n-m+1}\right)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g,h) = \epsilon$. Then $N(g(x)-h(x), \epsilon t) \geq \frac{t}{t+\varphi(x,\cdots,x)}$ for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned}
&N(Jg(x)-Jh(x), L\epsilon t) \\
&= N\left((n-m+1)g\left(\frac{x}{n-m+1}\right) - (n-m+1)h\left(\frac{x}{n-m+1}\right), L\epsilon t\right) \\
&= N\left(g\left(\frac{x}{n-m+1}\right) - h\left(\frac{x}{n-m+1}\right), \frac{L\epsilon t}{n-m+1}\right) \\
&\geq \frac{\frac{Lt}{n-m+1}}{\frac{Lt}{n-m+1} + \varphi\left(\frac{x}{n-m+1}, \cdots, \frac{x}{n-m+1}\right)} \geq \frac{\frac{Lt}{n-m+1}}{\frac{Lt}{n-m+1} + \frac{L\varphi(x,\cdots,x)}{n-m+1}} = \frac{t}{t+\varphi(x,\cdots,x)}
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g,h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that $d(Jg, Jh) \leq Ld(g,h)$ for all $g, h \in S$. It follows from (3.3) that

$$\begin{aligned}
N\left((n-m+1)f\left(\frac{x}{n-m+1}\right) - f(x), \frac{t}{\begin{pmatrix} n \\ m \end{pmatrix}}\right) &\geq \frac{t}{t+\varphi\left(\frac{x}{n-m+1}, \cdots, \frac{x}{n-m+1}\right)} \\
&\geq \frac{t}{t+\frac{L\varphi(x,\cdots,x)}{n-m+1}}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So

$$N\left((n-m+1)f\left(\frac{x}{n-m+1}\right) - f(x), \frac{Lt}{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix}}\right) \geq \frac{t}{t+\varphi(x,\cdots,x)}.$$

This implies that

$$d(f, Jf) \leq \frac{L}{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix}}.$$

By Theorem 2.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A\left(\frac{x}{n-m+1}\right) = \frac{A(x)}{n-m+1} \tag{3.4}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that A is a unique mapping satisfying (3.4) such that there exists $\mu \in (0, \infty)$ satisfying $N(f(x) - A(x), \mu t) \geq \frac{t}{t+\varphi(x,\cdots,x)}$, for all $x \in X$ and $t > 0$.

(2) $d(J^p f, A) \to 0$ as $p \to \infty$. This implies the equality

$$N\text{-}\lim_{p\to\infty} \frac{f\left(\frac{x}{(n-m+1)^p}\right)}{(n-m+1)^{-p}} = A(x) \tag{3.5}$$

for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{L}{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix} - (n-m+1)\begin{pmatrix} n \\ m \end{pmatrix} L}$$

This implies that the inequality (3.2) holds. Furthermore, it follows from (3.1) and (3.5) that

$$\begin{aligned}
&N\left(\sum_{\substack{1\leq i_1<\cdots<i_m\leq n\\ 1\leq k_l(\neq i_j,\forall j\in\{1,\cdots,m\})\leq n}} A\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) - \frac{(n-m+1)}{n}\begin{pmatrix} n \\ m \end{pmatrix}\sum_{i=1}^n A(x_i), t\right)\\
&= N - \lim_{p\to\infty}\Bigg((n-m+1)^p \sum_{\substack{1\leq i_1<\cdots<i_m\leq n\\ 1\leq k_l(\neq i_j,\forall j\in\{1,\cdots,m\})\leq n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m(n-m+1)^p} + \sum_{l=1}^{n-m}\frac{x_{k_l}}{(n-m+1)^p}\right)\\
&-\frac{(n-m+1)^{p+1}}{n}\begin{pmatrix} n \\ m \end{pmatrix}\sum_{i=1}^n f\left(\frac{x_i}{(n-m+1)^p}\right), t\Bigg)\\
&\geq \lim_{p\to\infty}\frac{\frac{t}{(n-m+1)^p}}{\frac{t}{(n-m+1)^p} + \varphi\left(\frac{x_1}{(n-m+1)^p}, \cdots, \frac{x_n}{(n-m+1)^p}\right)}\\
&\geq \lim_{p\to\infty}\frac{\frac{t}{(n-m+1)^p}}{\frac{t}{(n-m+1)^p} + \frac{L^n\varphi(x_1,\cdots,x_n)}{(n-m+1)^p}} \to 1
\end{aligned}$$

for all $x_1, \cdots, x_n \in X$, $t > 0$. Hence

$$\sum_{\substack{1\leq i_1<\cdots<i_m\leq n\\ 1\leq k_l(\neq i_j,\forall j\in\{1,\cdots,m\})\leq n}} A\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) = \frac{(n-m+1)}{n}\begin{pmatrix} n \\ m \end{pmatrix}\sum_{i=1}^n A(x_i)$$

for all $x_1, \cdots, x_n \in X$ and therefore A satisfies (1.1). So the mapping $A : X \to Y$ is an additive, as desired. This completes the proof. □

Corollary 3.2. Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|.\|$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying the following inequality

$$\begin{aligned}
&N\left(\sum_{\substack{1\leq i_1<\cdots<i_m\leq n\\ 1\leq k_l(\neq i_j,\forall j\in\{1,\cdots,m\})\leq n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) - \frac{(n-m+1)}{n}\begin{pmatrix} n \\ m \end{pmatrix}\sum_{i=1}^n f(x_i), t\right)\\
&\geq \frac{t}{t + \theta\left(\sum_{i=1}^n \|x_i\|^p\right)}
\end{aligned} \tag{3.6}$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Then, the limit $A(x) := N\text{-}\lim_{p\to\infty} \frac{f\left(\frac{x}{(n-m+1)^p}\right)}{(n-m+1)^{-p}}$ exists for each $x \in X$ and defines a unique $(m, n)-$Cauchy-Jensen additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix}\Big[(n-m+1)^p - (n-m+1)\Big]t}{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix}\Big[(n-m+1)^p - (n-m+1)\Big]t + n(n-m+1)\theta\|x\|^p}$$

for all $x \in X$ and $t > 0$.

Proof . The proof follows from Theorem 3.1 by taking $\varphi(x_1, \cdots, x_n) := \theta\left(\sum_{i=1}^n \|x_i\|^p\right)$ for all $x_1, \cdots, x_n \in X$. Then we can choose $L = (n-m+1)^{1-p}$ and we get the desired result. □

Theorem 3.3. Let $\varphi : X^n \to [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\varphi(x_1, \cdots, x_n) \leq (n-m+1)L\varphi\left(\frac{x_1}{n-m+1}, \cdots, \frac{x_n}{n-m+1}\right)$$

for all $x_1, x_2, \cdots, x_n \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (3.1).Then, the limit $A(x) := N\text{-}\lim_{p\to\infty} \frac{f((n-m+1)^p x)}{(n-m+1)^p}$ exists for each $x \in X$ and defines a unique $(m, n)-$Cauchy-Jensen additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix}(1-L)t}{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix}(1-L)t + \varphi(x, \cdots, x)} \tag{3.7}$$

for all $x \in X$ and all $t > 0$.

Proof . Let (S, d) be the generalized metric space defined as in the proof of Theorem 3.1. Consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{g((n-m+1)x)}{n-m+1}$, for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then $N(g(x) - h(x), \epsilon t) \geq \frac{t}{t+\varphi(x,\cdots,x)}$, for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned}
N(Jg(x) - Jh(x), L\epsilon t) &= N\left(\frac{g((n-m+1)x)}{n-m+1} - \frac{h((n-m+1)x)}{n-m+1}, L\epsilon t\right) \\
&= N\Big(g((n-m+1)x) - h((n-m+1)x), (n-m+1)L\epsilon t\Big) \\
&\geq \frac{(n-m+1)Lt}{(n-m+1)Lt + \varphi((n-m+1)x, , \cdots, (n-m+1)x)} \\
&\geq \frac{(n-m+1)Lt}{(n-m+1)Lt + (n-m+1)L\varphi(x, , \cdots, x)} \\
&= \frac{t}{t + \varphi(x, \cdots, x)}
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in S$. It follows from (3.3) that

$$N\left(f(x) - \frac{f((n-m+1)x)}{n-m+1}, \frac{t}{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix}}\right) \geq \frac{t}{t + \varphi(x, \cdots, x)} \tag{3.8}$$

for all $x \in X$ and $t > 0$. So

$$d(f, Jf) \leq \frac{1}{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix}}.$$

By Theorem 2.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$(n-m+1)A(x) = A((n-m+1)x) \tag{3.9}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that A is a unique mapping satisfying (3.9) such that there exists $\mu \in (0, \infty)$ satisfying $N(f(x) - A(x), \mu t) \geq \frac{t}{t+\varphi(x,\cdots,x)}$, for all $x \in X$ and $t > 0$.

(2) $d(J^p f, A) \to 0$ as $p \to \infty$. This implies the equality

$$A(x) = N\text{-}\lim_{p\to\infty} \frac{f((n-m+1)^p x)}{(n-m+1)^p}$$

for all $x \in X$.

(3) $d(f, A) \le \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \le \frac{1}{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix} - (n-m+1)\begin{pmatrix} n \\ m \end{pmatrix} L}.$$

This implies that the inequality (3.7) holds. The rest of the proof is similar to that of the proof of Theorem 3.1. □

Corollary 3.4. Let $\theta \ge 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|.\|$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (3.6). Then, the limit

$$A(x) := N\text{-}\lim_{p\to\infty} \frac{f((n-m+1)^p x)}{(n-m+1)^p}$$

exists for each $x \in X$ and defines a unique $(m, n)-$Cauchy-Jensen additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \ge \frac{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix}\Big[(n-m+1) - (n-m+1)^p\Big]t}{(n-m+1)\begin{pmatrix} n \\ m \end{pmatrix}\Big[(n-m+1) - (n-m+1)^p\Big]t + n(n-m+1)\theta\|x\|^p}$$

for all $x \in X$.

Proof . The proof follows from Theorem 3.2 by taking $\varphi(x_1, \cdots, x_n) := \theta\left(\sum_{i=1}^n \|x_i\|^p\right)$ for all $x_1, \cdots, x_n \in X$. Then we can choose $L = (n-m+1)^{p-1}$ and we get the desired result. □

4 Fuzzy stability of $(m, n)-$Cauchy-Jensen functional equation (1.1): a direct method

In this section, using direct method, we prove the Ulam-Hyers-Rassias stability of functional equation (1.1) in fuzzy Banach spaces. Throughout this section, we assume that X is a linear space, (Y, N) is a fuzzy Banach space and (Z, N') is a fuzzy normed spaces. Moreover, we assume that $N(x, .)$ is a left continuous function on $\mathbb{R}$.

Theorem 4.1. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\begin{aligned} &N\left(\sum_{\substack{1\le i_1<\cdots<i_m\le n \\ 1\le k_l(\neq i_j, \forall j\in\{1,\cdots,m\})\le n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) - \frac{(n-m+1)}{n}\begin{pmatrix} n \\ m \end{pmatrix}\sum_{i=1}^n f(x_i), t\right) \\ &\ge N'(\varphi(x_1, \ldots, x_n), t) \end{aligned} \tag{4.1}$$

for all $x_1, \ldots, x_n \in X$, $t > 0$ and $\varphi : X^n \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < \frac{1}{n-m+1}$ such that

$$N'\left(\varphi\left(\frac{x_1}{n-m+1}, \ldots, \frac{x_n}{n-m+1}\right), t\right) \ge N'\left(\varphi(x_1, \ldots, x_n), \frac{t}{|r|}\right), \tag{4.2}$$

for all $x_1, \ldots, x_n \in X$ and all $t > 0$. Then there exists a unique $(m, n)-$Cauchy-Jensen additive mapping $A : X \to Y$ satisfying (1.1) and the inequality

$$N(f(x) - A(x), t) \ge N'\left(\frac{|r|\varphi(x, \ldots, x)}{\begin{pmatrix} n \\ m \end{pmatrix}(1 - (n-m+1)|r|)}, t\right) \tag{4.3}$$

for all $x \in X$ and all $t > 0$.

Proof . It follows from (4.2) that

$$N'\left(\varphi\left(\frac{x_1}{(n-m+1)^j},\dots,\frac{x_n}{(n-m+1)^j}\right),t\right)\geq N'\left(\varphi(x_1,\dots,x_n),\frac{t}{|r|^j}\right)$$

for all $x_1,\cdots,x_n\in X$ and all $t>0$. Substituting $(x_1,\cdots,x_n)$ by $(x,\cdots,x)$ in (4.1), we obtain

$$N\left(\frac{f\left(\frac{x}{n-m+1}\right)}{(n-m+1)^{-1}}-f(x),\frac{t}{\begin{pmatrix}n\\m\end{pmatrix}}\right)\geq N'\left(\varphi\left(\frac{x}{n-m+1},\dots,\frac{x}{n-m+1}\right),t\right)$$

for all $x\in X$ and all $t>0$. Replacing x by $\frac{x}{(n-m+1)^j}$ in the above inequality, we have

$$\begin{aligned}&N\left(\frac{f\left(\frac{x}{(n-m+1)^{j+1}}\right)}{(n-m+1)^{-j-1}}-\frac{f\left(\frac{x}{(n-m+1)^{j}}\right)}{(n-m+1)^{-j}},\frac{(n-m+1)^jt}{\begin{pmatrix}n\\m\end{pmatrix}}\right)\\&\geq N'\left(\varphi\left(\frac{x}{(n-m+1)^{j+1}},\cdots,\frac{x}{(n-m+1)^{j+1}}\right),t\right)\\&\geq N'\left(\varphi(x,\cdots,x),\frac{t}{|r|^{j+1}}\right)\end{aligned}\tag{4.4}$$

for all $x\in X$, all $t>0$ and any integer $j\geq 0$. So,

$$\begin{aligned}&N\left(f(x)-\frac{f\left(\frac{x}{(n-m+1)^p}\right)}{(n-m+1)^{-p}},\frac{\sum_{j=0}^{p-1}(n-m+1)^j|r|^{j+1}t}{\begin{pmatrix}n\\m\end{pmatrix}}\right)\\&=N\left(\sum_{j=0}^{p-1}\left[\frac{f\left(\frac{x}{(n-m+1)^{j+1}}\right)}{(n-m+1)^{-j-1}}-\frac{f\left(\frac{x}{(n-m+1)^{j}}\right)}{(n-m+1)^{-j}}\right],\frac{\sum_{j=0}^{p-1}(n-m+1)^j|r|^{j+1}t}{\begin{pmatrix}n\\m\end{pmatrix}}\right)\\&\geq\min_{0\leq j\leq p-1}\left\{N\left(\frac{f\left(\frac{x}{(n-m+1)^{j+1}}\right)}{(n-m+1)^{-j-1}}-\frac{f\left(\frac{x}{(n-m+1)^{j}}\right)}{(n-m+1)^{-j}},\frac{(n-m+1)^j|r|^{j+1}t}{\begin{pmatrix}n\\m\end{pmatrix}}\right)\right\}\\&\geq N'(\varphi(x,\cdots,x),t)\end{aligned}\tag{4.5}$$

which yields

$$\begin{aligned}&N\left(\frac{f\left(\frac{x}{(n-m+1)^{p+q}}\right)}{(n-m+1)^{-p-q}}-\frac{f\left(\frac{x}{(n-m+1)^{q}}\right)}{(n-m+1)^{-q}},\frac{\sum_{j=0}^{p-1}(n-m+1)^{j+q}|r|^{j+1}t}{\begin{pmatrix}n\\m\end{pmatrix}}\right)\\&\geq N'\left(\varphi\left(\frac{x}{(n-m+1)^{q}},\cdots,\frac{x}{(n-m+1)^{q}}\right),t\right)\\&\geq N'\left(\varphi(x,\cdots,x),\frac{t}{|r|^{q}}\right)\end{aligned}\tag{4.6}$$

for all $x\in X$, $t>0$ and any integers $p>0$, $q\geq 0$. So

$$N\left(\frac{f\left(\frac{x}{(n-m+1)^{p+q}}\right)}{(n-m+1)^{-p-q}}-\frac{f\left(\frac{x}{(n-m+1)^{q}}\right)}{(n-m+1)^{-q}},\frac{\sum_{j=0}^{p-1}(n-m+1)^{j+q}|r|^{j+q+1}t}{\begin{pmatrix}n\\m\end{pmatrix}}\right)\geq N'\left(\varphi(x,\cdots,x),t\right)$$

for all $x \in X$, $t > 0$ and any integers $p > 0$, $q \geq 0$. Hence one obtains

$$N\left(\frac{f\left(\frac{x}{(n-m+1)^{p+q}}\right)}{(n-m+1)^{-p-q}} - \frac{f\left(\frac{x}{(n-m+1)^{q}}\right)}{(n-m+1)^{-q}}, t\right) \geq N'\left(\varphi(x,\cdots,x), \frac{\begin{pmatrix} n \\ m \end{pmatrix} t}{\sum_{j=0}^{p-1}(n-m+1)^{j+q}|r|^{j+q+1}}\right)$$

for all $x \in X$, $t > 0$ and any integers $p > 0$, $q \geq 0$. Since, the series $\sum_{j=0}^{+\infty}(n-m+1)^j|r|^{j+1}$ is convergent series, we see by taking the limit $q \to \infty$ in the last inequality that the sequence $\left\{\frac{f\left(\frac{x}{(n-m+1)^p}\right)}{(n-m+1)^{-p}}\right\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N) and so it converges in Y. Therefore a mapping $A : X \to Y$ defined by $A(x) := N - \lim_{p\to\infty} \frac{f\left(\frac{x}{(n-m+1)^p}\right)}{(n-m+1)^{-p}}$ is well defined for all $x \in X$. It means that

$$\lim_{p\to\infty} N\left(A(x) - \frac{f\left(\frac{x}{(n-m+1)^p}\right)}{(n-m+1)^{-p}}, t\right) = 1 \tag{4.7}$$

for all $x \in X$ and all $t > 0$. In addition, it follows from (4.5) that

$$N\left(f(x) - \frac{f\left(\frac{x}{(n-m+1)^p}\right)}{(n-m+1)^{-p}}, t\right) \geq N'\left(\varphi(x,\cdots,x), \frac{\begin{pmatrix} n \\ m \end{pmatrix} t}{\sum_{j=0}^{p-1}(n-m+1)^{j}|r|^{j+1}}\right)$$

for all $x \in X$ and all $t > 0$. So

$$\begin{aligned}
&N(f(x) - A(x), t) \\
&\geq \min\left\{N\left(f(x) - \frac{f\left(\frac{x}{(n-m+1)^p}\right)}{(n-m+1)^{-p}}, (1-\epsilon)t\right), N\left(A(x) - \frac{f\left(\frac{x}{(n-m+1)^p}\right)}{(n-m+1)^{-p}}, \epsilon t\right)\right\} \\
&\geq N'\left(\varphi(x,\cdots,x), \frac{\begin{pmatrix} n \\ m \end{pmatrix} \epsilon t}{\sum_{j=0}^{p-1}(n-m+1)^{j}|r|^{j+1}}\right) \geq N'\left(\varphi(x,\cdots,x), \frac{\begin{pmatrix} n \\ m \end{pmatrix} \epsilon(1-(n-m+1)|r|)t}{|r|}\right)
\end{aligned}$$

for sufficiently large n and for all $x \in X$, $t > 0$ and ϵ with $0 < \epsilon < 1$. Since ϵ is arbitrary and N' is left continuous, we obtain

$$N(f(x) - A(x), t) \geq N'\left(\varphi(x,\cdots,x), \frac{\begin{pmatrix} n \\ m \end{pmatrix}(1-(n-m+1)|r|)t}{|r|}\right),$$

for all $x \in X$ and $t > 0$. It follows from (4.1) that

$$\begin{aligned}
&N\Bigg((n-m+1)^p \sum_{\substack{1\leq i_1<\cdots<i_m\leq n \\ 1\leq k_l(\neq i_j, \forall j\in\{1,\cdots,m\})\leq n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m(n-m+1)^p} + \sum_{l=1}^{n-m}\frac{x_{k_l}}{(n-m+1)^p}\right) \\
&-\frac{(n-m+1)^{p+1}}{n}\begin{pmatrix} n \\ m \end{pmatrix}\sum_{i=1}^n f\left(\frac{x_i}{(n-m+1)^p}\right), t\Bigg) \\
&\geq N'\left(\varphi\left(\frac{x_1}{(n-m+1)^p}, \cdots, \frac{x_n}{(n-m+1)^p}\right), \frac{t}{(n-m+1)^p}\right) \\
&\geq N'\left(\varphi(x_1,\cdots,x_n), \frac{t}{(n-m+1)^p|r|^p}\right)
\end{aligned}$$

for all $x_1, \cdots, x_n \in X$, $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{p\to\infty} N'\left(\varphi(x_1, \cdots, x_n), \frac{t}{(n-m+1)^p|r|^p}\right) = 1$ and so

$$\lim_{p\to+\infty} N\Bigg((n-m+1)^p \sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l(\neq i_j, \forall j\in\{1,\cdots,m\})\le n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m(n-m+1)^p} + \sum_{l=1}^{n-m} \frac{x_{k_l}}{(n-m+1)^p}\right) - \frac{(n-m+1)^{p+1}}{n}\left(\begin{array}{c} n \\ m \end{array}\right)\sum_{i=1}^n f\left(\frac{x_i}{(n-m+1)^p}\right) - A(x), t\Bigg) = 1$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Therefore, we obtain in view of (4.7)

$$\begin{aligned}
&N\left(\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l(\neq i_j, \forall j\in\{1,\cdots,m\})\le n}} A\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) - \frac{(n-m+1)}{n}\left(\begin{array}{c} n \\ m \end{array}\right)\sum_{i=1}^n A(x_i), t\right)\\
&\ge \min\Bigg\{N\Bigg(\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l(\neq i_j, \forall j\in\{1,\cdots,m\})\le n}} A\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) - \frac{(n-m+1)}{n}\left(\begin{array}{c} n \\ m \end{array}\right)\sum_{i=1}^n A(x_i)\\
&-(n-m+1)^p \sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l(\neq i_j, \forall j\in\{1,\cdots,m\})\le n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m(n-m+1)^p} + \sum_{l=1}^{n-m} \frac{x_{k_l}}{(n-m+1)^p}\right)\\
&-\frac{(n-m+1)^{p+1}}{n}\left(\begin{array}{c} n \\ m \end{array}\right)\sum_{i=1}^n f\left(\frac{x_i}{(n-m+1)^p}\right), \frac{t}{2}\Bigg), N\Bigg((n-m+1)^p\\
&\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l(\neq i_j, \forall j\in\{1,\cdots,m\})\le n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m(n-m+1)^p} + \sum_{l=1}^{n-m} \frac{x_{k_l}}{(n-m+1)^p}\right)\\
&-\frac{(n-m+1)^{p+1}}{n}\left(\begin{array}{c} n \\ m \end{array}\right)\sum_{i=1}^n f\left(\frac{x_i}{(n-m+1)^p}\right), \frac{t}{2}\Bigg)\Bigg\}\\
&= N\Bigg((n-m+1)^p \sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l(\neq i_j, \forall j\in\{1,\ldots,m\})\le n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m(n-m+1)^p} + \sum_{l=1}^{n-m} \frac{x_{k_l}}{(n-m+1)^p}\right)\\
&-\frac{(n-m+1)^{p+1}}{n}\left(\begin{array}{c} n \\ m \end{array}\right)\sum_{i=1}^n f\left(\frac{x_i}{(n-m+1)^p}\right), \frac{t}{2}\Bigg) \quad \text{(for sufficiently large } p\text{)}\\
&\ge N'\left(\varphi(x_1, \ldots, x_n), \frac{t}{2(n-m+1)^p|r|^p}\right)\\
&\to 1 \quad \text{as } p \to \infty
\end{aligned}$$

which implies

$$\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l(\neq i_j, \forall j\in\{1,\cdots,m\})\le n}} A\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) = \frac{(n-m+1)}{n}\left(\begin{array}{c} n \\ m \end{array}\right)\sum_{i=1}^n A(x_i)$$

for all $x_1, \cdots, x_n \in X$. Thus $A : X \to Y$ is a mapping satisfying the equation (1.1) and the inequality (4.3). To prove the uniqueness, let there is another mapping $L : X \to Y$ which satisfies the inequality (4.3). Since $L\left(\frac{x}{(m+n-1)^p}\right) =$

$\frac{L(x)}{(m+n-1)^p}$ and $A\left(\frac{x}{(m+n-1)^p}\right) = \frac{A(x)}{(m+n-1)^p}$, for all $x \in X$, we have

$$\begin{aligned}
&N(A(x)-L(x),t)\\
&= N\left(\frac{A\left(\frac{x}{(m+n-1)^p}\right)}{(m+n-1)^{-p}} - \frac{L\left(\frac{x}{(m+n-1)^p}\right)}{(m+n-1)^{-p}}, t\right)\\
&\geq \min\left\{N\left(\frac{A\left(\frac{x}{(m+n-1)^p}\right)}{(m+n-1)^{-p}} - \frac{f\left(\frac{x}{(m+n-1)^p}\right)}{(m+n-1)^{-p}}, \frac{t}{2}\right), N\left(\frac{f\left(\frac{x}{(m+n-1)^p}\right)}{(m+n-1)^{-p}} - \frac{L\left(\frac{x}{(m+n-1)^p}\right)}{(m+n-1)^{-p}}, \frac{t}{2}\right)\right\}\\
&\geq N'\left(\varphi\left(\frac{x}{(m+n-1)^p}, \cdots, \frac{x}{(m+n-1)^p}\right), \frac{\begin{pmatrix} n \\ m \end{pmatrix}(1-(n-m+1)|r|)t}{2|r|(n-m+1)^p}\right)\\
&\geq N\left(\varphi(x,\cdots,x), \frac{\begin{pmatrix} n \\ m \end{pmatrix}(1-(n-m+1)|r|)t}{2|r|^{p+1}(n-m+1)^p}\right) \to 1 \text{ as } n \to \infty
\end{aligned}$$

for all $t > 0$. Therefore $A(x) = L(x)$ for all $x \in X$. This completes the proof. □

Corollary 4.2. Let X be a normed spaces and that $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exists real numbers $\theta \geq 0$ and $p > 1$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following inequality

$$\begin{aligned}
&N\left(\sum_{\substack{1\leq i_1<\cdots<i_m\leq n \\ 1\leq k_l(\neq i_j, \forall j\in\{1,\cdots,m\})\leq n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) - \frac{(n-m+1)}{n}\begin{pmatrix} n \\ m \end{pmatrix}\sum_{i=1}^n f(x_i), t\right)\\
&\geq N'\left(\theta\left(\sum_{i=1}^n \|x_i\|^p\right), t\right),
\end{aligned} \tag{4.8}$$

for all $x_1, \ldots, x_n \in X$ and $t > 0$. Then there is a unique $(m,n)-$Cauchy-Jensen additive mapping $A : X \to Y$ that satisfying (1.1) and the inequality

$$N(f(x) - A(x), t) \geq N'\left(\frac{n\theta\|x\|^p}{\begin{pmatrix} n \\ m \end{pmatrix}\left[(n-m+1)^p - (n-m+1)\right]}, t\right).$$

Proof . Let $\varphi(x_1, \cdots, x_n) := \theta\left(\sum_{i=1}^n \|x_i\|^p\right)$ and $|r| = (n-m+1)^{-p}$. Apply Theorem 4.1, we get desired results. □

Theorem 4.3. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (4.1) and $\varphi : X^n \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < n - m + 1$ such that

$$N'\left(\varphi(x_1, \ldots, x_n), |r|t\right) \geq N'\left(\varphi\left(\frac{x_1}{n-m+1}, \ldots, \frac{x_n}{n-m+1}\right), t\right) \tag{4.9}$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Then there exists a unique $(m,n)-$Cauchy-Jensen additive mapping $A : X \to Y$ that satisfying (1.1) and the following inequality

$$N(f(x) - A(x), t) \geq N'\left(\varphi(x, \ldots, x), \frac{(n-m+1-|r|)t}{\begin{pmatrix} n \\ m \end{pmatrix}}\right) \tag{4.10}$$

for all $x \in X$ and all $t > 0$.

Proof . It follows from (4.4) that

$$N\left(f(x)-\frac{f((n-m+1)x)}{n-m+1},\frac{t}{(n-m+1)\begin{pmatrix}n\\m\end{pmatrix}}\right)\geq N'(\varphi(x,\dots,x),t) \tag{4.11}$$

for all $x\in X$ and all $t>0$. Replacing x by $(n-m+1)^p x$ in (4.11), we obtain

$$\begin{aligned}&N\left(\frac{f((n-m+1)^{p+1}x)}{(n-m+1)^{p+1}}-\frac{f((n-m+1)^{p}x)}{(n-m+1)^{p}},\frac{t}{(n-m+1)^{p+1}\begin{pmatrix}n\\m\end{pmatrix}}\right)\\&\geq N'(\varphi((n-m+1)^p x,\dots,(n-m+1)^p x),t)\\&\geq N'\left(\varphi(x,\dots,x),\frac{t}{|r|^p}\right).\end{aligned} \tag{4.12}$$

So,

$$N\left(\frac{f((n-m+1)^{p+1}x)}{(n-m+1)^{p+1}}-\frac{f((n-m+1)^{p}x)}{(n-m+1)^{p}},\frac{|r|^p t}{(n-m+1)^{p+1}\begin{pmatrix}n\\m\end{pmatrix}}\right)\geq N'(\varphi(x,\dots,x),t)$$

for all $x\in X$ and all $t>0$. Proceeding as in the proof of Theorem 4.1, we obtain that

$$N\left(f(x)-\frac{f((n-m+1)^{p}x)}{(n-m+1)^{p}},\sum_{j=0}^{p-1}\frac{|r|^j t}{(n-m+1)^{j+1}\begin{pmatrix}n\\m\end{pmatrix}}\right)\geq N'(\varphi(x,\dots,x),t)$$

for all $x\in X$, all $t>0$ and any integer $n>0$. So,

$$\begin{aligned}N\left(f(x)-\frac{f((n-m+1)^{p}x)}{(n-m+1)^{p}},t\right) &\geq N'\left(\varphi(x,\dots,x),\frac{t}{\begin{pmatrix}n\\m\end{pmatrix}\sum_{j=0}^{p-1}\frac{|r|^j}{(n-m+1)^{j+1}}}\right)\\&\geq N'\left(\varphi(x,\dots,x),\frac{(n-m+1-|r|)t}{\begin{pmatrix}n\\m\end{pmatrix}}\right).\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. Let X be a normed spaces and that $(\mathbb{R},N')$ a fuzzy Banach space. Assume that there exists real number $\theta\geq 0$ and $0<p<1$ such that a mapping $f:X\to Y$ with $f(0)=0$ satisfies (4.8). Then there is a unique $(m,n)-$Cauchy-Jensen additive mapping $A:X\to Y$ that satisfying (1.1) and the inequality

$$N(f(x)-A(x),t)\geq N'\left(n\theta\|x\|^p,\frac{(n-m+1-(n-m+1)^p)t}{\begin{pmatrix}n\\m\end{pmatrix}}\right).$$

Proof . Let $\varphi(x_1,\dots,x_n):=\theta\left(\sum_{i=1}^n\|x_i\|^p\right)$ and $|r|=(n-m+1)^p$. Apply Theorem 4.3, we get desired results. □

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