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Fixed point theorems for (ϕ, F)-contraction in b-rectangular asymmetric metric spaces

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Abstract

In this work, we introduce the concept of b-rectangular asymmetric metric space, which is not necessarily Hausdorff and which generalizes the concept of metric space, rectangular metric space, rectangular asymmetric metric space and b-rectangular metric space. Also, we introduce the notion of (ϕ, F)-contraction and establish some new fixed point theorems for mappings in the setting of complete b-generalized asymmetric metric spaces. Our results generalize, improve and extend the corresponding results of Banach and Wardoski. Moreover, an illustrative example is presented to support the obtained results.

Keywords: (ϕ, F)-contraction, fixed point, generalized asymmetric metric space
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1 Introduction

It is widely known that the Banach contraction principle [3] is the first fixed point theorem and one of the most powerful and versatile results in the field of functional analysis. Various generalizations of it appeared in the literature, much mathematics steadied many interesting extensions and generalizations [2, 8, 12, 14, 20, 25] and the recent works of Wardowski [25, 26, 27].

A well known, several generalizations of standard metric spaces have appeared. In particular, asymmetric metric spaces were introduced by Wilson [24] and then studied by many authors [1, 16, 18, 21].

In 2000 , for the first time rectangular metric spaces were introduced by Branciari [4], in such a way that triangle inequality is replaced by the quadrilateral inequality

$$d(x, y) \leq d(x, z) + d(z, u) + d(u, y),$$

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for all pairwise distinct points x, y, z and u. Any metric space is a generalized metric space but in general, generalized metric space might not be a metric space. Various fixed point results were established on such spaces [6, 7, 13, 15, 23] and references therein.

Combining conditions used to define b-metric and rectangular metric spaces, George et al. [5] announced the notions of b-rectangular metric space. Various fixed point results were established on such spaces [10, 11, 22] and references therein.

Combining conditions used to define rectangular metric spaces and asymmetric metric spaces, Piri et al. [19] announced the notion of rectangular asymmetric metric space and formulated some first fixed point theorems for θ-contraction mapping in generalized asymmetric metric space.

In this paper, we introduce the concept of b-rectangular asymmetric metric space, which generalizes the concept of metric space, rectangular metric space, rectangular asymmetric metric space and $b-$ rectangular metric space. Also, we introduce the notion of (ϕ, F)-contraction and establish some fixed point results for mappings in the setting of complete b-generalized asymmetric metric spaces. Our results generalize, improve and extend the corresponding results. Moreover, an illustrative example is presented to support the obtained results.

2 Preliminaries

In the following, we recollect some definitions which will be useful in our main results.

Definition 2.1. [4] Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+$be a function such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all distinct points $x, y \in X$;
3. $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).

Then (X, d) is called a rectangular metric space.

Definition 2.2. [19] Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+$be a function such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).

Then (X, d) is called a rectangular asymmetric metric space.

Definition 2.3. [19] Let (X, d) be a generalized asymmetric metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$.

(i) We say that $\{x_n\}_{n\in\mathbb{N}}$ is forward (backward) convergent to x if

$$\lim_{n\to+\infty} d(x, x_n) = 0 \quad \left(\lim_{n\to+\infty} d(x_n, x) = 0\right).$$

(ii) We say that $\{x_n\}_{n\in\mathbb{N}}$ is forward (backward) Cauchy if

$$\lim_{n>m\to+\infty} d(x_n, x_m) = 0 \quad \left(\lim_{m>n\to+\infty} d(x_m, x_n) = 0\right).$$

Example 2.4. [9] Let $X = A \cup B$, where $A = \{0, 2\}$ and $B = \left\{\frac{1}{n}, n \in \mathbb{N}^*\right\}$, and $d : X \times X \to [0, +\infty[$ be defined by

$$\begin{cases} d(0,2) = d(2,0) = 1 \\ d\left(\frac{1}{n}, 0\right) = \frac{1}{n}, d\left(0, \frac{1}{n}\right) = 1 \\ d\left(\frac{1}{n}, 2\right) = 1, d\left(2, \frac{1}{n}\right) = \frac{1}{n} \\ d\left(\frac{1}{n}, \frac{1}{m}\right) = d\left(\frac{1}{m}, \frac{1}{n}\right) = 1. \end{cases}$$

for all $n, m \in \mathbb{N}^*, n \neq m$. Then (X, d) is a rectangular asymmetric metric space. However, we have the following:

1) (X,d) is not a metric space, since $d\left(\frac{1}{n},0\right) \neq d\left(0,\frac{1}{n}\right)$, for all $n>1$.

2) (X,d) is not a asymmetric metric space, since $d(2,0)=1>\frac{1}{2}=d\left(2,\frac{1}{4}\right)+d\left(\frac{1}{4},0\right)$.

3) (X,d) is not a rectangular metric space, since $d\left(\frac{1}{n},2\right) \neq d\left(2,\frac{1}{n}\right)$, for all $n>1$.

Remark 2.5. [9] Let (X,d) be as in Example 2.4, and $\left\{\frac{1}{n}\right\}_{n\in N^*}$ be a sequence in X. However, we have the following:

i) $\lim_{n\to+\infty} d\left(\frac{1}{n},0\right)=0, \lim_{n\to+\infty} d\left(\frac{1}{n},2\right)=1$ and $\lim_{n\to+\infty} d\left(0,\frac{1}{n}\right)=1, \lim_{n\to+\infty} d\left(2,\frac{1}{n}\right)=0$. Then the sequence $\left\{\frac{1}{n}\right\}$ is forward convergent to 2 and backward convergent to 0 . So the limit is not unique.

ii) $\lim_{n\to+\infty} d\left(\frac{1}{m},\frac{1}{n}\right) = \lim_{n\to+\infty} d\left(\frac{1}{m},\frac{1}{n}\right) = 1$. So forward (backward) convergence does not imply forward (backward) Cauchy.

Lemma 2.6. [19] Let (X,d) be a generalized asymmetric metric space and $\{x_n\}_n$ be a forwand (or backward) Cauchy sequence with pairwise disjoint elements in X. If $\{x_n\}_n$ is forward convergent to $x\in X$ and backward convergent to $y\in X$, then $x=y$.

Definition 2.7. [22] Let X be a nonempty set and $d: X\times X \to \mathbb{R}^+$be a function such that for all $x,y\in X$ and for all distinct points $u,v\in X$, each of them different from x and y, one has

i) $d(x,y)=0$ if and only if $x=y$;
ii) $d(x,y)=d(y,x)$;
iii) there exists a real number $s\geq 1$ such that $d(x,y)\leq s(d(x,u)+d(u,v)+d(v,y))$ (b-rectangular inequality).

Then (X,d) is called a b-rectangular metric space.

Lemma 2.8. [22] Let (X,d) be a b-rectangular metric space.

(a) Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n\to x$ and $y_n\to y$ as $n\to\infty$, with $x\neq y, x_n\neq x$ and $y_n\neq y$, for all $n\in\mathbb{N}$. Then we have

$$\frac{1}{s}d(x,y)\leq \lim_{n\to\infty}\inf d\left(x_n,y_n\right)\leq \lim_{n\to\infty}\sup d\left(x_n,y_n\right)\leq sd(x,y).$$

(b) If $y\in X$ and $\{x_n\}$ is a Cauchy sequence in X with $x_n\neq x_m$, for all $m,n\in\mathbb{N}, m\neq n$, converging to $x\neq y$, then

$$\frac{1}{s}d(x,y)\leq \lim_{n\to\infty}\inf d\left(x_n,y\right)\leq \lim_{n\to\infty}\sup d\left(x_n,y\right)\leq sd(x,y)$$

for all $x\in X$.

Lemma 2.9. [11] Let (X,d) be a b-rectangular asymmetric metric space and $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} d\left(x_n,x_{n+1}\right)=\lim_{n\to\infty} d\left(x_n,x_{n+2}\right)=0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon>0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that

$$\begin{gathered}
\varepsilon\leq \lim_{k\to\infty}\inf d\left(x_{m_{(k)}},x_{n_{(k)}}\right)\leq \lim_{k\to\infty}\sup d\left(x_{m_{(k)}},x_{n_{(k)}}\right)\leq s\varepsilon,\\
\varepsilon\leq \lim_{k\to\infty}\inf d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right)\leq \lim_{k\to\infty}\sup d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right)\leq s\varepsilon,\\
\varepsilon\leq \lim_{k\to\infty}\inf d\left(x_{m_{(k)}},x_{n_{(k)+1}}\right)\leq \lim_{k\to\infty}\sup d\left(x_{m_{(k)}},x_{n_{(k)+1}}\right)\leq s\varepsilon,\\
\frac{\varepsilon}{s}\leq \lim_{k\to\infty}\inf d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\leq \lim_{k\to\infty}\sup d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\leq s^2\varepsilon.
\end{gathered}$$

The following definition was introduced by Wardowski [25].

Definition 2.10. [25] Let $\mathcal{F}$ be the family of all functions $F:\mathbb{R}^+\to\mathbb{R}$ such that

(i) F is strictly increasing:
(ii) for each sequence $\{x_n\}_{n\in N}$ of positive numbers

$$\lim_{n\to 0} x_n = 0, \text{ if and only if } \lim_{n\to\infty} F(x_n) = -\infty;$$

(iii) there exists $k \in]0,1[$ such that $\lim_{x\to 0} x^k F(x) = 0$.

Recently, Piri and Kuman [17] extended the result of Wardowski [25] by changing the condition (iii) in Definition 2.9 as follow.

Definition 2.11. [17] Let Γ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that

(i) F is strictly increasing;
(ii) for each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to\infty} x_n = 0, \text{ if and only if } \lim_{n\to\infty} F(x_n) = -\infty;$$

(iii) F is continuous.

The following result introduced by Wardowski [26] will be used to prove our result.

Definition 2.12. [26]. Let $\mathbb{F}$ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ and $\phi :]0,+\infty[\to]0,+\infty[$ satisfy the following.

(i) F is strictly increasing:
(ii) For each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to\infty} x_n = 0, \text{ if and only if } \lim_{n\to\infty} F(x_n) = -\infty;$$

(iii) $\liminf_{s\to\alpha} \phi(s) > 0$ for all $\alpha > 0$;
(iv) There exists $k \in]0,1[$ such that

$$\lim_{x\to 0^+} x^k F(x) = 0.$$

Recently, Kari et al. [15] extended the result of Wardowski [26] by changing the condition (iV) in Definition 2.11 as follow.

Definition 2.13. Let Θ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ and Φ be the family of all functions $\phi :]0,+\infty[\to]0,+\infty[$ satisfying the following.

(i) F is strictly increasing;
(ii) For each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to\infty} x_n = 0, \text{ if and only if } \lim_{n\to\infty} F(x_n) = -\infty;$$

(iii) $\liminf \text{sia}^+ \phi(s) > 0$, for all $\alpha > 0$;
(iv) F is continuous.

Definition 2.14. [26] Let (X,d) be a metric space. A mapping $T : X \to X$ is called a (ϕ, F) contraction on (X,d), if there exist $F \in \mathrm{F}$ and ϕ such that

$$F(d(Tx,Ty)) + \phi(d(x,y)) \le F(d(x,y))$$

for all $x, y \in X$ for which $Tx \neq Ty$.

Theorem 2.15. [26] Let (X,d) be a complete metric space and $T : X \to X$ be an (F,ϕ)-contraction. Then T has a unique fixed point.

3 Main results

In the following, we define the b-rectangular asymmetric metric spaces. Also, we obtain some results in such spaces.

Definition 3.1. Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$:
(ii) there exists a real number $s \geq 1$ such that $d(x, y) \leq s(d(x, u) + d(u, v) + d(v, y))$ (b-rectangular inequality).

Then (X, d) is called a b-rectangular asymmetric metric space.

Example 3.2. Let $X = A \cup B$, where $A = \left\{0, \frac{1}{5}, \frac{1}{9}, \frac{1}{16}\right\}$ and $B = \left[\frac{1}{2}, 1\right]$. Define $d : X \times X \to [0, +\infty[$ as follows:

$$d(x, y) = 0 \Leftrightarrow x = y, \forall x, y \in X$$

and

$$\begin{cases} d\left(0, \dfrac{1}{9}\right) = d\left(\dfrac{1}{5}, \dfrac{1}{16}\right) = 0,1 \\ d\left(0, \dfrac{1}{5}\right) = d\left(\dfrac{1}{5}, 0\right) = d\left(\dfrac{1}{5}, \dfrac{1}{9}\right) = d\left(\dfrac{1}{9}, \dfrac{1}{5}\right) = 0,5 \\ d\left(0, \dfrac{1}{16}\right) = d\left(\dfrac{1}{9}, \dfrac{1}{16}\right) = 0,05 \\ d\left(\dfrac{1}{9}, 0\right) = d\left(\dfrac{1}{16}, \dfrac{1}{5}\right) = 0,04 \\ d\left(\dfrac{1}{16}, 0\right) = d\left(\dfrac{1}{16}, \dfrac{1}{9}\right) = 0.06 \\ d(x, y) = (|x - y|)^2 \text{ otherwise.} \end{cases}$$

Then (X, d) is a b-rectangular asymmetric metric space with coefficient $s = 3$. However we have the following:

1) (X, d) is not a metric space, since $d\left(\frac{1}{5}, \frac{1}{9}\right) = 0.5 > 0.16 = d\left(\frac{1}{5}, \frac{1}{16}\right) + d\left(\frac{1}{16}, \frac{1}{5}\right)$.
2) (X, d) is not a asymmetric metric space, since $d\left(0, \frac{1}{9}\right) = 0.1 \neq 0.4 = d\left(\frac{1}{9}, 0\right)$.
3) (X, d) is not a b-metric space for $s = 3$, since $d\left(\frac{1}{5}, \frac{1}{9}\right) = 0.5 > 0.48 = 3\left[d\left(\frac{1}{5}, \frac{1}{16}\right) + d\left(\frac{1}{16}, \frac{1}{9}\right)\right]$.
4) (X, d) is not a rectangular metric space, since $d\left(\frac{1}{5}, \frac{1}{9}\right) = 0.5 > 0.26 = d\left(\frac{1}{5}, \frac{1}{16}\right) + d\left(\frac{1}{16}, 0\right) + d\left(0, \frac{1}{9}\right)$.

Definition 3.3. Let (X, d) is a b-generalized asymmetric metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X, and $x \in X$.

(i) We say that $\{x_n\}_{n\in\mathbb{N}}$ is b-forward (b-backward) convergent to x if and only if

$$\lim_{n\to+\infty} d(x, x_n) = \lim_{n\to+\infty} d(x_n, x) = 0.$$

(ii) We say that $\{x_n\}_{n\in\mathbb{N}}$ is b-forward (b-backward) Cauchy if

$$\lim_{n>m\to+\infty} d(x_n, x_m) = \lim_{m>n\to+\infty} d(x_m, x_n) = 0.$$

Definition 3.4. Let (X, d) be a b-rectangular asymmetric metric space. X is said to be b-forward (b-backward) complete if every b-forward (b-backward) Cauchy sequence $\{x_n\}_n$ in X is b-forward (b-backward) convergent to $x \in X$.

Definition 3.5. Let (X, d) be a b-generalized asymmetric metric space. X is said to be complete if X is b-forward and b-backward complete.

Lemma 3.6. Let (X, d) be a b-generalized asymmetric metric space and $\{x_n\}_n$ be a b-forward (or b-backward) Cauchy sequence with pairwise disjoint elements in X. If $\{x_n\}_n$ is b-forward convergent to $x \in X$ and b-backward convergent to $y \in X$, then $x = y$.

Proof . Let $\varepsilon > 0$. First assume that $\{x_n\}_n$ is a b-forward Cauchy sequence. Then there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{c}{3s}$ for all $m \geq n \geq n_0$. Since $\{x_n\}_n$ b-forward converges to x, there exists $n_1 \in \mathbb{N}$ such that $d(x_n, x) < \frac{c}{3s}$ for all $n \geq n_1$. Also $\{x_n\}_n$ is b-forward convergent to y and so there exists $n_2 \in \mathbb{N}$ such that $d(y, x_n) < \frac{c}{3s}$, for all $n \geq n_2$. Then for all $n \geq \max\{n_0, n_1, n_2\}$,

$$d(x,y) \leq s\left[d(x, x_n) + d(x_n, x_{n+1})\, d(x_{n+1}, y)\right] < s\left(\frac{\varepsilon}{3s} + \frac{\varepsilon}{3s} + \frac{\varepsilon}{3s}\right) = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $d(x,y) = 0$, which implies $x = y$. When $\{x_n\}_n$ is a b-backward Cauchy sequence, the proof is similar to an earlier state. By a similar method as in Lemma 2.8, we conclude the following. □

Lemma 3.7. Let (X, d) be a b-rectangular asymmetric metric space.

(a) Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, with $x \neq y, x_n \neq x$ and $y_n \neq y$ for all $n \in \mathbb{N}$. Then we have

$$\frac{1}{s} d(x,y) \leq \lim_{n\to\infty} \inf d(x_n, y_n) \leq \lim_{n\to\infty} \sup d(x_n, y_n) \leq s d(x,y).$$

(b) If $y \in X$ and $\{x_n\}$ is a Cauchy sequence in X with $x_n \neq x_m$ for any $m, n \in \mathbb{N}, m \neq n$, converging to $x \neq y$, then

$$\frac{1}{s} d(x,y) \leq \lim_{n\to\infty} \inf d(x_n, y) \leq \lim_{n\to\infty} \sup d(x_n, y) \leq s d(x,y),$$

for all $x \in X$. Now, we present the concept of (F, ϕ)-contraction in a b-rectangular asymmetric metric spaces. Also, we prove some fixed point results for such mapping.

Definition 3.8. Let (X, d) be a rectangular asymmetric metric space and $T : X \to X$ be a mapping. T is said to be a (ϕ, F)-contraction of type $(\Im)$ if there exist $F \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$F\left[s^2 d(Tx, Ty)\right] + \phi[d(x,y)] \leq F[d(x,y)].$$

By a similar way to Lemma 2.9, we introduce the following lemma concerning b-rectangular asymmetric metric space.

Lemma 3.9. Let (X, d) be a b-rectangular asymmetric metric space and $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+2}) = 0 \text{ and } \lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} d(x_{n+2}, x_n) = 0.$$

If $\{x_n\}$ is not a backward Cauchy sequence and is not a forward Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that

$$\begin{aligned}
\varepsilon &\leq \lim_{k\to\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq \lim_{k\to\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq s\varepsilon,\\
\varepsilon &\leq \lim_{k\to\infty} \inf d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq \lim_{k\to\infty} \sup d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq s\varepsilon,\\
\varepsilon &\leq \lim_{k\to\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq \lim_{k\to\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq s\varepsilon,\\
\frac{\varepsilon}{s} &\leq \lim_{k\to\infty} \inf d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq \lim_{k\to\infty} \sup d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq s^2\varepsilon.
\end{aligned}$$

Theorem 3.10. Let (X, d) be a complete b-generalized asymmetric metric space and $T : X \to X$ be a mapping. Suppose that there exist $F \in \mathcal{S}$ and $\phi \in \Phi$ such that for any $x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$F\left[s^2 d(Tx, Ty)\right] + \phi(d(x,y)) \leq F[d(x,y)]. \tag{3.1}$$

Then T has a unique fixed point.

Proof . Let $x_0 \in X$ and define a sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0, \text{ for all } n \in \mathbb{N}.$$

If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$ or $d(x_{n_0+1}, x_{n_0}) = 0$, then the proof is finished. Suppose that $d(x_n, x_{n+1}) > 0$ and $d(x_{n+1}, x_n) > 0$, for all $n \in \mathbb{N}$.

Case 1. $d(x_n, x_{n+1}) > 0$. Letting $x = x_{n-1}$ and $y = x_n$ in 3.1, for all $n \in \mathbb{N}$, we have

$$F[d(x_n, x_{n+1})] + \phi[d(x_{n-1}, x_n)] \leq F\left[s^2 d(x_n, x_{n+1})\right] + \phi[d(x_{n-1}, x_n)] \leq F[d(x_{n-1}, x_n)].$$

Therefore,

$$F(d(x_n, x_{n+1})) < F(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)).$$

Since F is increasing,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{3.2}$$

Repeating this step, we conclude that

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq (F(d(x_{n-1}, x_n))) - \phi[d(x_{n-1}, x_n)] \\ &\leq F(d(x_{n-2}, x_{n-1})) - \phi[d(x_{n-1}, x_n)] - \phi[d(x_{n-2}, x_{n-1})] \\ &\leq \ldots \leq F(d(x_0, x_1)) - \sum_{i=0}^{n} \phi[d(x_i, x_{i+1})]. \end{aligned}$$

Since $\liminf_{s \to \alpha^+} \phi(s) > 0$, we have $\liminf_{n \to \infty} \phi(d(x_{n-1}, x_n)) > 0$. By the definition of the limit, there exist $n_0 \in \mathbb{N}$ and $A > 0$ such that for all $n \geq n_0, \phi(d(x_{n-1}, x_n)) > A$. Thus

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq F(d(x_0, x_1)) - \sum_{i=0}^{n_0-1} \phi(d(x_i, x_{i+1})) - \sum_{i=n_0-1}^{n} \phi(d(x_i, x_{i+1})) \\ &\leq F(d(x_0, x_1)) - \sum_{i=n_0-1}^{n} A \\ &= F(d(x_0, x_1)) - (n - n_0) A \end{aligned}$$

for all $n \geq n_0$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$\lim_{n \to \infty} F(d(x_n, x_{n+1})) \leq \lim_{n \to \infty} [F(d(x_0, x_1)) - (n - n_0) A],$$

that is, $\lim_{n \to \infty} F(d(x_n, x_{n+1})) = -\infty$. By the condition (ii) of Definition 2.13, we conclude that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.$$

Case 2. $d(x_{n+1}, x_n) > 0$. Letting $x = x_n$ and $y = x_{n-1}$ in 3.1, for all $n \in \mathbb{N}$, we have

$$F[d(x_{n+1}, x_n)] + \phi[d(x_n, x_{n-1})] \leq F\left[s^2 d(x_{n+1}, x_n)\right] + \phi[d(x_n, x_{n-1})] \leq F[d(x_n, x_{n-1})].$$

Therefore,

$$F(d(x_{n+1}, x_n)) < F(d(x_n, x_{n-1})) - \phi(d(x_n, x_{n-1})).$$

Since F is increasing,

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}). \tag{3.3}$$

Repeating this step, we conclude that

$$\begin{aligned} F(d(x_{n+1}, x_n)) &\leq (F(d(x_n, x_{n-1}))) - \phi[d(x_{n-1}, x_n)] \\ &\leq F(d(x_{n-1}, x_{n-2})) - \phi[d(x_n, x_{n-1})] - \phi[d(x_{n-1}, x_{n-2})] \\ &\leq \ldots \leq F(d(x_1, x_0)) - \sum_{i=0}^{n} \phi[d(x_{i+1}, x_i)]. \end{aligned}$$

Since $\liminf_{s\to\alpha^+}\phi(s)>0$, we have $\liminf_{n\to\infty}\phi\left(d\left(x_n,x_{n-1}\right)\right)>0$. By the definition of the limit, there exist $n_1\in\mathbb{N}$ and $B>0$ such that for all $n\geq n_1, \phi\left(D\left(x_{n-1},x_n\right)\right)>B$. Thus

$$\begin{aligned} F\left(d\left(x_{n+1},x_n\right)\right) &\leq F\left(d\left(x_1,x_0\right)\right)-\sum_{i=0}^{n_1-1}\phi\left(d\left(x_{i+1},x_i\right)\right)-\sum_{i=n_1-1}^{n}\phi\left(d\left(x_{i+1},x_i\right)\right)\\ &\leq F\left(d\left(x_0,x_1\right)\right)-\sum_{i=n_1-1}^{n}B\\ &=F\left(d\left(x_1,x_0\right)\right)-\left(n-n_1\right)B, \end{aligned}$$

for all $n\geq n_1$. Taking the limit as $n\to\infty$ in the above inequality, we get

$$\lim_{n\to\infty}F\left(d\left(x_{n+1},x_n\right)\right)\leq\lim_{n\to\infty}\left[F\left(d\left(x_1,x_0\right)\right)-\left(n-n_1\right)B\right],$$

that is, $\lim_{n\to\infty}F\left(d\left(x_{n+1},x_n\right)\right)=-\infty$. By the condition (ii) of Definition 2.13, we conclude that

$$\lim_{n\to\infty}d\left(x_{n+1},x_n\right)=0.$$

Next, we shall prove that

$$\lim_{n\to\infty}d\left(x_n,x_{n+2}\right)=0 \text{ and } \lim_{n\to\infty}d\left(x_{n+2},x_n\right)=0.$$

We assume that $x_n\neq x_m$ for all $n,m\in\mathbb{N}, n\neq m$. Indeed, suppose that $x_n=x_m$ for some $n=m+k$ with $k>0$. Then we have $x_{n+1}=Tx_n=Tx_m=x_{m+1}$. By 3.2 and 3.3, we have

$$d\left(x_n,x_{n+1}\right)<d\left(x_{n-1},x_n\right)$$

and

$$d\left(x_{n+1},x_n\right)<d\left(x_n,x_{n-1}\right).$$

Continuing this process, we can that

$$d\left(x_m,x_{m+1}\right)<d\left(x_m,x_{m+1}\right)$$

and

$$d\left(x_{m+1},x_m\right)<d\left(x_{m+1},x_m\right),$$

which is a contradiction. Therefore,

$$d\left(x_m,x_n\right)>0 \text{ and } d\left(x_n,x_m\right)>0$$

for all $n,m\in\mathbb{N}, n\neq m$.

Case 3. $d\left(x_n,x_{n+2}\right)>0$.

Applying 3.1 with $x=x_{n-1}$ and $y=x_{n+1}$, we have

$$F\left[s^2d\left(x_n,x_{n+2}\right)\right]+\phi\left[d\left(x_{n-1},x_{n+1}\right)\right]\leq F\left[d\left(x_n,x_{n+2}\right)\right]+\phi\left[d\left(x_{n-1},x_{n+1}\right)\right]\leq F\left[d\left(x_{n+1},x_{n-1}\right)\right].$$

Therefore,

$$F\left(d\left(x_n,x_{n+2}\right)\right)<F\left(d\left(x_{n-1},x_{n+1}\right)\right)-\phi\left(d\left(x_{n-1},x_{n+1}\right)\right).$$

Since F is increasing,

$$d\left(x_n,x_{n+2}\right)<d\left(x_{n-1},x_{n+1}\right).$$

Repeating this step, we conclude that

$$\begin{aligned} F\left(d\left(x_n,x_{n+2}\right)\right) &\leq\left(F\left(d\left(x_{n-1},x_{n+1}\right)\right)\right)-\phi\left[d\left(x_{n-1},x_{n+1}\right)\right]\\ &\leq F\left(d\left(x_{n-2},x_n\right)\right)-\phi\left[d\left(x_{n+1},x_{n-1}\right)\right]-\phi\left[d\left(x_{n-2},x_n\right)\right]\\ &\leq\ldots\leq F\left(d\left(x_0,x_1\right)\right)-\sum_{i=0}^{n}\phi\left[d\left(x_{i-1},x_{i+1}\right)\right]. \end{aligned}$$

Since $\liminf_{s\to\alpha^+}\phi(s)>0$, we have $\liminf_{n\to\infty}\phi\left(d\left(x_{n-1},x_{n+1}\right)\right)>0$. By the definition of the limit, there exist $n_0\in\mathbb{N}$ and $C>0$ such that for all $n\geq n_0, \phi\left(d\left(x_{n-1},x_{n+1}\right)\right)>C$. Thus

$$
\begin{aligned}
F\left(d\left(x_n,x_{n+2}\right)\right) &\leq F\left(d\left(x_0,x_2\right)\right)-\sum_{i=0}^{n_0-1}\phi\left(d\left(x_{i+1},x_i\right)\right)-\sum_{i=n_0-1}^{n}\phi\left(d\left(x_i,x_{i+2}\right)\right)\\
&\leq F\left(d\left(x_0,x_1\right)\right)-\sum_{i=n_0-1}^{n}C\\
&=F\left(d\left(x_1,x_0\right)\right)-\left(n-n_0\right)C
\end{aligned}
$$

for all $n\geq n_0$. Taking the limit as $n\to\infty$ in the above inequality, we get

$$\lim_{n\to\infty}F\left(d\left(x_n,x_{n+2}\right)\right)\leq\lim_{n\to\infty}\left[F\left(d\left(x_0,x_2\right)\right)-\left(n-n_0\right)C\right],$$

that is, $\lim_{n\to\infty}F\left(d\left(x_{n+1},x_n\right)\right)=-\infty$. By the condition (ii) of Definition 2.13, we conclude that

$$\lim_{n\to\infty}d\left(x_n,x_{n+2}\right)=0.$$

Case 4. $d\left(x_{n+2},x_n\right)>0$. Applying 3.1 with $x=x_{n+1}$ and $y=x_{n-1}$, we conclude that

$$\lim_{n\to\infty}d\left(x_{n+2},x_n\right)=0.$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Firstly we show $\{x_n\}_{n\in\mathbb{N}}$ is a b-forward Cauchy sequence, i.e. $\lim_{n,m\to\infty}d\left(x_n,x_m\right)=0$ for all $n,m\in\mathbb{N}$.

Suppose to the contrary. By Lemma 3.9 , there is an $\varepsilon>0$ such that for an integer k there exist two sequences $\left\{n_{(k)}\right\}$ and $\left\{m_{(k)}\right\}$ such that $d\left(x_{m_{(k)}},x_{n_{(k)+1}}\right)\geq\varepsilon$ and $d\left(x_{m_{(k)}},x_{n_{(k)-1}}\right)<\varepsilon$ and

i) $\varepsilon\leq\lim_{k\to\infty}\inf d\left(x_{m_{(k)}},x_{n_{(k)}}\right)\leq\lim_{k\to\infty}\sup d\left(x_{m_{(k)}},x_{n_{(k)}}\right)\leq s\varepsilon$,
ii) $\varepsilon\leq\lim_{k\to\infty}\inf d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right)\leq\lim_{k\to\infty}\sup d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right)\leq s\varepsilon$,
iii) $\varepsilon\leq\lim_{k\to\infty}\inf d\left(x_{m_{(k)}},x_{n_{(k)+1}}\right)\leq\lim_{k\to\infty}\sup d\left(x_{m_{(k)}},x_{n_{(k)+1}}\right)\leq s\varepsilon$,
vi) $\frac{s}{s}\leq\lim_{k\to\infty}\inf d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\leq\lim_{k\to\infty}\sup d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\leq s^2\varepsilon$.

Applying 3.1 with $x=x_{m_{(k)}}$ and $y=x_{n_{(k)}}$, we obtain

$$F\left[s^2d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\right]\leq F\left(d\left(x_{m_{(k)}},x_{n_{(k)}}\right)\right)-\phi\left(d\left(x_{m_{(k)}},x_{n_{(k)}}\right)\right).$$

Letting $k\to\infty$ in the above inequality and using (i) and (vi), we obtain

$$
\begin{aligned}
F\left(\frac{\varepsilon}{s}s^2\right)&=F(\varepsilon s)\\
&\leq F\left(s^2\lim_{k\to\infty}\sup d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\right)\\
&=\lim_{k\to\infty}\sup F\left(s^2d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\right)\\
&\leq\lim_{k\to\infty}\sup F\left(d\left(x_{m_{(k)}},x_{n_{(k)}}\right)\right)-\lim_{k\to\infty}\sup\phi\left(d\left(x_{m_{(k)}},x_{n_{(k)}}\right)\right)\\
&\leq F(s\varepsilon).
\end{aligned}
$$

Therefore,

$$F(s\varepsilon)<F(s\varepsilon).$$

Since F is increasing, we get

$$s\varepsilon<s\varepsilon.$$

It is a contradiction. Thus

$$\lim_{n\to m\to\infty}d\left(x_m,x_n\right)=0.$$

Hence $\{x_n\}$ is a b-forward Cauchy sequence in X. By completeness of (X,d), there exists $z\in X$ such that

$$\lim_{n\to\infty} d(z,x_n)=0.$$

Secondly we show $\{x_n\}$ is a b-backward-Cauchy sequence. Suppose to the contrary. By Lemma 3.9 , there is an $\varepsilon>0$ such that for an integer k there exist two sequences $\{n_{(k)}\}$ and $\{m_{(k)}\}$ such that $d\left(x_{m_{(k)}},x_{n_{(k)+1}}\right)\geq\varepsilon$ and $d\left(x_{n_{(k)-1}},x_{m_{(k)}}\right)<\varepsilon$ and

i) $\varepsilon\leq\lim_{k\to\infty}\inf d\left(x_{n_{(k)}},x_{m_{(k)}}\right)\leq\lim_{k\to\infty}\sup d\left(x_{n_{(k)}},x_{m_{(k)}}\right)\leq s\varepsilon$,
ii) $\varepsilon\leq\lim_{k\to\infty}\inf d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right)\leq\lim_{k\to\infty}\sup d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right)\leq s\varepsilon$,
iii) $\varepsilon\leq\lim_{k\to\infty}\inf d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right)\leq\lim_{k\to\infty}\sup d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right)\leq s\varepsilon$,
vi) $\frac{\varepsilon}{s}\leq\lim_{k\to\infty}\inf d\left(x_{n_{(k)+1}},x_{m_{(k)+1}}\right)\leq\lim_{k\to\infty}\sup d\left(x_{n_{(k)+1}},x_{m_{(k)+1}}\right)\leq s^2\varepsilon$.

Applying 3.1 with $x=x_{n_{(k)}}$ and $y=x_{m_{(k)}}$, we obtain

$$F\left[s^2 d\left(x_{n_{(k)+1}},x_{m_{(k)+1}}\right)\right]\leq F\left(d\left(x_{n_{(k)}},x_{m_{(k)}}\right)\right)-\phi\left(d\left(x_{n_{(k)}},x_{m_{(k)}}\right)\right).$$

Letting $k\to\infty$ in the above inequality and using (vi), we obtain

$$\begin{aligned}
F\left(\frac{\varepsilon}{s}s^2\right)&=F(\varepsilon s)\\
&\leq F\left(s^2\lim_{k\to\infty}\sup d\left(x_{n_{(k)+1}},x_{m_{(k)+1}}\right)\right)\\
&=\lim_{k\to\infty}\sup F\left(s^2 d\left(x_{n_{(k)+1}},x_{m_{(k)+1}}\right)\right)\\
&\leq\lim_{k\to\infty}\sup F\left(d\left(x_{n_{(k)}},x_{m_{(k)}}\right)\right)-\lim_{k\to\infty}\sup\phi\left(d\left(x_{n_{(k)}},x_{m_{(k)}}\right)\right)\\
&\leq F(s\varepsilon).
\end{aligned}$$

Therefore,

$$F(s\varepsilon)<F(s\varepsilon).$$

Since F is increasing, we get $s\varepsilon<s\varepsilon$. It is a contradiction. Thus

$$\lim_{n\to m\to\infty} d(x_n,x_m)=0.$$

Hence $\{x_n\}$ is a b-backward Cauchy sequence in X. By completeness of (X,d), there exists $u\in X$ such that $\lim_{n\to\infty} d(x_n,u)=0$. It follows from Lemma 3.6 that $z=u$. Now, we show that $d(Tz,z)=0$ or $d(z,Tz)=0$. Assume that

$$d(z,Tz)>0 \text{ and } d(Tz,z)>0.$$

We shall show that z is a fixed point of T.

i) Assume that $d(z,Tz)>0$. Since $x_n\to z$ as $n\to\infty$, from Lemma 3.7, we conclude that

$$\frac{1}{s}d(z,Tz)\leq\lim_{n\to\infty}\sup d(Tx_n,Tz)\leq sd(z,Tz). \tag{3.4}$$

Now, applying 3.1 with $x=x_n$ and $y=z$, we have

$$F\left(s^2 d(Tx_n,Tz)\right)\leq F\left(d(x_n,z)\right)-\phi\left(d(x_n,z)\right),\forall n\in\mathbb{N}. \tag{3.5}$$

Letting $n\to\infty$ in 3.5, using 3.4 and continuity of F, we obtain

$$\begin{aligned}
F\left[s^2\frac{1}{s}d(z,Tz)\right]&=F[sd(z,Tz)]\\
&\leq F\left[s^2\lim_{n\to\infty}\sup d(Tx_n,Tz)\right]\\
&=\lim_{n\to\infty}\sup F\left[s^2 d(Tx_n,Tz)\right]\\
&\leq\lim_{n\to\infty}\sup F\left(d(x_n,z)\right)-\lim_{n\to\infty}\phi\left(d(x_n,z)\right)\\
&<\lim_{n\to\infty}\sup F\left(d(x_n,z)\right)\\
&=F\left(\lim_{n\to\infty}\sup d(x_n,z)\right).
\end{aligned}$$

Since F is increasing, we get $sd(z, Tz) \leq 0$, which implies that $s = 0$. It is a contradiction and so $d(z, Tz) = 0$.

ii) Assume that $d(Tz, z) > 0$. Since $x_n \to z$ as $n \to \infty$, from Lemma 3.7, we conclude that

$$\frac{1}{s} d(Tz, z) \leq \lim_{n\to\infty} \sup d\left(Tz, Tx_n\right) \leq sd(Tz, z). \tag{3.6}$$

Now, applying 3.1 with $x = z$ and $y = x_n$, we have

$$F\left(s^2 d\left(Tz, Tx_n\right)\right) \leq F\left(d\left(z, x_n\right)\right) - \phi\left(d\left(z, x_n\right)\right), \forall n \in \mathbb{N}. \tag{3.7}$$

Letting $n \to \infty$ in 3.6, using 3.7 and continuity of F, we obtain

$$\begin{aligned}
F\left[s^2 \frac{1}{s} d(Tz, z)\right] &= F[sd(Tz, z)] \\
&\leq F\left[s^2 \lim_{n\to\infty} \sup d\left(Tz, Tx_n\right)\right] \\
&= \lim_{n\to\infty} \sup F\left[s^2 d\left(Tz, Tx_n\right)\right] \\
&\leq \lim_{n\to\infty} \sup F\left(d\left(z, x_n\right)\right) - \lim_{n\to\infty} \phi\left(d\left(z, x_n\right)\right) \\
&< \lim_{n\to\infty} \sup F\left(d\left(z, x_n\right)\right) \\
&= F\left(\lim_{n\to\infty} \sup d\left(z, x_n\right)\right).
\end{aligned}$$

Since F is increasing, we get $sd(Tz, z) \leq 0$, which implies that $s = 0$. This is a contradiction. So $d(Tz, z) = 0$. Hence $Tz = z$. Uniqueness. Now, suppose that $z, u \in X$ are two fixed points of T such that $u \neq z$. Then we have

$$d(z, u) = d(Tz, Tu) > 0.$$

Applying 3.1 with $x = z$ and $y = u$, we have

$$F(d(z, u)) = F(d(Tz, Tu)) \leq F\left(s^2 d(Tz, Tu)\right) \leq F(d(z, u)) - \phi(d(z, u)).$$

Therefore, we have $d(z, u) < d(z, u)$. It is a contradiction. Thus $u = z$. □

Corollary 3.11. Let $d(X, d)$ be a complete b-rectangular asymmetric metric space with parameter $s > 1$ and T be a self mapping on X. If for all $x, y \in X$ we have

$$d(Tx, Ty) > 0 \Rightarrow s^2 d(Tx, Ty) \leq e^{\frac{-1}{T+(x,y)}} d(x, y),$$

then T has a unique fixed point.

Proof . Since $d(Tx, Ty) > 0$, we can take natural logarithm to get

$$\begin{aligned}
\ln\left(s^2 d(Tx, Ty)\right) &\leq \ln\left[e^{\frac{-1}{1+d(x,y)}} d(x,y)\right] \\
&= \frac{-1}{1 + d(x, y)} + \ln[d(x, y)].
\end{aligned}$$

Hence

$$F\left[s^2 d(Tx, Ty)\right] + \phi(d(x, y)) \leq F(d(x, y))$$

with $F(t) = \ln(t)$ and $\phi(t) = \frac{1}{1+t}$. As in the proof of Theorem 3.10, T has a unique fixed point. □

Corollary 3.12. Let $d(X, d)$ be a complete b-rectangular asymmetric metric space with parameter $s > 1$ and T be a self mapping on X. Suppose that there exists $k \in]0, 1[$ such that for all $x, y \in X$ we have

$$d(Tx, Ty) > 0 \Rightarrow s^2 d(Tx, Ty) \leq kd(x, y).$$

Then T has a unique fixed point.

Proof . Since $d(Tx,Ty)>0$, we can take natural logarithm to get

$$\begin{aligned}\ln\left(s^2 d(Tx,Ty)\right) &\leq \ln[kd(x,y)]\\ &= \ln(k)+\ln[d(x,y)].\end{aligned}$$

Hence

$$F\left[s^2 d(Tx,Ty)\right]+\phi(d(x,y))\leq F(d(x,y))$$

with $F(t)=\ln(t)$ and $\phi(t)=\ln\left(\frac{1}{k}\right)$. As in the proof of Theorem 3.10, T has a unique fixed point. □

Example 3.13. Let $X=A\cup B$, where $A=\left\{0,\frac{1}{3},\frac{1}{4},\frac{1}{5}\right\}$ and $B=\left[\frac{3}{4},2\right]$. Define $d:X\times X\to[0,+\infty[$ as follows:

$$\begin{cases} d\left(0,\dfrac{1}{4}\right)=d\left(\dfrac{1}{3},\dfrac{1}{5}\right)=0,1\\ d\left(0,\dfrac{1}{3}\right)=d\left(\dfrac{1}{3},0\right)=d\left(\dfrac{1}{3},\dfrac{1}{4}\right)=d\left(\dfrac{1}{4},\dfrac{1}{3}\right)=0,5\\ d\left(0,\dfrac{1}{5}\right)=d\left(\dfrac{1}{4},\dfrac{1}{5}\right)=0,05\\ d\left(\dfrac{1}{4},0\right)=d\left(\dfrac{1}{5},\dfrac{1}{3}\right)=0,04\\ d\left(\dfrac{1}{5},0\right)=d\left(\dfrac{1}{5},\dfrac{1}{4}\right)=0.06\\ d(x,y)=0\Leftrightarrow x=y\\ d(x,y)=(|x-y|)^2 \text{ otherwise.}\end{cases}$$

Then (X,d) is a b-rectangular asymmetric metric space with coefficient $s=3$. Define the mapping $T:X\to X$ by

$$T(x)=\begin{cases}\sqrt{x} \text{ if } x\in\left[\frac{3}{4},2\right]\\ 1 \text{ if } x\in A.\end{cases}$$

Then $T(x)\in\left[\frac{3}{4},2\right]$. Let $F(t)=\ln(\sqrt{t})$ for all $t\in]0,+\infty\left[,\phi(t)=\frac{1}{2+t}\right.$. It obvious that $F\in\mathcal{S}$ and $\phi\in\Phi$. Consider the following possibilities: Case 1: $x,y\in\left[\frac{3}{4},2\right]$ with $x\neq y$. Assume that $x>y$.

$$d(Tx,Ty)=(\sqrt{x}-\sqrt{y})^2 \qquad \text{and} \qquad d(x,y)=(x-y)^2.$$

Therefore,

$$F(d(Tx,Ty))=\ln(\sqrt{x}-\sqrt{y}) \qquad \text{and} \qquad \phi(d(x,y))=\left[\frac{1}{2+(x-y)}\right].$$

On the other hand,

$$\begin{aligned}F(d(Tx,Ty))+\phi(d(x,y))-F(d(x,y)) &= \ln(\sqrt{x}-\sqrt{y})+\left[\frac{1}{2+(x-y)}\right]-\ln(x-y).\\ &= \ln\left(\frac{\sqrt{x}-\sqrt{y}}{x-y}\right)+\left[\frac{1}{2+(x-y)}\right]\\ &= \ln\left(\frac{1}{\sqrt{x}+\sqrt{y}}\right)+\left[\frac{1}{2+(x-y)}\right]\\ &= -\ln(\sqrt{x}+\sqrt{y})+\left[\frac{1}{2+(x-y)}\right].\end{aligned}$$

Since $x,y\in\left[\frac{3}{4},2\right]$, we have

$$-\ln(\sqrt{x}+\sqrt{y})\leq-\ln(\sqrt{3}) \qquad \text{and} \qquad \left[\frac{1}{2+(x-y)}\right]\leq\ln(\sqrt{3}).$$

Thus, $F(d(Tx,Ty)) + \phi(d(x,y)) \leq F(d(x,y))$.

Case 2: $x \in \left[\frac{3}{4}, 2\right], y \in A$ or $y \in \left[\frac{3}{4}, 2\right], x \in A$. Then $T(x) = \sqrt{x}, T(y) = 1$, then $d(Tx,Ty) = (|\sqrt{x} - 1|)^2$. So we have

$$F(d(Tx,Ty)) = \ln(\sqrt{x} - 1)$$

and

$$\begin{aligned} d(x,y) &= (x-y)^2 \\ &\geq \left(x - \frac{1}{3}\right)^2 \\ &\geq (x-1)^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} F(d(Tx,Ty)) + \phi(d(x,y)) - F(d(x,y)) &= \ln(\sqrt{x}-1) + \left[\frac{1}{2+(x-y)}\right] - \ln(x-y) \\ &\leq \ln(\sqrt{x}-1) + \left[\frac{1}{2+(x-y)}\right] - \ln(x-1) \\ &= \ln\left(\frac{\sqrt{x}-1}{x-1}\right) + \left[\frac{1}{2+(x-y)}\right] \\ &= \ln\left(\frac{1}{\sqrt{x}+1}\right) + \left[\frac{1}{2+(x-y)}\right] \\ &= -\ln(\sqrt{x}+1) + \left[\frac{1}{2+(x-y)}\right]. \end{aligned}$$

Since $x \in]\frac{3}{4}, 2]$, we have

$$-\ln(\sqrt{x}+1) \leq -\ln(2) \qquad \text{and} \qquad \left[\frac{1}{2+(x-y)}\right] \leq \frac{1}{2} \leq \ln(2).$$

Thus

$$F(d(Tx,Ty)) + \phi(d(x,y)) \leq F(d(x,y)).$$

Hence T satisfies all the assumptions of Theorem 3.10 and $z = 1$ is the unique fixed point of T.

4 Conclusion

In this paper, we introduced the concept of b-rectangular asymmetric metric space, which is not necessarily Hausdorff and which generalizes the concept of metric space, rectangular metric space, rectangular asymmetric metric space and b-rectangular metric space. Also, we introduced the notion of (ϕ, F)-contraction and establish some new fixed point theorems for mappings in the setting of complete b-generalized asymmetric metric spaces. Our results generalize, improve and extend the corresponding results of Banach and Wardoski. Moreover, an illustrative example was presented to support the obtained results.

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