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Nonlinear ∗-derivation-type maps on sum of triple products on ∗-algebras

João Carlos da Motta Ferreira*, Maria das Graças Bruno Marietto

Center for Mathematics, Computing and Cognition, Federal University of ABC, Avenida dos Estados, 5001, 09210-580, Santo André, Brazil

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Abstract

Let $\mathcal{A}$ be a unital complex ∗-algebra having a nontrivial projection. In this paper, we proved that every ∗-derivation-type map $\Phi : \mathcal{A} \to \mathcal{A}$ on sum of triple products $\alpha_1 abc + \alpha_2 a^* cb^* + \alpha_3 ba^* c + \alpha_4 cab^* + \alpha_5 bca + \alpha_6 cb^* a^*$, where the scalars $\{\alpha_k\}_{k=1}^{6}$ are rational numbers satisfying some conditions, is an additive ∗-derivation. Some applications of the results obtained are also presented.

Keywords: Additive ∗-derivations, ∗-algebras, prime algebras, von Neumann algebras
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1 Introduction

Throughout this paper, $\mathcal{A}$ will denote a complex ∗-algebra. Given elements $a, b \in \mathcal{A}$ and a rational number η, we call the product $a\blacklozenge_\eta b = ab + \eta ba^*$ of *Jordan* η*-∗-product*. In particular, the product $a\blacklozenge_{-1} b = ab - ba^*$, usually denoted by $[a, b]_*$, is called of *skew Lie product* and $a\blacklozenge_1 b = ab + ba^*$, usually denoted by $a \bullet b$, is called of *Jordan ∗-product*.

A map $\Phi : \mathcal{A} \to \mathcal{A}$ is called ∗*-derivation-type on mixed product* $a\blacklozenge_\eta b\blacklozenge_\nu c$, where $a\blacklozenge_\eta b\blacklozenge_\nu c = (a\blacklozenge_\eta b)\blacklozenge_\nu c$ and η, ν are rational numbers, if

$$\Phi(a\blacklozenge_\eta b\blacklozenge_\nu c) = \Phi(a)\blacklozenge_\eta b\blacklozenge_\nu c + a\blacklozenge_\eta \Phi(b)\blacklozenge_\nu c + a\blacklozenge_\eta b\blacklozenge_\nu \Phi(c),$$

for all $a, b, c \in \mathcal{A}$.

In recent years, some research activities have been addressed to study ∗-derivation-type on mixed product $a\blacklozenge_\eta b\blacklozenge_\nu c$ (for example, see the works [2], [6], [7] and [8]). In this sense, Darvish et al. [2] studied the structure of the nonlinear ∗-derivation-type on mixed product $a\blacklozenge_1 b\blacklozenge_1 c$ on prime ∗-algebras whose result was refined by Li et al. [8]. Also, Li and Zhang [6] studied the structure of the nonlinear ∗-derivation-type on mixed product $a\blacklozenge_1 b\blacklozenge_{-1} c$, on ∗-algebras satisfying some mild conditions and Li et al. [7] studied the structure of the nonlinear ∗-derivation-type on mixed product $a\blacklozenge_{-1} b\blacklozenge_{-1} c$, between factors. Based on these presented cases, it is natural to inquire if similar or different characterizations can be obtained: for other rational values of η and ν or under more general hypotheses on the algebra or the map than those which have been taken into account previously (for example, this last question was addressed in [3]). The discussion of these questions, in this paper, will be carried out from the study of a more general map.

*Corresponding author
Email addresses: joao.cmferreira@ufabc.edu.br (João Carlos da Motta Ferreira), graca.marietto@ufabc.edu.br (Maria das Graças Bruno Marietto)

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A map $\Phi : \mathcal{A} \to \mathcal{A}$ is called *$*$-derivation-type on sum of triple products*

$$\alpha_1 abc + \alpha_2 a^* cb^* + \alpha_3 ba^* c + \alpha_4 cab^* + \alpha_5 bca + \alpha_6 cb^* a^*,$$

where $\{\alpha_k\}_{k=1}^6$ are arbitrary rational numbers, if

$$\begin{aligned}
&\Phi(\alpha_1 abc + \alpha_2 a^* cb^* + \alpha_3 ba^* c + \alpha_4 cab^* + \alpha_5 bca + \alpha_6 cb^* a^*)\\
=&\alpha_1 \Phi(a)bc + \alpha_2 \Phi(a)^* cb^* + \alpha_3 b\Phi(a)^* c + \alpha_4 c\Phi(a)b^* + \alpha_5 bc\Phi(a) + \alpha_6 cb^* \Phi(a)^* + \alpha_1 a\Phi(b)c + \alpha_2 a^* c\Phi(b)^*\\
&+ \alpha_3 \Phi(b)a^* c + \alpha_4 ca\Phi(b)^* + \alpha_5 \Phi(b)ca + \alpha_6 c\Phi(b)^* a^* + \alpha_1 ab\Phi(c) + \alpha_2 a^* \Phi(c)b^* + \alpha_3 ba^* \Phi(c)\\
&+ \alpha_4 \Phi(c)ab^* + \alpha_5 b\Phi(c)a + \alpha_6 \Phi(c)b^* a^*, \qquad (1.1)
\end{aligned}$$

for all $a, b, c \in \mathcal{A}$. This kind of map is related to $*$-derivation-type on mixed product $a\blacklozenge_\eta b\blacklozenge_\nu c$. In fact, it is easy to verify that all $*$-derivation-type on mixed product $a\blacklozenge_\eta b\blacklozenge_\nu c$ is a $*$-derivation-type on sum of triple products $\alpha_1 abc + \alpha_2 a^* cb^* + \alpha_3 ba^* c + \alpha_4 cab^* + \alpha_5 bca + \alpha_6 cb^* a^*$. More precisely, each $*$-derivation-type on mixed product $a\blacklozenge_\eta b\blacklozenge_\nu c$ is a $*$-derivation-type on sum of triple products $1abc + 0a^* cb^* + \eta ba^* c + \nu\eta cab^* + 0bca + \nu cb^* a^*$. The advantage of this approach is that the obtained results can be used to determine characterizations of particular types of maps and can also lead to new view points for other possible characterizations. Thus, the main purpose of this work will be to characterize such $*$-derivation-type map. On account of these considerations and inspired by the work of Ferreira and Marietto [3], the purpose of this paper is to study the structure of nonlinear $*$-derivation-type maps on sum of triple products $\alpha_1 abc + \alpha_2 a^* cb^* + \alpha_3 ba^* c + \alpha_4 cab^* + \alpha_5 bca + \alpha_6 cb^* a^*$, where $\{\alpha_k\}_{k=1}^6$ are rational numbers satisfying certain conditions, from unital $*$-algebras having a nontrivial projection. As an application of our main result, we will discuss the structure of $*$-derivation-type on mixed product $a\blacklozenge_\eta b\blacklozenge_\nu c$, by considering other rational values of η and ν, beyond of the ones mentioned in [2], [6], [7] and [8].

2 The main results and their proofs

Our main result in this paper reads as follows:

Theorem 2.1. Let $\{\alpha_k\}_{k=1}^6$ be rational numbers satisfying the three following conditions $\alpha_1+\alpha_3+\alpha_5 \neq 0$, $\alpha_2+\alpha_4+\alpha_6 \neq 0$ and $(\alpha_1+\alpha_3+\alpha_5)^2 - (\alpha_2+\alpha_4+\alpha_6)^2 \neq 0$, $\mathcal{A}$ a unital $*$-algebra with $1_\mathcal{A}$ its multiplicative identity having a nontrivial projection p_1 (write $p_2 = 1_\mathcal{A} - p_1$) such that the following two properties

$$x\mathcal{A}p_l = 0 \text{ implies } x = 0 \ (l = 1, 2) \qquad (\spadesuit)$$

are verified and $\Phi : \mathcal{A} \to \mathcal{A}$ a $*$-derivation-type on sum of triple products $\alpha_1 abc + \alpha_2 a^* cb^* + \alpha_3 ba^* c + \alpha_4 cab^* + \alpha_5 bca + \alpha_6 cb^* a^*$. Then Φ is additive. In addition, if $\alpha_1 - \alpha_2 + \alpha_5 - \alpha_6 \neq 0$, then Φ is an additive $*$-derivation.

The following well known result will be used throughout this paper: Let p_1 be a nontrivial projection of $\mathcal{A}$ and write $p_2 = 1_\mathcal{A} - p_1$. Then $\mathcal{A}$ has a Peirce decomposition $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$, relative to p_1, where $\mathcal{A}_{ij} = p_i\mathcal{A}p_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations: $\mathcal{A}_{ij}\mathcal{A}_{kl} \subseteq \delta_{jk}\mathcal{A}_{il}$, where δ_{jk} is the *Kronecker delta function*. More details about the Peirce decomposition and its properties, can be found in references [4] and [9].

We organize the proof of Theorem 2.1 in a series of claims. We begin with a claim, whose proof is easy and is omitted.

Claim 1. $\Phi(0) = 0$.

Claim 2. Φ is additive.

We organize the proof of Claim 2 in a series of steps.

Step 1. For all $a_{ii} \in \mathcal{A}_{ii}$, $a_{ij} \in \mathcal{A}_{ij}$, $a_{ji} \in \mathcal{A}_{ji}$, $a_{jj} \in \mathcal{A}_{jj}$ $(i \neq j; i, j = 1, 2)$ and $r, s \in \mathcal{A}$, write $t = \Phi(\sum_{i,j=1,2} a_{ij}) - \sum_{i,j=1,2} \Phi(a_{ij})$. The following two assertions are true: (i) if

$$\begin{aligned}
&\Phi(\alpha_1 r(\textstyle\sum_{i,j=1,2} a_{ij})s + \alpha_2 r^* s(\sum_{i,j=1,2} a_{ij})^* + \alpha_3(\sum_{i,j=1,2} a_{ij})r^* s\\
&+ \alpha_4 sr(\textstyle\sum_{i,j=1,2} a_{ij})^* + \alpha_5(\sum_{i,j=1,2} a_{ij})sr + \alpha_6 s(\sum_{i,j=1,2} a_{ij})^* r^*)\\
=& \textstyle\sum_{i,j=1,2} \Phi(\alpha_1 r a_{ij} s + \alpha_2 r^* s a_{ij}^* + \alpha_3 a_{ij} r^* s + \alpha_4 sr a_{ij}^* + \alpha_5 a_{ij} sr + \alpha_6 s a_{ij}^* r^*), \qquad (2.1)
\end{aligned}$$

then

$$\alpha_1 rts + \alpha_2 r^* st^* + \alpha_3 tr^* s + \alpha_4 srt^* + \alpha_5 tsr + \alpha_6 st^* r^* = 0 \tag{2.2}$$

and (ii) if

$$\begin{aligned}&\Phi(\alpha_1 rs(\textstyle\sum_{i,j=1,2} a_{ij}) + \alpha_2 r^*(\sum_{i,j=1,2} a_{ij})s^* + \alpha_3 sr^*(\sum_{i,j=1,2} a_{ij})\\ &+ \alpha_4(\textstyle\sum_{i,j=1,2} a_{ij})rs^* + \alpha_5 s(\sum_{i,j=1,2} a_{ij})r + \alpha_6(\sum_{i,j=1,2} a_{ij})s^* r^*)\\ =&\sum_{i,j=1,2}\Phi(\alpha_1 rsa_{ij} + \alpha_2 r^* a_{ij}s^* + \alpha_3 sr^* a_{ij} + \alpha_4 a_{ij}rs^* + \alpha_5 sa_{ij}r + \alpha_6 a_{ij}s^* r^*),\end{aligned} \tag{2.3}$$

then

$$\alpha_1 rst + \alpha_2 r^* ts^* + \alpha_3 sr^* t + \alpha_4 trs^* + \alpha_5 str + \alpha_6 ts^* r^* = 0. \tag{2.4}$$

Proof . By identity (1.1) we have

$$\begin{aligned}&\alpha_1\Phi(r)(\textstyle\sum_{i,j=1,2} a_{ij})s + \alpha_2\Phi(r)^* s(\sum_{i,j=1,2} a_{ij})^* + \alpha_3(\sum_{i,j=1,2} a_{ij})\Phi(r)^* s\\ &+ \alpha_4 s\Phi(r)(\textstyle\sum_{i,j=1,2} a_{ij})^* + \alpha_5(\sum_{i,j=1,2} a_{ij})s\Phi(r) + \alpha_6 s(\sum_{i,j=1,2} a_{ij})^*\Phi(r)^*\\ &+ \alpha_1 r\Phi(\textstyle\sum_{i,j=1,2} a_{ij})s + \alpha_2 r^* s\Phi(\sum_{i,j=1,2} a_{ij})^* + \alpha_3\Phi(\sum_{i,j=1,2} a_{ij})r^* s\\ &+ \alpha_4 sr\Phi(\textstyle\sum_{i,j=1,2} a_{ij})^* + \alpha_5\Phi(\sum_{i,j=1,2} a_{ij})sr + \alpha_6 s\Phi(\sum_{i,j=1,2} a_{ij})^* r^*\\ &+ \alpha_1 r(\textstyle\sum_{i,j=1,2} a_{ij})\Phi(s) + \alpha_2 r^*\Phi(s)(\sum_{i,j=1,2} a_{ij})^* + \alpha_3(\sum_{i,j=1,2} a_{ij})r^*\Phi(s)\\ &+ \alpha_4\Phi(s)r(\textstyle\sum_{i,j=1,2} a_{ij})^* + \alpha_5(\sum_{i,j=1,2} a_{ij})\Phi(s)r + \alpha_6\Phi(s)(\sum_{i,j=1,2} a_{ij})^* r^*\\ =&\Phi(\alpha_1 r(\textstyle\sum_{i,j=1,2} a_{ij})s + \alpha_2 r^* s(\sum_{i,j=1,2} a_{ij})^* + \alpha_3(\sum_{i,j=1,2} a_{ij})r^* s\\ &+ \alpha_4 sr(\textstyle\sum_{i,j=1,2} a_{ij})^* + \alpha_5(\sum_{i,j=1,2} a_{ij})sr + \alpha_6 s(\sum_{i,j=1,2} a_{ij})^* r^*)\\ =&\textstyle\sum_{i,j=1,2}\Phi(\alpha_1 ra_{ij}s + \alpha_2 r^* sa_{ij}^* + \alpha_3 a_{ij}r^* s + \alpha_4 sra_{ij}^* + \alpha_5 a_{ij}sr + \alpha_6 sa_{ij}^* r^*)\\ =&\textstyle\sum_{i,j=1,2}(\alpha_1\Phi(r)a_{ij}s + \alpha_2\Phi(r)^* sa_{ij}^* + \alpha_3 a_{ij}\Phi(r)^* s + \alpha_4 s\Phi(r)a_{ij}^* + \alpha_5 a_{ij}s\Phi(r)\\ &+ \alpha_6 sa_{ij}^*\Phi(r)^* + \alpha_1 r\Phi(a_{ij})s + \alpha_2 r^* s\Phi(a_{ij})^* + \alpha_3\Phi(a_{ij})r^* s + \alpha_4 sr\Phi(a_{ij})^*\\ &+ \alpha_5\Phi(a_{ij})sr + \alpha_6 s\Phi(a_{ij})^* r^* + \alpha_1 ra_{ij}\Phi(s) + \alpha_2 r^*\Phi(s)a_{ij}^* + \alpha_3 a_{ij}r^*\Phi(s)\\ &+ \alpha_4\Phi(s)ra_{ij}^* + \alpha_5 a_{ij}\Phi(s)r + \alpha_6\Phi(s)a_{ij}^* r^*)\\ =&\alpha_1\Phi(r)(\textstyle\sum_{i,j=1,2} a_{ij})s + \alpha_2\Phi(r)^* s(\sum_{i,j=1,2} a_{ij})^* + \alpha_3(\sum_{i,j=1,2} a_{ij})\Phi(r)^* s\\ &+ \alpha_4 s\Phi(r)(\textstyle\sum_{i,j=1,2} a_{ij})^* + \alpha_5(\sum_{i,j=1,2} a_{ij})s\Phi(r) + \alpha_6 s(\sum_{i,j=1,2} a_{ij})^*\Phi(r)^*\\ &+ \alpha_1 r(\textstyle\sum_{i,j=1,2}\Phi(a_{ij}))s + \alpha_2 r^* s(\sum_{i,j=1,2}\Phi(a_{ij}))^* + \alpha_3(\sum_{i,j=1,2}\Phi(a_{ij}))r^* s\\ &+ \alpha_4 sr(\textstyle\sum_{i,j=1,2}\Phi(a_{ij}))^* + \alpha_5(\sum_{i,j=1,2}\Phi(a_{ij}))sr + \alpha_6 s(\sum_{i,j=1,2}\Phi(a_{ij}))^* r^*\\ &+ \alpha_1 r(\textstyle\sum_{i,j=1,2} a_{ij})\Phi(s) + \alpha_2 r^*\Phi(s)(\sum_{i,j=1,2} a_{ij})^* + \alpha_3(\sum_{i,j=1,2} a_{ij})r^*\Phi(s)\\ &+ \alpha_4\Phi(s)r(\textstyle\sum_{i,j=1,2} a_{ij})^* + \alpha_5(\sum_{i,j=1,2} a_{ij})\Phi(s)r + \alpha_6\Phi(s)(\sum_{i,j=1,2} a_{ij})^* r^*\end{aligned}$$

which implies that

$$\begin{aligned}&\alpha_1 r\Phi(\textstyle\sum_{i,j=1,2} a_{ij})s + \alpha_2 r^* s\Phi(\sum_{i,j=1,2} a_{ij})^* + \alpha_3\Phi(\sum_{i,j=1,2} a_{ij})r^* s + \alpha_4 sr\Phi(\sum_{i,j=1,2} a_{ij})^*\\ &+ \alpha_5\Phi(\textstyle\sum_{i,j=1,2} a_{ij})sr + \alpha_6 s\Phi(\sum_{i,j=1,2} a_{ij})^* r^*\\ =&\alpha_1 r(\textstyle\sum_{i,j=1,2}\Phi(a_{ij}))s + \alpha_2 r^* s(\sum_{i,j=1,2}\Phi(a_{ij}))^* + \alpha_3(\sum_{i,j=1,2}\Phi(a_{ij}))r^* s + \alpha_4 sr(\sum_{i,j=1,2}\Phi(a_{ij}))^*\\ &+ \alpha_5(\textstyle\sum_{i,j=1,2}\Phi(a_{ij}))sr + \alpha_6 s(\sum_{i,j=1,2}\Phi(a_{ij}))^* r^*.\end{aligned}$$

Thus

$$\begin{aligned}&\alpha_1 r(\Phi(\textstyle\sum_{i,j=1,2} a_{ij}) - \sum_{i,j=1,2}\Phi(a_{ij}))s + \alpha_2 r^* s(\Phi(\sum_{i,j=1,2} a_{ij}) - \sum_{i,j=1,2}\Phi(a_{ij}))^*\\ &+ \alpha_3(\Phi(\textstyle\sum_{i,j=1,2} a_{ij}) - \sum_{i,j=1,2}\Phi(a_{ij}))r^* s + \alpha_4 sr(\Phi(\sum_{i,j=1,2} a_{ij}) - \sum_{i,j=1,2}\Phi(a_{ij}))^*\end{aligned}$$

$$+\alpha_5(\Phi(\textstyle\sum_{i,j=1,2} a_{ij}) - \sum_{i,j=1,2}\Phi(a_{ij}))sr + \alpha_6 s(\Phi(\sum_{i,j=1,2} a_{ij}) - \sum_{i,j=1,2}\Phi(a_{ij}))^* r^* = 0.$$

It therefore follows that

$$\alpha_1 rts + \alpha_2 r^* st^* + \alpha_3 tr^* s + \alpha_4 srt^* + \alpha_5 tsr + \alpha_6 st^* r^* = 0.$$

Similarly, we prove the case (ii) using the identity (1.1). □

Step 2. For all $a_{ij} \in \mathcal{A}_{ij}$ and $a_{ji} \in \mathcal{A}_{ji}$ $(i \neq j; i,j = 1,2)$ we have $\Phi(a_{ij} + a_{ji}) = \Phi(a_{ij}) + \Phi(a_{ji})$.

Proof . Take $t = \Phi(a_{ij} + a_{ji}) - \Phi(a_{ij}) - \Phi(a_{ji})$ and $t = t_{ii} + t_{ij} + t_{ji} + t_{jj}$ in the Peirce decomposition of $\mathcal{A}$, relative to p_1. Then, for $s = p_i$ or $s = p_j$,

$$\begin{aligned}
&\Phi(\alpha_1 1_{\mathcal{A}}(a_{ij} + a_{ji})s + \alpha_2 1_{\mathcal{A}}^* s(a_{ij} + a_{ji})^* + \alpha_3(a_{ij} + a_{ji})1_{\mathcal{A}}^* s \\
&+ \alpha_4 s1_{\mathcal{A}}(a_{ij} + a_{ji})^* + \alpha_5(a_{ij} + a_{ji})s1_{\mathcal{A}} + \alpha_6 s(a_{ij} + a_{ji})^* 1_{\mathcal{A}}^*) \\
=&\Phi(\alpha_1 1_{\mathcal{A}} a_{ij} s + \alpha_2 1_{\mathcal{A}}^* s a_{ij}^* + \alpha_3 a_{ij} 1_{\mathcal{A}}^* s + \alpha_4 s1_{\mathcal{A}} a_{ij}^* + \alpha_5 a_{ij} s1_{\mathcal{A}} + \alpha_6 s a_{ij}^* 1_{\mathcal{A}}^*) \\
&+ \Phi(\alpha_1 1_{\mathcal{A}} a_{ji} s + \alpha_2 1_{\mathcal{A}}^* s a_{ji}^* + \alpha_3 a_{ji} 1_{\mathcal{A}}^* s + \alpha_4 s1_{\mathcal{A}} a_{ji}^* + \alpha_5 a_{ji} s1_{\mathcal{A}} + \alpha_6 s a_{ji}^* 1_{\mathcal{A}}^*).
\end{aligned}$$

It follows from Step 1(i) that

$$\alpha_1 1_{\mathcal{A}} ts + \alpha_2 1_{\mathcal{A}}^* st^* + \alpha_3 t1_{\mathcal{A}}^* s + \alpha_4 s1_{\mathcal{A}} t^* + \alpha_5 ts1_{\mathcal{A}} + \alpha_6 st^* 1_{\mathcal{A}}^* = 0. \tag{2.5}$$

If $s = p_i$, then from (2.5), we get

$$(\alpha_1 + \alpha_3 + \alpha_5)(t_{ii} + t_{ji}) + (\alpha_2 + \alpha_4 + \alpha_6)(t_{ii} + t_{ji})^* = 0. \tag{2.6}$$

By applying involution to the identity (2.6), we get

$$(\alpha_2 + \alpha_4 + \alpha_6)(t_{ii} + t_{ji}) + (\alpha_1 + \alpha_3 + \alpha_5)(t_{ii} + t_{ji})^* = 0. \tag{2.7}$$

Multiplying (2.6) by the scalar $\alpha_1 + \alpha_3 + \alpha_5$, (2.7) by the scalar $\alpha_2 + \alpha_4 + \alpha_6$ and subtracting the resulting identities, we arrive at $((\alpha_1 + \alpha_3 + \alpha_5)^2 - (\alpha_2 + \alpha_4 + \alpha_6)^2)(t_{ii} + t_{ji}) = 0$ which shows that $t_{ii} + t_{ji} = 0$. If $s = p_j$, then using similar reasoning to the previous case, we conclude that $t_{ij} + t_{jj} = 0$. This shows that $t = 0$. □

In the following two steps, we consider γ a rational number satisfying the condition $\alpha_2 + \alpha_4 + \alpha_6 = \gamma(\alpha_1 + \alpha_3 + \alpha_5)$.

Step 3. For all $a_{ii} \in \mathcal{A}_{ii}$, $a_{ij} \in \mathcal{A}_{ij}$ and $a_{ji} \in \mathcal{A}_{ji}$ $(i \neq j; i,j = 1,2)$ we have $\Phi(a_{ii} + a_{ij}) = \Phi(a_{ii}) + \Phi(a_{ij})$ and $\Phi(a_{ii} + a_{ji}) = \Phi(a_{ii}) + \Phi(a_{ji})$.

Proof . Take $t = \Phi(a_{ii} + a_{ij}) - \Phi(a_{ii}) - \Phi(a_{ij})$ and $t = t_{ii} + t_{ij} + t_{ji} + t_{jj}$ in the Peirce decomposition of $\mathcal{A}$, relative to p_1. Then, for $s = p_i$ or $s = p_j$,

$$\begin{aligned}
&\Phi(\alpha_1 1_{\mathcal{A}}(a_{ii} + a_{ij})s + \alpha_2 1_{\mathcal{A}}^* s(a_{ii} + a_{ij})^* + \alpha_3(a_{ii} + a_{ij})1_{\mathcal{A}}^* s \\
&+ \alpha_4 s1_{\mathcal{A}}(a_{ii} + a_{ij})^* + \alpha_5(a_{ii} + a_{ij})s1_{\mathcal{A}} + \alpha_6 s(a_{ii} + a_{ij})^* 1_{\mathcal{A}}^*) \\
=&\Phi(\alpha_1 1_{\mathcal{A}} a_{ii} s + \alpha_2 1_{\mathcal{A}}^* s a_{ii}^* + \alpha_3 a_{ii} 1_{\mathcal{A}}^* s + \alpha_4 s1_{\mathcal{A}} a_{ii}^* + \alpha_5 a_{ii} s1_{\mathcal{A}} + \alpha_6 s a_{ii}^* 1_{\mathcal{A}}^*) \\
&+ \Phi(\alpha_1 1_{\mathcal{A}} a_{ij} s + \alpha_2 1_{\mathcal{A}}^* s a_{ij}^* + \alpha_3 a_{ij} 1_{\mathcal{A}}^* s + \alpha_4 s1_{\mathcal{A}} a_{ij}^* + \alpha_5 a_{ij} s1_{\mathcal{A}} + \alpha_6 s a_{ij}^* 1_{\mathcal{A}}^*).
\end{aligned}$$

It follows from Step 1(i) that

$$\alpha_1 1_{\mathcal{A}} ts + \alpha_2 1_{\mathcal{A}}^* st^* + \alpha_3 t1_{\mathcal{A}}^* s + \alpha_4 s1_{\mathcal{A}} t^* + \alpha_5 ts1_{\mathcal{A}} + \alpha_6 st^* 1_{\mathcal{A}}^* = 0. \tag{2.8}$$

If $s = p_i$, then (2.8) becomes the identity

$$(\alpha_1 + \alpha_3 + \alpha_5)(t_{ii} + t_{ji}) + (\alpha_2 + \alpha_4 + \alpha_6)(t_{ii} + t_{ji})^* = 0 \tag{2.9}$$

By applying involution to the identity (2.9), we get

$$(\alpha_2+\alpha_4+\alpha_6)(t_{ii}+t_{ji})+(\alpha_1+\alpha_3+\alpha_5)(t_{ii}+t_{ji})^*=0. \tag{2.10}$$

Multiplying (2.9) by the scalar $\alpha_1+\alpha_3+\alpha_5$, (2.10) by the scalar $\alpha_2+\alpha_4+\alpha_6$ and subtracting the resulting identities, we arrive at $((\alpha_1+\alpha_3+\alpha_5)^2-(\alpha_2+\alpha_4+\alpha_6)^2)(t_{ii}+t_{ji})=0$ which yields $t_{ii}+t_{ji}=0$. If $s=p_j$, then using similar reasoning to the previous case, we arrive at $t_{ij}+t_{jj}=0$. This shows that $t=0$. Next, take $u=\Phi(a_{ii}+a_{ji})-\Phi(a_{ii})-\Phi(a_{ji})$ and $u=u_{ii}+u_{ij}+u_{ji}+u_{jj}$ in the Peirce decomposition of $\mathcal{A}$, relative to p_1. Then

$$\begin{aligned}&\Phi(\alpha_1 1_{\mathcal{A}}(a_{ii}+a_{ji})p_j+\alpha_2 1_{\mathcal{A}}^*p_j(a_{ii}+a_{ji})^*+\alpha_3(a_{ii}+a_{ji})1_{\mathcal{A}}^*p_j\\&+\alpha_4p_j1_{\mathcal{A}}(a_{ii}+a_{ji})^*+\alpha_5(a_{ii}+a_{ji})p_j1_{\mathcal{A}}+\alpha_6p_j(a_{ii}+a_{ji})^*1_{\mathcal{A}}^*)\\=&\Phi(\alpha_1 1_{\mathcal{A}}a_{ii}p_j+\alpha_2 1_{\mathcal{A}}^*p_ja_{ii}^*+\alpha_3a_{ii}1_{\mathcal{A}}^*p_j+\alpha_4p_j1_{\mathcal{A}}a_{ii}^*+\alpha_5a_{ii}p_j1_{\mathcal{A}}+\alpha_6p_ja_{ii}^*1_{\mathcal{A}}^*)\\&+\Phi(\alpha_1 1_{\mathcal{A}}a_{ji}p_j+\alpha_2 1_{\mathcal{A}}^*p_ja_{ji}^*+\alpha_3a_{ji}1_{\mathcal{A}}^*p_j+\alpha_4p_j1_{\mathcal{A}}a_{ji}^*+\alpha_5a_{ji}p_j1_{\mathcal{A}}+\alpha_6p_ja_{ji}^*1_{\mathcal{A}}^*).\end{aligned}$$

This implies that

$$\alpha_1 1_{\mathcal{A}}up_j+\alpha_2 1_{\mathcal{A}}^*p_ju^*+\alpha_3u1_{\mathcal{A}}^*p_j+\alpha_4p_j1_{\mathcal{A}}u^*+\alpha_5up_j1_{\mathcal{A}}+\alpha_6p_ju^*1_{\mathcal{A}}^*=0$$

which shows that

$$(\alpha_1+\alpha_3+\alpha_5)(u_{ij}+u_{jj})+(\alpha_2+\alpha_4+\alpha_6)(u_{ij}+u_{jj})^*=0. \tag{2.11}$$

By applying involution to the identity (2.11), we get

$$(\alpha_2+\alpha_4+\alpha_6)(u_{ij}+u_{jj})+(\alpha_1+\alpha_3+\alpha_5)(u_{ij}+u_{jj})^*=0. \tag{2.12}$$

Multiplying (2.11) by the scalar $\alpha_1+\alpha_3+\alpha_5$, (2.12) by the scalar $\alpha_2+\alpha_4+\alpha_6$ and subtracting the resulting identities, we arrive at $((\alpha_1+\alpha_3+\alpha_5)^2-(\alpha_2+\alpha_4+\alpha_6)^2)(u_{ij}+u_{jj})=0$ which yields $u_{ij}+u_{jj}=0$. Next, as

$$\begin{aligned}&\Phi(\alpha_1 1_{\mathcal{A}}p_j(a_{ii}+a_{ji})+\alpha_2 1_{\mathcal{A}}^*(a_{ii}+a_{ji})p_j^*+\alpha_3p_j1_{\mathcal{A}}^*(a_{ii}+a_{ji})\\&+\alpha_4(a_{ii}+a_{ji})1_{\mathcal{A}}p_j^*+\alpha_5p_j(a_{ii}+a_{ji})1_{\mathcal{A}}+\alpha_6(a_{ii}+a_{ji})p_j^*1_{\mathcal{A}}^*)\\=&\Phi(\alpha_1 1_{\mathcal{A}}p_ja_{ii}+\alpha_2 1_{\mathcal{A}}^*a_{ii}p_j^*+\alpha_3p_j1_{\mathcal{A}}^*a_{ii}+\alpha_4a_{ii}1_{\mathcal{A}}p_j^*+\alpha_5p_ja_{ii}1_{\mathcal{A}}+\alpha_6a_{ii}p_j^*1_{\mathcal{A}}^*)\\&+\Phi(\alpha_1 1_{\mathcal{A}}p_ja_{ji}+\alpha_2 1_{\mathcal{A}}^*a_{ji}p_j^*+\alpha_3p_j1_{\mathcal{A}}^*a_{ji}+\alpha_4a_{ji}1_{\mathcal{A}}p_j^*+\alpha_5p_ja_{ji}1_{\mathcal{A}}+\alpha_6a_{ji}p_j^*1_{\mathcal{A}}^*),\end{aligned}$$

then $\alpha_1 1_{\mathcal{A}}p_ju+\alpha_2 1_{\mathcal{A}}^*up_j^*+\alpha_3p_j1_{\mathcal{A}}^*u+\alpha_4u1_{\mathcal{A}}p_j^*+\alpha_5p_ju1_{\mathcal{A}}+\alpha_6up_j^*1_{\mathcal{A}}^*=0$, by Step 1(ii). From this we get $(\alpha_1+\alpha_3+\alpha_5)u_{ji}=0$ which leads to $u_{ji}=0$. Finally, as

$$\begin{aligned}&\alpha_1 1_{\mathcal{A}}(p_i-\gamma p_j)a_{ji}+\alpha_2 1_{\mathcal{A}}^*a_{ji}(p_i-\gamma p_j)^*+\alpha_3(p_i-\gamma p_j)1_{\mathcal{A}}^*a_{ji}\\&+\alpha_4a_{ji}1_{\mathcal{A}}(p_i-\gamma p_j)^*+\alpha_5(p_i-\gamma p_j)a_{ji}1_{\mathcal{A}}+\alpha_6a_{ji}(p_i-\gamma p_j)^*1_{\mathcal{A}}^*=0,\end{aligned}$$

then

$$\begin{aligned}&\Phi(\alpha_1 1_{\mathcal{A}}(p_i-\gamma p_j)(a_{ii}+a_{ji})+\alpha_2 1_{\mathcal{A}}^*(a_{ii}+a_{ji})(p_i-\gamma p_j)^*+\alpha_3(p_i-\gamma p_j)1_{\mathcal{A}}^*(a_{ii}+a_{ji})+\alpha_4(a_{ii}+a_{ji})1_{\mathcal{A}}(p_i-\gamma p_j)^*\\&+\alpha_5(p_i-\gamma p_j)(a_{ii}+a_{ji})1_{\mathcal{A}}+\alpha_6(a_{ii}+a_{ji})(p_i-\gamma p_j)^*1_{\mathcal{A}}^*)\\=&\Phi(\alpha_1 1_{\mathcal{A}}(p_i-\gamma p_j)a_{ii}+\alpha_2 1_{\mathcal{A}}^*a_{ii}(p_i-\gamma p_j)^*+\alpha_3(p_i-\gamma p_j)1_{\mathcal{A}}^*a_{ii}+\alpha_4a_{ii}1_{\mathcal{A}}(p_i-\gamma p_j)^*+\alpha_5(p_i-\gamma p_j)a_{ii}1_{\mathcal{A}}\\&+\alpha_6a_{ii}(p_i-\gamma p_j)^*1_{\mathcal{A}}^*)+\Phi(\alpha_1 1_{\mathcal{A}}(p_i-\gamma p_j)a_{ji}+\alpha_2 1_{\mathcal{A}}^*a_{ji}(p_i-\gamma p_j)^*+\alpha_3(p_i-\gamma p_j)1_{\mathcal{A}}^*a_{ji}\\&+\alpha_4a_{ji}1_{\mathcal{A}}(p_i-\gamma p_j)^*+\alpha_5(p_i-\gamma p_j)a_{ji}1_{\mathcal{A}}+\alpha_6a_{ji}(p_i-\gamma p_j)^*1_{\mathcal{A}}^*)\end{aligned}$$

which leads to

$$\alpha_1 1_{\mathcal{A}}(p_i-\gamma p_j)u+\alpha_2 1_{\mathcal{A}}^*u(p_i-\gamma p_j)^*+\alpha_3(p_i-\gamma p_j)1_{\mathcal{A}}^*u+\alpha_4u1_{\mathcal{A}}(p_i-\gamma p_j)^*+\alpha_5(p_i-\gamma p_j)u1_{\mathcal{A}}+\alpha_6u(p_i-\gamma p_j)^*1_{\mathcal{A}}^*=0.$$

From this we get $(\sum_{k=1}^6\alpha_k)u_{ii}=0$. Therefore $u_{ii}=0$. The Step is proved. □

Step 4. For all $a_{ii}\in\mathcal{A}_{ii}$, $a_{ij}\in\mathcal{A}_{ij}$, $a_{ji}\in\mathcal{A}_{ji}$ and $a_{jj}\in\mathcal{A}_{jj}$ $(i\neq j; i,j=1,2)$ we have $\Phi(a_{ii}+a_{ij}+a_{ji})=\Phi(a_{ii})+\Phi(a_{ij})+\Phi(a_{ji})$ and $\Phi(a_{ij}+a_{ji}+a_{jj})=\Phi(a_{ij})+\Phi(a_{ji})+\Phi(a_{jj})$.

Proof . Take $t = \Phi(a_{ii} + a_{ij} + a_{ji}) - \Phi(a_{ii}) - \Phi(a_{ij}) - \Phi(a_{ji})$ and $t = t_{ii} + t_{ij} + t_{ji} + t_{jj}$ in the Peirce decomposition of $\mathcal{A}$, relative to p_1. Then, by Step 2 we have

$$\begin{aligned}
&\Phi(\alpha_1 1_{\mathcal{A}} p_j(a_{ii} + a_{ij} + a_{ji}) + \alpha_2 1_{\mathcal{A}}^*(a_{ii} + a_{ij} + a_{ji})p_j^* + \alpha_3 p_j 1_{\mathcal{A}}^*(a_{ii} + a_{ij} + a_{ji})\\
&+ \alpha_4(a_{ii} + a_{ij} + a_{ji})1_{\mathcal{A}} p_j^* + \alpha_5 p_j(a_{ii} + a_{ij} + a_{ji})1_{\mathcal{A}} + \alpha_6(a_{ii} + a_{ij} + a_{ji})p_j^* 1_{\mathcal{A}}^*)\\
=&\Phi(\alpha_1 1_{\mathcal{A}} p_j a_{ii} + \alpha_2 1_{\mathcal{A}}^* a_{ii} p_j^* + \alpha_3 p_j 1_{\mathcal{A}}^* a_{ii} + \alpha_4 a_{ii} 1_{\mathcal{A}} p_j^* + \alpha_5 p_j a_{ii} 1_{\mathcal{A}} + \alpha_6 a_{ii} p_j^* 1_{\mathcal{A}}^*)\\
&+ \Phi(\alpha_1 1_{\mathcal{A}} p_j a_{ij} + \alpha_2 1_{\mathcal{A}}^* a_{ij} p_j^* + \alpha_3 p_j 1_{\mathcal{A}}^* a_{ij} + \alpha_4 a_{ij} 1_{\mathcal{A}} p_j^* + \alpha_5 p_j a_{ij} 1_{\mathcal{A}} + \alpha_6 a_{ij} p_j^* 1_{\mathcal{A}}^*)\\
&+ \Phi(\alpha_1 1_{\mathcal{A}} p_j a_{ji} + \alpha_2 1_{\mathcal{A}}^* a_{ji} p_j^* + \alpha_3 p_j 1_{\mathcal{A}}^* a_{ji} + \alpha_4 a_{ji} 1_{\mathcal{A}} p_j^* + \alpha_5 p_j a_{ji} 1_{\mathcal{A}} + \alpha_6 a_{ji} p_j^* 1_{\mathcal{A}}^*)
\end{aligned}$$

from which we get $\alpha_1 1_{\mathcal{A}} p_j t + \alpha_2 1_{\mathcal{A}}^* t p_j^* + \alpha_3 p_j 1_{\mathcal{A}}^* t + \alpha_4 t 1_{\mathcal{A}} p_j^* + \alpha_5 p_j t 1_{\mathcal{A}} + \alpha_6 t p_j^* 1_{\mathcal{A}}^* = 0$, by Step 1(ii). It therefore follows that $(\alpha_1 + \alpha_3 + \alpha_5)(t_{ji} + t_{jj}) + (\alpha_2 + \alpha_4 + \alpha_6)(t_{ij} + t_{jj}) = 0$ which results that $t_{ij} = 0$, $t_{ji} = 0$ and $t_{jj} = 0$, by directness of the Peirce decomposition and by hypotheses on the rational scalars. Next, as

$$\begin{aligned}
&\alpha_1 1_{\mathcal{A}}(p_i - \gamma p_j)a_{ji} + \alpha_2 1_{\mathcal{A}}^* a_{ji}(p_i - \gamma p_j)^* + \alpha_3(p_i - \gamma p_j)1_{\mathcal{A}}^* a_{ji}\\
&+ \alpha_4 a_{ji} 1_{\mathcal{A}}(p_i - \gamma p_j)^* + \alpha_5(p_i - \gamma p_j)a_{ji} 1_{\mathcal{A}} + \alpha_6 a_{ji}(p_i - \gamma p_j)^* 1_{\mathcal{A}}^* = 0,
\end{aligned}$$

then, by Step 3, we have

$$\begin{aligned}
&\Phi(\alpha_1 1_{\mathcal{A}}(p_i - \gamma p_j)(a_{ii} + a_{ij} + a_{ji}) + \alpha_2 1_{\mathcal{A}}^*(a_{ii} + a_{ij} + a_{ji})(p_i - \gamma p_j)^* + \alpha_3(p_i - \gamma p_j)1_{\mathcal{A}}^*(a_{ii} + a_{ij} + a_{ji})\\
&+ \alpha_4(a_{ii} + a_{ij} + a_{ji})1_{\mathcal{A}}(p_i - \gamma p_j)^* + \alpha_5(p_i - \gamma p_j)(a_{ii} + a_{ij} + a_{ji})1_{\mathcal{A}} + \alpha_6(a_{ii} + a_{ij} + a_{ji})(p_i - \gamma p_j)^* 1_{\mathcal{A}}^*)\\
=&\Phi(\alpha_1 1_{\mathcal{A}}(p_i - \gamma p_j)a_{ii} + \alpha_2 1_{\mathcal{A}}^* a_{ii}(p_i - \gamma p_j)^* + \alpha_3(p_i - \gamma p_j)1_{\mathcal{A}}^* a_{ii} + \alpha_4 a_{ii} 1_{\mathcal{A}}(p_i - \gamma p_j)^* + \alpha_5(p_i - \gamma p_j)a_{ii} 1_{\mathcal{A}}\\
&+ \alpha_6 a_{ii}(p_i - \gamma p_j)^* 1_{\mathcal{A}}^*) + \Phi(\alpha_1 1_{\mathcal{A}}(p_i - \gamma p_j)a_{ij} + \alpha_2 1_{\mathcal{A}}^* a_{ij}(p_i - \gamma p_j)^* + \alpha_3(p_i - \gamma p_j)1_{\mathcal{A}}^* a_{ij} + \alpha_4 a_{ij} 1_{\mathcal{A}}(p_i - \gamma p_j)^*\\
&+ \alpha_5(p_i - \gamma p_j)a_{ij} 1_{\mathcal{A}} + \alpha_6 a_{ij}(p_i - \gamma p_j)^* 1_{\mathcal{A}}^*) + \Phi(\alpha_1 1_{\mathcal{A}}(p_i - \gamma p_j)a_{ji} + \alpha_2 1_{\mathcal{A}}^* a_{ji}(p_i - \gamma p_j)^* + \alpha_3(p_i - \gamma p_j)1_{\mathcal{A}}^* a_{ji}\\
&+ \alpha_4 a_{ji} 1_{\mathcal{A}}(p_i - \gamma p_j)^* + \alpha_5(p_i - \gamma p_j)a_{ji} 1_{\mathcal{A}} + \alpha_6 a_{ji}(p_i - \gamma p_j)^* 1_{\mathcal{A}}^*)
\end{aligned}$$

from which we get

$$\alpha_1 1_{\mathcal{A}}(p_i - \gamma p_j)t + \alpha_2 1_{\mathcal{A}}^* t(p_i - \gamma p_j)^* + \alpha_3(p_i - \gamma p_j)1_{\mathcal{A}}^* t + \alpha_4 t 1_{\mathcal{A}}(p_i - \gamma p_j)^* + \alpha_5(p_i - \gamma p_j)t 1_{\mathcal{A}} + \alpha_6 t(p_i - \gamma p_j)^* 1_{\mathcal{A}}^* = 0.$$

This implies that $(\sum_{k=1}^{6} \alpha_k)t_{ii} = 0$ which yields $t_{ii} = 0$. Similarly, we prove the other case. □

Step 5. For all $a_{ii} \in \mathcal{A}_{ii}$, $a_{ij} \in \mathcal{A}_{ij}$, $a_{ji} \in \mathcal{A}_{ji}$ and $a_{jj} \in \mathcal{A}_{jj}$ $(i \neq j; i, j = 1, 2)$, we have $\Phi(\sum_{i,j=1,2} a_{ij}) = \sum_{i,j=1,2} \Phi(a_{ij})$.

Proof . Take $t = \Phi(\sum_{i,j=1,2} a_{ij}) - \sum_{i,j=1,2} \Phi(a_{ij})$ $(= \Phi(a_{ii} + a_{ij} + a_{ji} + a_{jj}) - \Phi(a_{ii}) - \Phi(a_{ij}) - \Phi(a_{ji}) - \Phi(a_{jj}))$ and $t = t_{ii} + t_{ij} + t_{ji} + t_{jj}$ in the Peirce decomposition of $\mathcal{A}$, relative to p_1. Then, for $s = p_i$ or $s = p_j$, by Step 4 we have

$$\begin{aligned}
&\Phi(\alpha_1 1_{\mathcal{A}} s(\textstyle\sum_{i,j=1,2} a_{ij}) + \alpha_2 1_{\mathcal{A}}^*(\sum_{i,j=1,2} a_{ij})s^* + \alpha_3 s 1_{\mathcal{A}}^*(\sum_{i,j=1,2} a_{ij})\\
&+ \alpha_4(\textstyle\sum_{i,j=1,2} a_{ij})1_{\mathcal{A}} s^* + \alpha_5 s(\sum_{i,j=1,2} a_{ij})1_{\mathcal{A}} + \alpha_6(\sum_{i,j=1,2} a_{ij})s^* 1_{\mathcal{A}}^*\\
=& \textstyle\sum_{i,j=1,2} \Phi(\alpha_1 1_{\mathcal{A}} s a_{ij} + \alpha_2 1_{\mathcal{A}}^* a_{ij} s^* + \alpha_3 s 1_{\mathcal{A}}^* a_{ij} + \alpha_4 a_{ij} 1_{\mathcal{A}} s^* + \alpha_5 s a_{ij} 1_{\mathcal{A}} + \alpha_6 a_{ij} s^* 1_{\mathcal{A}}^*)
\end{aligned}$$

which implies the identity

$$\alpha_1 1_{\mathcal{A}} st + \alpha_2 1_{\mathcal{A}}^* ts^* + \alpha_3 s 1_{\mathcal{A}}^* t + \alpha_4 t 1_{\mathcal{A}} s^* + \alpha_5 st 1_{\mathcal{A}} + \alpha_6 ts^* 1_{\mathcal{A}}^* = 0 \tag{2.13}$$

If $s = p_i$, then (2.13) becomes the identity

$$(\alpha_1 + \alpha_3 + \alpha_5)(t_{ii} + t_{ij}) + (\alpha_2 + \alpha_4 + \alpha_6)(t_{ii} + t_{ji}) = 0. \tag{2.14}$$

If $s = p_j$, then (2.13) becomes the identity

$$(\alpha_1 + \alpha_3 + \alpha_5)(t_{ji} + t_{jj}) + (\alpha_2 + \alpha_4 + \alpha_6)(t_{ij} + t_{jj}) = 0. \tag{2.15}$$

Adding (2.14) and (2.15), we get $(\sum_{k=1}^{6} \alpha_k)(t_{ii} + t_{ij} + t_{ji} + t_{jj}) = 0$. This shows that $t = 0$. □

Step 6. For all $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$ $(i \neq j; i, j = 1, 2)$ we have $\Phi(a_{ij} + b_{ij}) = \Phi(a_{ij}) + \Phi(b_{ij})$.

Proof . First, note that the following identity holds

$$\begin{aligned}(\alpha_1+\alpha_3+\alpha_5)(a_{ij}+b_{ij})+(\alpha_2+\alpha_4+\alpha_6)(a_{ij}^*+b_{ij}a_{ij}^*) =&\alpha_1 1_{\mathcal{A}}(p_i+a_{ij})(p_j+b_{ij})+\alpha_2 1_{\mathcal{A}}^*(p_j+b_{ij})(p_i+a_{ij})^*\\ &+\alpha_3(p_i+a_{ij})1_{\mathcal{A}}^*(p_j+b_{ij})+\alpha_4(p_j+b_{ij})1_{\mathcal{A}}(p_i+a_{ij})^*\\ &+\alpha_5(p_j+a_{ij})(p_i+b_{ij})1_{\mathcal{A}}+\alpha_6(p_j+b_{ij})(p_i+a_{ij})^*1_{\mathcal{A}}^*,\end{aligned}$$

for all $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$. It follows from Step 5 that

$$\begin{aligned}&\Phi((\alpha_1+\alpha_3+\alpha_5)(a_{ij}+b_{ij}))+\Phi((\alpha_2+\alpha_4+\alpha_6)(a_{ij}^*+b_{ij}a_{ij}^*))\\
=&\Phi((\alpha_1+\alpha_3+\alpha_5)(a_{ij}+b_{ij})+(\alpha_2+\alpha_4+\alpha_6)(a_{ij}^*+b_{ij}a_{ij}^*))\\
=&\Phi(\alpha_1 1_{\mathcal{A}}(p_i+a_{ij})(p_j+b_{ij})+\alpha_2 1_{\mathcal{A}}^*(p_j+b_{ij})(p_i+a_{ij})^*+\alpha_3(p_i+a_{ij})1_{\mathcal{A}}^*(p_j+b_{ij})\\
&+\alpha_4(p_j+b_{ij})1_{\mathcal{A}}(p_i+a_{ij})^*+\alpha_5(p_j+a_{ij})(p_i+b_{ij})1_{\mathcal{A}}+\alpha_6(p_j+b_{ij})(p_i+a_{ij})^*1_{\mathcal{A}}^*)\\
=&\alpha_1\Phi(1_{\mathcal{A}})(p_i+a_{ij})(p_j+b_{ij})+\alpha_2\Phi(1_{\mathcal{A}})^*(p_j+b_{ij})(p_i+a_{ij})^*+\alpha_3(p_i+a_{ij})\Phi(1_{\mathcal{A}})^*(p_j+b_{ij})\\
&+\alpha_4(p_j+b_{ij})\Phi(1_{\mathcal{A}})(p_i+a_{ij})^*+\alpha_5(p_i+a_{ij})(p_j+b_{ij})\Phi(1_{\mathcal{A}})+\alpha_6(p_j+b_{ij})(p_i+a_{ij})^*\Phi(1_{\mathcal{A}})^*\\
&+\alpha_1 1_{\mathcal{A}}\Phi(p_i+a_{ij})(p_j+b_{ij})+\alpha_2 1_{\mathcal{A}}^*(p_j+b_{ij})\Phi(p_i+a_{ij})^*+\alpha_3\Phi(p_i+a_{ij})1_{\mathcal{A}}^*(p_j+b_{ij})\\
&+\alpha_4(p_j+b_{ij})1_{\mathcal{A}}\Phi(p_i+a_{ij})^*+\alpha_5\Phi(p_i+a_{ij})(p_j+b_{ij})1_{\mathcal{A}}+\alpha_6(p_j+b_{ij})\Phi(p_i+a_{ij})^*1_{\mathcal{A}}^*\\
&+\alpha_1 1_{\mathcal{A}}(p_i+a_{ij})\Phi(p_j+b_{ij})+\alpha_2 1_{\mathcal{A}}^*\Phi(p_j+b_{ij})(p_i+a_{ij})^*+\alpha_3(p_i+a_{ij})1_{\mathcal{A}}^*\Phi(p_j+b_{ij})\\
&+\alpha_4\Phi(p_j+b_{ij})1_{\mathcal{A}}(p_i+a_{ij})^*+\alpha_5(p_i+a_{ij})\Phi(p_j+b_{ij})1_{\mathcal{A}}+\alpha_6\Phi(p_j+b_{ij})(p_i+a_{ij})^*1_{\mathcal{A}}^*\\
=&\alpha_1\Phi(1_{\mathcal{A}})p_ip_j+\alpha_2\Phi(1_{\mathcal{A}})^*p_jp_i^*+\alpha_3p_i\Phi(1_{\mathcal{A}})^*p_j+\alpha_4p_j\Phi(1_{\mathcal{A}})p_i^*+\alpha_5p_ip_j\Phi(1_{\mathcal{A}})+\alpha_6p_jp_i^*\Phi(1_{\mathcal{A}})^*\\
&+\alpha_1\Phi(1_{\mathcal{A}})a_{ij}p_j+\alpha_2\Phi(1_{\mathcal{A}})^*p_ja_{ij}^*+\alpha_3a_{ij}\Phi(1_{\mathcal{A}})^*p_j+\alpha_4p_j\Phi(1_{\mathcal{A}})a_{ij}^*+\alpha_5a_{ij}p_j\Phi(1_{\mathcal{A}})+\alpha_6p_ja_{ij}^*\Phi(1_{\mathcal{A}})^*\\
&+\alpha_1\Phi(1_{\mathcal{A}})p_ib_{ij}+\alpha_2\Phi(1_{\mathcal{A}})^*b_{ij}p_i^*+\alpha_3p_i\Phi(1_{\mathcal{A}})^*b_{ij}+\alpha_4b_{ij}\Phi(1_{\mathcal{A}})p_i^*\\
&+\alpha_5p_ib_{ij}\Phi(1_{\mathcal{A}})+\alpha_6b_{ij}p_i^*\Phi(1_{\mathcal{A}})^*+\alpha_1\Phi(1_{\mathcal{A}})a_{ij}b_{ij}+\alpha_2\Phi(1_{\mathcal{A}})^*b_{ij}a_{ij}^*\\
&+\alpha_3a_{ij}\Phi(1_{\mathcal{A}})^*b_{ij}+\alpha_4b_{ij}\Phi(1_{\mathcal{A}})a_{ij}^*+\alpha_5a_{ij}b_{ij}\Phi(1_{\mathcal{A}})+\alpha_6b_{ij}a_{ij}^*\Phi(1_{\mathcal{A}})^*\\
&+\alpha_1 1_{\mathcal{A}}\Phi(p_i)p_j+\alpha_2 1_{\mathcal{A}}^*p_j\Phi(p_i)^*+\alpha_3\Phi(p_i)1_{\mathcal{A}}^*p_j+\alpha_4p_j1_{\mathcal{A}}\Phi(p_i)^*\\
&+\alpha_5\Phi(p_i)p_j1_{\mathcal{A}}+\alpha_6p_j\Phi(p_i)^*1_{\mathcal{A}}^*+\alpha_1 1_{\mathcal{A}}\Phi(a_{ij})p_j+\alpha_2 1_{\mathcal{A}}^*p_j\Phi(a_{ij})^*\\
&+\alpha_3\Phi(a_{ij})1_{\mathcal{A}}^*p_j+\alpha_4p_j1_{\mathcal{A}}\Phi(a_{ij})^*+\alpha_5\Phi(a_{ij})p_j1_{\mathcal{A}}+\alpha_6p_j\Phi(a_{ij})^*1_{\mathcal{A}}^*\\
&+\alpha_1 1_{\mathcal{A}}\Phi(p_i)b_{ij}+\alpha_2 1_{\mathcal{A}}^*b_{ij}\Phi(p_i)^*+\alpha_3\Phi(p_i)1_{\mathcal{A}}^*b_{ij}+\alpha_4b_{ij}1_{\mathcal{A}}\Phi(p_i)^*\\
&+\alpha_5\Phi(p_i)b_{ij}1_{\mathcal{A}}+\alpha_6b_{ij}\Phi(p_i)^*1_{\mathcal{A}}^*+\alpha_1 1_{\mathcal{A}}\Phi(a_{ij})b_{ij}+\alpha_2 1_{\mathcal{A}}^*b_{ij}\Phi(a_{ij})^*\\
&+\alpha_3\Phi(a_{ij})1_{\mathcal{A}}^*b_{ij}+\alpha_4b_{ij}1_{\mathcal{A}}\Phi(a_{ij})^*+\alpha_5\Phi(a_{ij})b_{ij}1_{\mathcal{A}}+\alpha_6b_{ij}\Phi(a_{ij})^*1_{\mathcal{A}}^*\\
&+\alpha_1 1_{\mathcal{A}}p_i\Phi(p_j)+\alpha_2 1_{\mathcal{A}}^*\Phi(p_j)p_i^*+\alpha_3p_i1_{\mathcal{A}}^*\Phi(p_j)+\alpha_4\Phi(p_j)1_{\mathcal{A}}p_i^*\\
&+\alpha_5p_i\Phi(p_j)1_{\mathcal{A}}+\alpha_6\Phi(p_j)p_i^*1_{\mathcal{A}}^*+\alpha_1 1_{\mathcal{A}}a_{ij}\Phi(p_j)+\alpha_2 1_{\mathcal{A}}^*\Phi(p_j)a_{ij}^*\\
&+\alpha_3a_{ij}1_{\mathcal{A}}^*\Phi(p_j)+\alpha_4\Phi(p_j)1_{\mathcal{A}}a_{ij}^*+\alpha_5a_{ij}\Phi(p_j)1_{\mathcal{A}}+\alpha_6\Phi(p_j)a_{ij}^*1_{\mathcal{A}}^*\\
&+\alpha_1 1_{\mathcal{A}}p_i\Phi(b_{ij})+\alpha_2 1_{\mathcal{A}}^*\Phi(b_{ij})p_i^*+\alpha_3p_i1_{\mathcal{A}}^*\Phi(b_{ij})+\alpha_4\Phi(b_{ij})1_{\mathcal{A}}p_i^*\\
&+\alpha_5p_i\Phi(b_{ij})1_{\mathcal{A}}+\alpha_6\Phi(b_{ij})p_i^*1_{\mathcal{A}}^*+\alpha_1 1_{\mathcal{A}}a_{ij}\Phi(b_{ij})+\alpha_2 1_{\mathcal{A}}^*\Phi(b_{ij})a_{ij}^*\\
&+\alpha_3a_{ij}1_{\mathcal{A}}^*\Phi(b_{ij})+\alpha_4\Phi(b_{ij})1_{\mathcal{A}}a_{ij}^*+\alpha_5a_{ij}\Phi(b_{ij})1_{\mathcal{A}}+\alpha_6\Phi(b_{ij})a_{ij}^*1_{\mathcal{A}}^*\\
=&\Phi(\alpha_1 1_{\mathcal{A}}p_ip_j+\alpha_2 1_{\mathcal{A}}^*p_jp_i^*+\alpha_3p_i1_{\mathcal{A}}^*p_j+\alpha_4p_j1_{\mathcal{A}}p_i^*+\alpha_5p_ip_j1_{\mathcal{A}}+\alpha_6p_jp_i^*1_{\mathcal{A}}^*)\\
&+\Phi(\alpha_1 1_{\mathcal{A}}a_{ij}p_j+\alpha_2 1_{\mathcal{A}}^*p_ja_{ij}^*+\alpha_3a_{ij}1_{\mathcal{A}}^*p_j+\alpha_4p_j1_{\mathcal{A}}a_{ij}^*+\alpha_5a_{ij}p_j1_{\mathcal{A}}+\alpha_6p_ja_{ij}^*1_{\mathcal{A}}^*)\\
&+\Phi(\alpha_1 1_{\mathcal{A}}p_ib_{ij}+\alpha_2 1_{\mathcal{A}}^*b_{ij}p_i^*+\alpha_3p_i1_{\mathcal{A}}^*b_{ij}+\alpha_4b_{ij}1_{\mathcal{A}}p_i^*+\alpha_5p_ib_{ij}1_{\mathcal{A}}+\alpha_6b_{ij}p_i^*1_{\mathcal{A}}^*)\\
&+\Phi(\alpha_1 1_{\mathcal{A}}a_{ij}b_{ij}+\alpha_2 1_{\mathcal{A}}^*b_{ij}a_{ij}^*+\alpha_3a_{ij}1_{\mathcal{A}}^*b_{ij}+\alpha_4b_{ij}1_{\mathcal{A}}a_{ij}^*+\alpha_5a_{ij}b_{ij}1_{\mathcal{A}}+\alpha_6b_{ij}a_{ij}^*1_{\mathcal{A}}^*)\\
=&\Phi((\alpha_1+\alpha_3+\alpha_5)a_{ij})+\Phi((\alpha_2+\alpha_4+\alpha_6)a_{ij}^*)+\Phi((\alpha_1+\alpha_3+\alpha_5)b_{ij})+\Phi((\alpha_2+\alpha_4+\alpha_6)b_{ij}a_{ij}^*).\end{aligned}$$

This shows that

$$\Phi((\alpha_1+\alpha_3+\alpha_5)(a_{ij}+b_{ij}))=\Phi((\alpha_1+\alpha_3+\alpha_5)a_{ij})+\Phi((\alpha_1+\alpha_3+\alpha_5)b_{ij}).$$

This last result allows us to conclude that $\Phi(a_{ij}+b_{ij}) = \Phi(a_{ij}) + \Phi(b_{ij})$, for all $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$. □

Step 7. For all $a_{ii}, b_{ii} \in \mathcal{A}_{ii}$ $(i = 1,2)$, we have $\Phi(a_{ii}+b_{ii}) = \Phi(a_{ii}) + \Phi(b_{ii})$.

Proof . Take $t = \Phi(a_{ii}+b_{ii}) - \Phi(a_{ii}) - \Phi(b_{ii})$ and $t = t_{ii}+t_{ij}+t_{ji}+t_{jj}$ $(i \neq j)$ in the Peirce decomposition of $\mathcal{A}$, relative to p_1. Since

$$\begin{aligned}&\Phi(\alpha_1 1_{\mathcal{A}}(a_{ii}+b_{ii})p_j + \alpha_2 1_{\mathcal{A}}^* p_j (a_{ii}+b_{ii})^* + \alpha_3 (a_{ii}+b_{ii}) 1_{\mathcal{A}}^* p_j \\ &\quad + \alpha_4 p_j 1_{\mathcal{A}} (a_{ii}+b_{ii})^* + \alpha_5 (a_{ii}+b_{ii}) p_j 1_{\mathcal{A}} + \alpha_6 p_j (a_{ii}+b_{ii})^* 1_{\mathcal{A}}^*) \\ =&\Phi(\alpha_1 1_{\mathcal{A}} a_{ii} p_j + \alpha_2 1_{\mathcal{A}}^* p_j a_{ii}^* + \alpha_3 a_{ii} 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} a_{ii}^* + \alpha_5 a_{ii} p_j 1_{\mathcal{A}} + \alpha_6 p_j a_{ii}^* 1_{\mathcal{A}}^*) \\ &\quad + \Phi(\alpha_1 1_{\mathcal{A}} b_{ii} p_j + \alpha_2 1_{\mathcal{A}}^* p_j b_{ii}^* + \alpha_3 b_{ii} 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} b_{ii}^* + \alpha_5 b_{ii} p_j 1_{\mathcal{A}} + \alpha_6 p_j b_{ii}^* 1_{\mathcal{A}}^*),\end{aligned}$$

it follows that

$$\alpha_1 1_{\mathcal{A}} t p_j + \alpha_2 1_{\mathcal{A}}^* p_j t^* + \alpha_3 t 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} t^* + \alpha_5 t p_j 1_{\mathcal{A}} + \alpha_6 p_j t^* 1_{\mathcal{A}}^* = 0$$

from which we get

$$(\alpha_1+\alpha_3+\alpha_5)(t_{ij}+t_{jj}) + (\alpha_2+\alpha_4+\alpha_6)(t_{ij}+t_{jj})^* = 0. \tag{2.16}$$

By applying involution to (2.16), we deduce the identity

$$(\alpha_2+\alpha_4+\alpha_6)(t_{ij}+t_{jj}) + (\alpha_1+\alpha_3+\alpha_5)(t_{ij}+t_{jj})^* = 0. \tag{2.17}$$

Multiplying (2.16) by $\alpha_1+\alpha_3+\alpha_5$, (2.17) by $\alpha_2+\alpha_4+\alpha_6$ and subtracting the resulting identities, we obtain $((\alpha_1+\alpha_3+\alpha_5)^2 - (\alpha_2+\alpha_4+\alpha_6)^2)(t_{ij}+t_{jj}) = 0$. This shows that $t_{ij}+t_{jj} = 0$. Next, for all $s_{ij} \in \mathcal{A}_{ij}$, we have

$$\begin{aligned}&\Phi(\alpha_1 1_{\mathcal{A}} s_{ij}(a_{ii}+b_{ii}) + \alpha_2 1_{\mathcal{A}}^*(a_{ii}+b_{ii}) s_{ij}^* + \alpha_3 s_{ij} 1_{\mathcal{A}}^* (a_{ii}+b_{ii}) \\ &\quad + \alpha_4 (a_{ii}+b_{ii}) 1_{\mathcal{A}} s_{ij}^* + \alpha_5 s_{ij}(a_{ii}+b_{ii}) 1_{\mathcal{A}} + \alpha_6 (a_{ii}+b_{ii}) s_{ij}^* 1_{\mathcal{A}}^*) \\ =&\Phi(\alpha_1 1_{\mathcal{A}} s_{ij} a_{ii} + \alpha_2 1_{\mathcal{A}}^* a_{ii} s_{ij}^* + \alpha_3 s_{ij} 1_{\mathcal{A}}^* a_{ii} + \alpha_4 a_{ii} 1_{\mathcal{A}} s_{ij}^* + \alpha_5 s_{ij} a_{ii} 1_{\mathcal{A}} + \alpha_6 a_{ii} s_{ij}^* 1_{\mathcal{A}}^*) \\ &\quad + \Phi(\alpha_1 1_{\mathcal{A}} s_{ij} b_{ii} + \alpha_2 1_{\mathcal{A}}^* b_{ii} s_{ij}^* + \alpha_3 s_{ij} 1_{\mathcal{A}}^* b_{ii} + \alpha_4 b_{ii} 1_{\mathcal{A}} s_{ij}^* + \alpha_5 s_{ij} b_{ii} 1_{\mathcal{A}} + \alpha_6 b_{ii} s_{ij}^* 1_{\mathcal{A}}^*).\end{aligned}$$

It follows that

$$\alpha_1 1_{\mathcal{A}} s_{ij} t + \alpha_2 1_{\mathcal{A}}^* t s_{ij}^* + \alpha_3 s_{ij} 1_{\mathcal{A}}^* t + \alpha_4 t 1_{\mathcal{A}} s_{ij}^* + \alpha_5 s_{ij} t 1_{\mathcal{A}} + \alpha_6 t s_{ij}^* 1_{\mathcal{A}}^* = 0$$

from which we get $(\alpha_1+\alpha_3+\alpha_5)s_{ij}t_{ji} = 0$. As consequence we have $t_{ji} = 0$, in view of the properties in $(\spadesuit)$, of Theorem 2.1, and of the hypotheses on the rational scalars. Also, by Steps 5 and 6, for all $s_{ji} \in \mathcal{A}_{ji}$ we have

$$\begin{aligned}&\Phi(\alpha_1 1_{\mathcal{A}} s_{ji}(a_{ii}+b_{ii}) + \alpha_2 1_{\mathcal{A}}^*(a_{ii}+b_{ii}) s_{ji}^* + \alpha_3 s_{ji} 1_{\mathcal{A}}^* (a_{ii}+b_{ii}) \\ &\quad + \alpha_4 (a_{ii}+b_{ii}) 1_{\mathcal{A}} s_{ji}^* + \alpha_5 s_{ji}(a_{ii}+b_{ii}) 1_{\mathcal{A}} + \alpha_6 (a_{ii}+b_{ii}) s_{ji}^* 1_{\mathcal{A}}^*) \\ =&\Phi(\alpha_1 1_{\mathcal{A}} s_{ji} a_{ii} + \alpha_2 1_{\mathcal{A}}^* a_{ii} s_{ji}^* + \alpha_3 s_{ji} 1_{\mathcal{A}}^* a_{ii} + \alpha_4 a_{ii} 1_{\mathcal{A}} s_{ji}^* + \alpha_5 s_{ji} a_{ii} 1_{\mathcal{A}} + \alpha_6 a_{ii} s_{ji}^* 1_{\mathcal{A}}^*) \\ &\quad + \Phi(\alpha_1 1_{\mathcal{A}} s_{ji} b_{ii} + \alpha_2 1_{\mathcal{A}}^* b_{ii} s_{ji}^* + \alpha_3 s_{ji} 1_{\mathcal{A}}^* b_{ii} + \alpha_4 b_{ii} 1_{\mathcal{A}} s_{ji}^* + \alpha_5 s_{ji} b_{ii} 1_{\mathcal{A}} + \alpha_6 b_{ii} s_{ji}^* 1_{\mathcal{A}}^*).\end{aligned}$$

It follows that

$$\alpha_1 1_{\mathcal{A}} s_{ji} t + \alpha_2 1_{\mathcal{A}}^* t s_{ji}^* + \alpha_3 s_{ji} 1_{\mathcal{A}}^* t + \alpha_4 t 1_{\mathcal{A}} s_{ji}^* + \alpha_5 s_{ji} t 1_{\mathcal{A}} + \alpha_6 t s_{ji}^* 1_{\mathcal{A}}^* = 0$$

from which we get $(\alpha_1+\alpha_3+\alpha_5)s_{ji}t_{ii} + (\alpha_2+\alpha_4+\alpha_6)t_{ii}s_{ji}^* = 0$. From this, we deduce that $t_{ii} = 0$, by directness of the Peirce decomposition, by properties in $(\spadesuit)$, of Theorem 2.1, and by hypotheses on the rational scalars. This shows that $t = 0$. □

Therefore, it follows from Steps 5, 6 and 7 that Φ is additive. The Claim 2 is proved. In the remainder of this paper, we assume in addition that $\alpha_1 - \alpha_2 + \alpha_5 - \alpha_6 \neq 0$.

Claim 3. $\Phi(1_{\mathcal{A}}) = 0$ and $\Phi(b^*) = \Phi(b)^*$, for all $b \in \mathcal{A}$.

Proof . First, note that

$$\begin{aligned}(\textstyle\sum_{k=1}^{6}\alpha_k)\Phi(1_{\mathcal{A}}) =&\Phi(\alpha_1 1_{\mathcal{A}}1_{\mathcal{A}}1_{\mathcal{A}} + \alpha_2 1_{\mathcal{A}}^* 1_{\mathcal{A}}1_{\mathcal{A}}^* + \alpha_3 1_{\mathcal{A}}1_{\mathcal{A}}^*1_{\mathcal{A}} + \alpha_4 1_{\mathcal{A}}1_{\mathcal{A}}1_{\mathcal{A}}^* + \alpha_5 1_{\mathcal{A}}1_{\mathcal{A}}1_{\mathcal{A}} + \alpha_6 1_{\mathcal{A}}1_{\mathcal{A}}^*1_{\mathcal{A}}^*)\\
=&\alpha_1\Phi(1_{\mathcal{A}})1_{\mathcal{A}}1_{\mathcal{A}} + \alpha_2\Phi(1_{\mathcal{A}})^*1_{\mathcal{A}}1_{\mathcal{A}}^* + \alpha_3 1_{\mathcal{A}}\Phi(1_{\mathcal{A}})^*1_{\mathcal{A}} + \alpha_4 1_{\mathcal{A}}\Phi(1_{\mathcal{A}})1_{\mathcal{A}}^* + \alpha_5 1_{\mathcal{A}}1_{\mathcal{A}}\Phi(1_{\mathcal{A}})\\
&+ \alpha_6 1_{\mathcal{A}}1_{\mathcal{A}}^*\Phi(1_{\mathcal{A}})^* + \alpha_1 1_{\mathcal{A}}\Phi(1_{\mathcal{A}})1_{\mathcal{A}} + \alpha_2 1_{\mathcal{A}}^*1_{\mathcal{A}}\Phi(1_{\mathcal{A}})^* + \alpha_3\Phi(1_{\mathcal{A}})1_{\mathcal{A}}^*1_{\mathcal{A}} + \alpha_4 1_{\mathcal{A}}1_{\mathcal{A}}\Phi(1_{\mathcal{A}})^*\\
&+ \alpha_5\Phi(1_{\mathcal{A}})1_{\mathcal{A}}1_{\mathcal{A}} + \alpha_6 1_{\mathcal{A}}\Phi(1_{\mathcal{A}})^*1_{\mathcal{A}}^* + \alpha_1 1_{\mathcal{A}}1_{\mathcal{A}}\Phi(1_{\mathcal{A}}) + \alpha_2 1_{\mathcal{A}}^*\Phi(1_{\mathcal{A}})1_{\mathcal{A}}^*\\
&+ \alpha_3 1_{\mathcal{A}}1_{\mathcal{A}}^*\Phi(1_{\mathcal{A}}) + \alpha_4\Phi(1_{\mathcal{A}})1_{\mathcal{A}}1_{\mathcal{A}}^* + \alpha_5 1_{\mathcal{A}}\Phi(1_{\mathcal{A}})1_{\mathcal{A}} + \alpha_6\Phi(1_{\mathcal{A}})1_{\mathcal{A}}^*1_{\mathcal{A}}^*\\
=&(3\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 3\alpha_5 + \alpha_6)\Phi(1_{\mathcal{A}}) + (2\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_6)\Phi(1_{\mathcal{A}})^*. \qquad (2.18)\end{aligned}$$

Hence, the identity (2.18) becomes

$$(2\alpha_1 + \alpha_3 + \alpha_4 + 2\alpha_5)\Phi(1_{\mathcal{A}}) + (2\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_6)\Phi(1_{\mathcal{A}})^* = 0 \qquad (2.19)$$

By applying involution to the identity (2.19), we get

$$(2\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_6)\Phi(1_{\mathcal{A}}) + (2\alpha_1 + \alpha_3 + \alpha_4 + 2\alpha_5)\Phi(1_{\mathcal{A}})^* = 0. \qquad (2.20)$$

Adding (2.19) and (2.20) we obtain $2(\sum_{k=1}^{6}\alpha_k)(\Phi(1_{\mathcal{A}}) + \Phi(1_{\mathcal{A}})^*) = 0$, which yields

$$\Phi(1_{\mathcal{A}})^* = -\Phi(1_{\mathcal{A}}). \qquad (2.21)$$

Replacing (2.21) in (2.19) we arrive at $2(\alpha_1-\alpha_2+\alpha_5-\alpha_6)\Phi(1_{\mathcal{A}}) = 0$ which shows that $\Phi(1_{\mathcal{A}}) = 0$. As a consequence, for all $b \in \mathcal{A}$, we have

$$\begin{aligned}&\Phi(\alpha_1 1_{\mathcal{A}}b1_{\mathcal{A}} + \alpha_2 1_{\mathcal{A}}^*1_{\mathcal{A}}b^* + \alpha_3 b1_{\mathcal{A}}^*1_{\mathcal{A}} + \alpha_4 1_{\mathcal{A}}1_{\mathcal{A}}b^* + \alpha_5 b1_{\mathcal{A}}1_{\mathcal{A}} + \alpha_6 1_{\mathcal{A}}b^*1_{\mathcal{A}}^*)\\
=&\alpha_1\Phi(1_{\mathcal{A}})b1_{\mathcal{A}} + \alpha_2\Phi(1_{\mathcal{A}})^*1_{\mathcal{A}}b^* + \alpha_3 b\Phi(1_{\mathcal{A}})^*1_{\mathcal{A}} + \alpha_4 1_{\mathcal{A}}\Phi(1_{\mathcal{A}})b^* + \alpha_5 b1_{\mathcal{A}}\Phi(1_{\mathcal{A}})\\
&+ \alpha_6 1_{\mathcal{A}}b^*\Phi(1_{\mathcal{A}})^* + \alpha_1 1_{\mathcal{A}}\Phi(b)1_{\mathcal{A}} + \alpha_2 1_{\mathcal{A}}^*1_{\mathcal{A}}\Phi(b)^* + \alpha_3\Phi(b)1_{\mathcal{A}}^*1_{\mathcal{A}} + \alpha_4 1_{\mathcal{A}}1_{\mathcal{A}}\Phi(b)^* + \alpha_5\Phi(b)1_{\mathcal{A}}1_{\mathcal{A}}\\
&+ \alpha_6 1_{\mathcal{A}}\Phi(b)^*1_{\mathcal{A}}^* + \alpha_1 1_{\mathcal{A}}b\Phi(1_{\mathcal{A}}) + \alpha_2 1_{\mathcal{A}}^*\Phi(1_{\mathcal{A}})b^* + \alpha_3 b1_{\mathcal{A}}^*\Phi(1_{\mathcal{A}})\\
&+ \alpha_4\Phi(1_{\mathcal{A}})1_{\mathcal{A}}b^* + \alpha_5 b\Phi(1_{\mathcal{A}})1_{\mathcal{A}} + \alpha_6\Phi(1_{\mathcal{A}})b^*1_{\mathcal{A}}^*\end{aligned}$$

which result that $(\alpha_1 + \alpha_3 + \alpha_5)\Phi(b) + (\alpha_2 + \alpha_4 + \alpha_6)\Phi(b^*) = (\alpha_1 + \alpha_3 + \alpha_5)\Phi(b) + (\alpha_2 + \alpha_4 + \alpha_6)\Phi(b)^*$. Therefore $\Phi(b^*) = \Phi(b)^*$. □

Claim 4. Φ is a multiplicative derivation.

Proof . For all $b, c \in \mathcal{A}$, replace a by $1_{\mathcal{A}}$ in the identity (1.1). Then

$$\begin{aligned}&\Phi(\alpha_1 1_{\mathcal{A}}bc + \alpha_2 1_{\mathcal{A}}^*cb^* + \alpha_3 b1_{\mathcal{A}}^*c + \alpha_4 c1_{\mathcal{A}}b^* + \alpha_5 bc1_{\mathcal{A}} + \alpha_6 cb^*1_{\mathcal{A}}^*)\\
=&\alpha_1\Phi(1_{\mathcal{A}})bc + \alpha_2\Phi(1_{\mathcal{A}})^*cb^* + \alpha_3 b\Phi(1_{\mathcal{A}})^*c + \alpha_4 c\Phi(1_{\mathcal{A}})b^* + \alpha_5 bc\Phi(1_{\mathcal{A}}) + \alpha_6 cb^*\Phi(1_{\mathcal{A}})^* + \alpha_1 1_{\mathcal{A}}\Phi(b)c + \alpha_2 1_{\mathcal{A}}^*c\Phi(b)^*\\
&+ \alpha_3\Phi(b)1_{\mathcal{A}}^*c + \alpha_4 c1_{\mathcal{A}}\Phi(b)^* + \alpha_5\Phi(b)c1_{\mathcal{A}} + \alpha_6 c\Phi(b)^*1_{\mathcal{A}}^* + \alpha_1 1_{\mathcal{A}}b\Phi(c) + \alpha_2 1_{\mathcal{A}}^*\Phi(c)b^* + \alpha_3 b1_{\mathcal{A}}^*\Phi(c)\\
&+ \alpha_4\Phi(c)1_{\mathcal{A}}b^* + \alpha_5 b\Phi(c)1_{\mathcal{A}} + \alpha_6\Phi(c)b^*1_{\mathcal{A}}^*\end{aligned}$$

which implies that

$$\begin{aligned}\Phi((\alpha_1 + \alpha_3 + \alpha_5)bc + (\alpha_2 + \alpha_4 + \alpha_6)cb^*) =&(\alpha_1 + \alpha_3 + \alpha_5)(\Phi(b)c + b\Phi(c))\\
&+ (\alpha_2 + \alpha_4 + \alpha_6)(\Phi(c)b^* + c\Phi(b)^*). \qquad (2.22)\end{aligned}$$

Next, replacing in (2.22) c by c^*, we get

$$\begin{aligned}\Phi((\alpha_1 + \alpha_3 + \alpha_5)bc^* + (\alpha_2 + \alpha_4 + \alpha_6)c^*b^*) =&(\alpha_1 + \alpha_3 + \alpha_5)(\Phi(b)c^* + b\Phi(c^*))\\
&+ (\alpha_2 + \alpha_4 + \alpha_6)(\Phi(c^*)b^* + c^*\Phi(b)^*) \qquad (2.23)\end{aligned}$$

and applying involution to the identity (2.23), we arrive at

$$\Phi((\alpha_2+\alpha_4+\alpha_6)bc+(\alpha_1+\alpha_3+\alpha_5)cb^*) = (\alpha_2+\alpha_4+\alpha_6)(\Phi(b)c+b\Phi(c)) + (\alpha_1+\alpha_3+\alpha_5)(\Phi(c)b^*+c\Phi(b)^*). \quad (2.24)$$

It follows from (2.22) and (2.24) that

$$((\alpha_1+\alpha_3+\alpha_5)^2-(\alpha_2+\alpha_4+\alpha_6)^2)(\Phi(bc)-\Phi(b)c-b\Phi(c))=0$$

which shows that $\Phi(bc)=\Phi(b)c+b\Phi(c)$. Therefore, Φ is a multiplicative derivation. □

By Claims 2, 3 and 4, we have proved that Φ is an additive $*$-derivation. This completes the proof of Theorem 2.1.

In view of this, we have the following corollaries.

Corollary 2.2. Let $\mathcal{A}$ be a unital $*$-algebra with $1_{\mathcal{A}}$ its multiplicative identity having a nontrivial projection p_1 (write $p_2=1_{\mathcal{A}}-p_1$) such that the following two properties

$$x\mathcal{A}p_l=0 \text{ implies } x=0 \ (l=1,2) \quad (\spadesuit)$$

are verified and $\Phi:\mathcal{A}\to\mathcal{A}$ a $*$-derivation-type on mixed product $a\blacklozenge_\eta b\blacklozenge_\nu c$, where η,ν are rational numbers satisfying the conditions $\eta\neq-1$ and $|\nu|\neq 0,1$. Then Φ is an additive $*$-derivation.

An algebra $\mathcal{A}$ is called *prime* if satisfies the property $a\mathcal{A}b=0$, for $a,b\in\mathcal{A}$, implies either $a=0$ or $b=0$. It is easy to see that prime $*$-algebras satisfy the two properties in $(\spadesuit)$, of the Theorem 2.1 (resp., of the Corollary 2.2).

Corollary 2.3. Let $\mathcal{A}$ be a unital prime $*$-algebra with $1_{\mathcal{A}}$ its multiplicative identity having a nontrivial projection and $\Phi:\mathcal{A}\to\mathcal{A}$ a $*$-derivation-type on mixed product $a\blacklozenge_\eta b\blacklozenge_\nu c$, where η,ν are rational numbers satisfying the conditions $\eta\neq-1$ and $|\nu|\neq 0,1$. Then Φ is an additive $*$-derivation.

3 Applications

In this section, we present applications of Corollaries 2.2 and 2.3 to von Neumann algebras. In [1] and [5] the authors showed that if a von Neumann algebra has no central summands of type I_1, then $\mathcal{A}$ satisfy the two properties in $(\spadesuit)$. Thus, we have the following corollary:

Corollary 3.1. Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1 and $\Phi:\mathcal{A}\to\mathcal{A}$ a $*$-derivation-type on mixed product $a\blacklozenge_\eta b\blacklozenge_\nu c$, where η,ν are rational numbers satisfying the conditions $\eta\neq-1$ and $|\nu|\neq 0,1$. Then Φ is an additive $*$-derivation.

A von Neumann algebra $\mathcal{A}$ is a weakly closed self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator $1_{\mathcal{A}}$. A von Neumann algebra whose centre is $\mathbb{C}1_{\mathcal{A}}$ is called a *factor*. It is well known that a factor von Neumann algebra is a prime algebra [10]. Therefore, we have the following corollary:

Corollary 3.2. Let $\mathcal{A}$ be a factor von Neumann algebra with $\dim\mathcal{A}\geq 2$ and $\Phi:\mathcal{A}\to\mathcal{A}$ a $*$-derivation-type on mixed product $a\blacklozenge_\eta b\blacklozenge_\nu c$, where η,ν are rational numbers satisfying the conditions $\eta\neq-1$ and $|\nu|\neq 0,1$. Then Φ is an additive $*$-derivation.

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