Int. J. Nonlinear Anal. Appl. In Press, 1–10
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.34951.5218

A class of bi-univalent functions defined by (p, q)-derivative operator subordinate to (m, n)-Lucas polynomials

S. R. Swamy*, M. D. Mary, V. Ushakumari

Department of Information Science and Engineering, Acharya Institute of Technology, Bengaluru- 560 107, Karnataka, India

(Communicated by Mugur Alexandru Acu)

Abstract

We propose a category of normalized analytic functions given by $g(\zeta) = \zeta + \sum_{j=2}^{\infty} d_j \zeta^j$ that are bi-univalent in the disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$ defined by (p,q)-derivative operator, subordinate to (m, n)-Lucas polynomials. For members of this family, we determine estimates for the coefficients $|d_2|$ and $|d_3|$ and the Fekete-Szegö result. New implications of the primary result, as well as pertinent links to previously published findings, are also provided.

Keywords: Bi-univalent function, (p, q)-derivative operator, (m, n)-Lucas polynomial, Fekete-Szegö problem
2020 MSC: Primary 30C45; Secondary 11B39

1 Preliminaries

Quantum calculus is essential because it is applied in many branches of mathematics, computer science, physics, and other fields. The quantum calculus's extension to the (p, q)-calculus was taken into consideration by the researchers. The (p, q)-calculus, which includes the (p, q)-number, is first examined around the same time (1991) and subsequently on its own by [7, 10, 15, 43]. Fibonacci oscillators were studied with the presentation of the (p, q)-number in [7]. The investigation of the (p, q)-number in [10] allows for the construction of a (p, q)-Harmonic oscillator. In [15], the (p, q)-number was explored to unify or generalize various forms of q-oscillator algebras. The (p, q)-numbers are investigated in [43] to calculate the (p, q)-Stirling numbers. Consequently, many mathematical, computer science, physical, chemical and other related problems require knowledge of (p, q)-calculus. Expanding upon the previously mentioned papers, numerous scientists have studied the (p, q)-calculus in a variety of research fields since 1991. A syntax for embedding the q-series into a (p, q)-series was given by the results in [22]. Additionally, they looked into the (p, q)-hypergeometric series and discovered some outcomes that matched (p, q)-extensions of the well-known q-identities. The q-identities are extended correspondingly to yield the (p, q)-series (see, e.g., [6]). We give some elementary definitions of the terms used in this paper related to (p, q)–calculus. The (p, q)-bracket number is given by $[j]_{p,q} = p^{j-1} + p^{j-2}q + ... + p^2q^{j-3} + pq^{j-2} + q^{j-1} = \frac{p^j - q^j}{p-q}$ $(p \neq q)$, which is an extension of q-number (see [21]), that is $[j]_q = \frac{1-q^j}{1-q}$ $(q \neq 1)$. Note that $[j]_{p,q}$ is symmetric and if p=1, then $[j]_{p,q} = [j]_q$.

Let $\mathbb{N} = \mathbb{N}_0 \backslash \{0\} := \{1, 2, 3, ...\}$ and $\mathbb{R}$ be the family of real numbers. let $\mathfrak{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, where $\mathbb{C}$ is the complex numbers set.

*Corresponding author
Email addresses: swamy2704@acharya.ac.in (S. R. Swamy), mary2462@acharya.ac.in (M. D. Mary), ushakumari2487@acharya.ac.in (V. Ushakumari)

Received: July 2024 *Accepted:* August 2024

Definition 1.1. [41] Consider a function g defined on $\mathbb{C}$ and let $0 < q < p \leq 1$. Then the (p,q)-derivative of g is defined by

$$D_{p,q}g(\zeta) = \frac{g(p\zeta) - g(q\zeta)}{(p-q)\zeta} \; (\zeta \neq 0),$$

and $D_{p,q}g(0) = g'(0)$, provided $g'(0)$ exists.

We note that $D_{p,q}\zeta^j = [j]_{p,q}\zeta^{j-1}$ *and* $D_{p,q}ln(\zeta) = \frac{ln(p/q)}{(p-q)\zeta}$. Also, $[j]_{p,q} \to j$, *if* $p = 1$ *and* $q \to 1^-$. Therefore, $D_{p,q}g(\zeta) \to g'(\zeta)$ as $p = 1$ *and* $q \to 1^-$. Any function's (p,q)-derivative is a linear operator. More accurately $D_{p,q}(cg_1(\zeta) + dg_2(\zeta)) = cD_{p,q}g_1(\zeta) + cD_{p,q}g_2(\zeta)$, where c and d are constants. The (p,q)-derivative satisfies the product rules and quotient rules (see [30]). Exponential functions are used to introduce the (p,q)-analogues of many functions, including sine, cosine, and tangent similar to how their Euler expressions. Durani et al. [17] have examined the (p,q)-derivatives of these functions. For further details on (p,q)-calculus, see, among other sources, ([11, 17, 41]).

Let us take a normalized regular function g in $\mathfrak{D}$ given by

$$g(\zeta) = \zeta + \sum_{j=2}^{\infty} d_j\zeta^j, \tag{1.1}$$

and let $\mathcal{A}$ be the class of all such functions. Let $\mathcal{S} = \{g \in \mathcal{A} : g\, is\, univalent\, in\, \mathfrak{D}\}$. If $g \in\mathcal{A}$ is of the form (1.1), then

$$D_{p,q}g(\zeta) = 1 + \sum_{j=2}^{\infty}[j]_{p,q}d_j\zeta^{j-1}, \qquad (\zeta \in \mathfrak{D}). \tag{1.2}$$

The renowned Koebe theorem (see [18]) states that each function $g \in \mathcal{S}$ has an inverse and is defined as

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \cdots \tag{1.3}$$

satisfying $\zeta = g^{-1}(g(\zeta))$ and $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$, $r_0(g) \geq 1/4$, ζ, $\omega \in \mathfrak{D}$. The notion of bi-univalent functions was first presented by Levin in his work [24]. These are analytic functions, denoted by g, where both g and $g^{-1}$are univalent in $\mathfrak{D}$. The set of all bi-univalent functions of the type (1.1) is symbolized by Σ. $\frac{1}{2}log\left(\frac{1+\zeta}{1-\zeta}\right)$, $-log(1-\zeta)$ and $\frac{\zeta}{1-\zeta}$ are some of the functions in the Σ family. However, $\zeta - \frac{\zeta^2}{2}$, $\frac{\zeta}{1-\zeta^2}$, and the Koebe function do not belong in Σ, even though they are in $\mathcal{S}$. For a concise analysis and to discover some of the remarkable characteristics of the family Σ, see [8, 9, 40] and the citation provided in these papers. The article by Srivastava et al. [32] gave rise to the recent momentum of studies of the bi-univalent function family. Numerous scholars have looked into several fascinating special families of Σ since this article brought the subject back to life (see [12, 13, 20]).

The (p,q)-calculus was used to study several subclasses of the class $\mathcal{S}$ and the class Σ. In [33], the subordination principle is used to define the (p,q)-starlike and (p,q)-convex functions classes. Novel subclasses of the class Σ associated with (p,q)-differential operators have also been presented and examined in several studies (refer to [3, 5, 16, 28, 27, 42]).

The (m,n)-Lucas polynomials $L_j(\varkappa)$ (or $L_j(m(\varkappa), n(\varkappa), \varkappa)$) are defined by the below mentioned recurrence relation (see [23]):

$$L_0(\varkappa) = 2,\, L_1(\varkappa) = m(\varkappa), \quad L_j(\varkappa) = m(\varkappa)L_{j-1}(\varkappa) + n(\varkappa)L_{j-2}(\varkappa), \tag{1.4}$$

where $j \in \mathbb{N}\backslash\{1\}$, $m(\varkappa)$ and $n(\varkappa)$ are real polynomials. The generating function (GF) of the (m,n)-Lucas polynomials $L_j(\varkappa)$ is given by

$$\mathcal{G}(\varkappa, \zeta) := \sum_{j=0}^{\infty} L_j(\varkappa)\zeta^j = \frac{2 - m(x)\zeta}{1 - m(\varkappa)\zeta - n(\varkappa)\zeta^2}. \tag{1.5}$$

One can easily find from (1.4) that $L_2(\varkappa) = m^2(\varkappa) + 2n(\varkappa)$, $L_3(\varkappa) = m^3(\varkappa) + 3m(x)n(\varkappa)$ and so on. For specific selections of $m(\varkappa)$ and $n(\varkappa)$, the (m,n)-Lucas polynomials $L_j(\varkappa)$ (or $L_j(m(\varkappa), n(\varkappa), \varkappa)$) leads to various known polynomials (see [4]). For members of some subfamilies of Σ linked to (m,n)-Lucas polynomials, fascinating findings about coefficient estimations and Fekete- Szegö result has been found in [1, 25, 29, 36, 37, 38, 39].

For functions g_1 and g_2 regular in $\mathfrak{D}$, g_1 is said to subordinate g_2, if there is a Schwarz function ψ in $\mathfrak{D}$, satisfying $\psi(0) = 0$, $|\psi(\zeta)| < 1$ and $g_1(\zeta) = g_2(\psi(\zeta))$, $\zeta \in \mathfrak{D}$. The notation $g_1 \prec g_2$ indicates this subordination. If $g_2 \in \mathcal{S}$, then $g_1(\zeta) \prec g_2(\zeta)$ is equivalent to $g_1(0) = g_2(0)$ and $g_1(\mathfrak{D}) \subset g_2(\mathfrak{D})$.

Definition 1.2. The (p,q)-analogue of Swamy differential operator for $g \in \mathcal{A}$ is defined as follows:

$$\begin{aligned}
&W_{p,q}^{\nu,\mu,0} g(\zeta) = g(\zeta),\\
&W_{p,q}^{\nu,\mu,1} g(\zeta) = \frac{\nu g(\zeta) + \mu z D_{p,q} g(\zeta)}{\nu+\mu},\\
&\vdots\\
&W_{p,q}^{\nu,\mu,k} g(\zeta) = W_{p,q}^{\nu,\mu}(W_{p,q}^{\nu,\mu,k-1} g(\zeta)),
\end{aligned}$$

where $\zeta \in \mathfrak{D}, \mu \geq 0$, ν a real number with $\nu+\mu>0$, $\mathrm{k} \in \mathbb{N}$ and $0<q<p\leq 1$.

Remark 1.3. i) We observe that $W_{p,q}^{\nu,\mu,k} : \mathcal{A} \to \mathcal{A}$ is a linear operator. We have

$$W_{p,q}^{\nu,\mu,k} g(\zeta) = \zeta + \sum_{j=2}^{\infty} \left(\frac{\nu + \mu[j]_{p,q}}{\nu+\mu}\right)^k d_j \zeta^j, \tag{1.6}$$

for $g(\zeta)$ given by (1.1).

ii) If we let $\nu = 0$ and $\mu = 1$, then $W_{p,q}^{\nu,\mu,k} g(\zeta)$ reduces to the (p,q)-analogue of Salagean operator discussed in [31].

iii) If we take $\nu = 1-\mu, \mu \geq 0$, then $A_{p,q}^{\mu,k} (= W_{p,q}^{1-\mu,\mu,k}) : \mathcal{A} \to \mathcal{A}$ is a linear operator. For $g(\zeta)$ given by (1.1), we have

$$A_{p,q}^{\mu,k} g(\zeta) = \zeta + \sum_{j=2}^{\infty} \left(1 + \mu([j]_{p,q} - 1)\right)^k d_j \zeta^j, \tag{1.7}$$

which is (p,q)-analogue of Al-Oboudi differential operator.

iv) If we put $\nu = l+1-\mu, l>-1, \mu \geq 0$, then $C_{p,q}^{l,\mu,k} (= W_{p,q}^{l+1-\mu,\mu,k}) : \mathcal{A} \to \mathcal{A}$ is another linear operator. For $g(\zeta)$ given by (1.1), we have

$$C_{p,q}^{l,\mu,k} g(\zeta) = \zeta + \sum_{j=2}^{\infty} \left(\frac{l+1+\mu([j]_{p,q}-1)}{l+1}\right)^k d_j \zeta^j, \tag{1.8}$$

which is (p,q)-analogue of Catas differential operator.

v) Swamy operator [35], Al-Oboudi operator [2], and Cătaş operator [14] are obtained by taking $q \to 1^- \, and \, p = 1$ in (1.6), (1.7), and (1.8), respectively.

We propose a subfamily of Σ using the (p,q)-analogue of the Swamy derivative operator, subordinate to (m,n)-Lucas polynomials $L_j(\varkappa)$ as in (1.4) with GF (1.5).In this paper, $0<q<p\leq 1$ is always satisfied by the parameters $p \, and \, q$.This paper also makes use of the inverse functions $g^{-1}(\omega) = f(\omega)$ and $\mathcal{G}(\varkappa,\zeta)$, which are as in (1.3) and (1.5), respectively.

Definition 1.4. Any function $g \in \Sigma$ is said to be in the family $\mathfrak{S}_{\Sigma,p,q}^{\tau,k}(\varkappa,\nu,\mu)$, if

$$\frac{1}{2}\left\{\frac{\zeta(W_{p,q}^{\nu,\mu,k} g(\zeta))'}{g(\zeta)} + \left(\frac{\zeta(W_{p,q}^{\nu,\mu,k} g(\zeta))'}{g(\zeta)}\right)^{\frac{1}{\tau}}\right\} \prec \mathcal{G}(\varkappa,\zeta) - 1,\, \zeta \in \mathfrak{D}$$

and

$$\frac{1}{2}\left\{\frac{\omega(W_{p,q}^{\nu,\mu,k} f(\omega))'}{f(\omega)} + \left(\frac{\omega(W_{p,q}^{\nu,\mu,k} f(\omega))'}{f(\omega)}\right)^{\frac{1}{\tau}}\right\} \prec \mathcal{G}(\varkappa,\omega) - 1,\, \omega \in \mathfrak{D},$$

where $0<\tau\leq 1, \mu \geq 0, \nu$ a real number with $\nu+\mu>0$, and $k \in \mathbb{N}$.

For particular chioces of $p, q, \tau, \mathrm{and}\, \nu$, the family $\mathfrak{S}_{\Sigma,p,q}^{\tau,k}(\varkappa,\nu,\mu)$ includes many new subfamilies of Σ as mentioned below:

Example 1.1. $S^{\tau,k}_{\Sigma,p,q}(\varkappa,\mu) \equiv \mathfrak{S}^{\tau,k}_{\Sigma,p,q}(\varkappa,1-\mu,\mu), 0<\tau\le 1, \mu\ge 0$, and $k\in\mathbb{N}$ is the set of members $g\in\Sigma$ that satisfy

$$\frac{1}{2}\left\{\frac{\zeta(A^{\mu,k}_{p,q}g(\zeta))'}{g(\zeta)}+\left(\frac{\zeta(A^{\mu,k}_{p,q}g(\zeta))'}{g(\zeta)}\right)^{\frac{1}{\tau}}\right\}\prec \mathcal{G}(\varkappa,\zeta)-1,\ \zeta\in\mathfrak{D}$$

and

$$\frac{1}{2}\left\{\frac{\omega(A^{\mu,k}_{p,q}f(\omega))'}{f(\omega)}+\left(\frac{\omega(A^{\mu,k}_{p,q}f(\omega))'}{f(\omega)}\right)^{\frac{1}{\tau}}\right\}\prec \mathcal{G}(\varkappa,\omega)-1,\ \omega\in\mathfrak{D}.$$

Example 1.2. $T^{\tau,k}_{\Sigma,p,q}(\varkappa,l,\mu) \equiv \mathfrak{S}^{\tau,k}_{\Sigma,p,q}(\varkappa,l+1-\mu,\mu), \mu\ge 0, 0<\tau\le 1, l>-1$, and $k\in\mathbb{N}$ is the set of members $g\in\Sigma$ that satisfy

$$\frac{1}{2}\left\{\frac{\zeta(C^{l,\mu,k}_{p,q}g(\zeta))'}{g(\zeta)}+\left(\frac{\zeta(C^{l,\mu,k}_{p,q}g(\zeta))'}{g(\zeta)}\right)^{\frac{1}{\tau}}\right\}\prec \mathcal{G}(\varkappa,\zeta)-1,\ \zeta\in\mathfrak{D}$$

and

$$\frac{1}{2}\left\{\frac{\omega(C^{l,\mu,k}_{p,q}f(\omega))'}{f(\omega)}+\left(\frac{\omega(C^{l,\mu,k}_{p,q}f(\omega))'}{f(\omega)}\right)^{\frac{1}{\tau}}\right\}\prec \mathcal{G}(\varkappa,\omega)-1,\ \omega\in\mathfrak{D}.$$

Example 1.3. $\mathfrak{R}^{k}_{\Sigma,p,q}(x,\nu,\mu) \equiv \mathfrak{S}^{1,k}_{\Sigma,p,q}(\varkappa,\nu,\mu),\ \mu\ge 0, \nu$ a real number with $\nu+\mu>0$, and $k\in\mathbb{N}$ is the set of members $g\in\Sigma$ that satisfy

$$\frac{\zeta(W^{\nu,\mu,k}_{p,q}g(\zeta))'}{g(\zeta)}\prec \mathcal{G}(\varkappa,\zeta)-1,\ \zeta\in\mathfrak{D}$$

and

$$\frac{\omega(W^{\nu,\mu,k}_{p,q}f(\omega))'}{f(\omega)}\prec \mathcal{G}(\varkappa,\omega)-1,\ \omega\in\mathfrak{D}.$$

Example 1.4. If $p=1$ and $q\to 1^-$ in the set $\mathfrak{S}^{\tau,k}_{\Sigma,p,q}(\varkappa,\nu,\mu)$, then we obtain a subset $\mathfrak{Y}^{\tau,k}_{\Sigma}(\varkappa,\nu,\mu)$, which is the collection of members of $g\in\Sigma$ that satisfy

$$\frac{1}{2}\left\{\frac{\zeta(\mathfrak{W}^{\nu,\mu,k}g(\zeta))'}{g(\zeta)}+\left(\frac{\zeta(\mathfrak{W}^{\nu,\mu,k}g(\zeta))'}{g(\zeta)}\right)^{\frac{1}{\tau}}\right\}\prec \mathcal{G}(\varkappa,\zeta)-1,\ \zeta\in\mathfrak{D}$$

and

$$\frac{1}{2}\left\{\frac{\omega(\mathfrak{W}^{\nu,\mu,k}f(\omega))'}{f(\omega)}+\left(\frac{\omega(\mathfrak{W}^{\nu,\mu,k}f(\omega))'}{f(\omega)}\right)^{\frac{1}{\tau}}\right\}\prec \mathcal{G}(\varkappa,\omega)-1,\ \omega\in\mathfrak{D},$$

where $\mathfrak{W}^{\nu,\mu,k}\equiv W^{\nu,\mu,k}_{p=1,q\to 1^-}$, $\mu\ge 0,\nu$ a real number with $\nu+\mu>0, k\in\mathbb{N}$, and $0<\tau\le 1$.

Fekete-Szegö inequality [19] and estimates for $|d_2|$ and $|d_3|$ are found in Section 2 for functions in $\mathfrak{S}^{\tau}_{\Sigma,p,q}(\varkappa,\nu,\mu)$. Along with relevant links to the earlier results, there are also a few intriguing implications of the main result.

2 Main Results

Initially, we compute the Fekete-Szegö inequality for functions in $\mathfrak{S}^{\tau,k}_{\Sigma,p,q}(\varkappa,\nu,\mu)$, as well as the bounds for $|d_2|$ and $|d_3|$.

Theorem 2.1. Let $0<\tau\le 1, \mu\ge 0, \nu$ a real number such that $\nu+\mu>0$ and $k\in\mathbb{N}$. If a function $g\in\mathfrak{S}^{\tau,k}_{\Sigma,p,q}(\varkappa,\nu,\mu)$, then

(i)

$$|d_2|\le\frac{2\tau\sqrt{|m^3(\varkappa)|}}{\sqrt{|(2\tau(\tau+1)(N-M)-\tau(\tau+3)M^2)m^2(\varkappa)-2M^2n(\varkappa)(1+\tau)^2)|}},\tag{2.1}$$

(ii)

$$|d_3| \leq \frac{4\tau^2 m^2(\varkappa)}{(1+\tau)^2 M^2} + \frac{2\tau|m(\varkappa)|}{(1+\tau)N}, \tag{2.2}$$

and for $\xi \in \mathbb{R}$,

(iii)

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{2\tau|m(\varkappa)|}{(1+\tau)N} & ; |1-\xi| \leq \mathbb{J} \\ \frac{4\tau^2|m^3(\varkappa)|\,|1-\xi|}{|(2\tau(\tau+1)(N-M)-\tau(\tau+3)M^2)m^2(\varkappa)-2M^2n(\varkappa)(1+\tau)^2)|} & ; |1-\xi| \geq \mathbb{J}, \end{cases} \tag{2.3}$$

where

$$\mathbb{J} = \frac{|(2\tau(\tau+1)(N-M)-\tau(\tau+3)M^2)m^2(\varkappa)-2M^2n(\varkappa)(1+\tau)^2|}{2\tau(\tau+1)Nm^2(\varkappa)}, \tag{2.4}$$

$$M = \left(2\left(\frac{\nu+\mu[2]_{p,q}}{\nu+\mu}\right)^k - 1\right), \tag{2.5}$$

and

$$N = \left(3\left(\frac{\nu+\mu[3]_{p,q}}{\nu+\mu}\right)^k - 1\right). \tag{2.6}$$

Proof . Let $g \in \mathfrak{S}^{\tau,k}_{\Sigma,p,q}(\varkappa,\nu,\mu)$. Then, we can write because of Definition 1.4:

$$\frac{1}{2}\left\{\frac{\zeta(W^{\nu,\mu,k}_{p,q}g(\zeta))'}{g(\zeta)} + \left(\frac{\zeta(W^{\nu,\mu,k}_{p,q}g(\zeta))'}{g(\zeta)}\right)^{\frac{1}{\delta}}\right\} = \mathcal{G}(\varkappa, r(\zeta)) - 1$$

and

$$\frac{1}{2}\left\{\frac{\omega(W^{\nu,\mu,k}_{p,q}f(\omega))'}{f(\omega)} + \left(\frac{\omega(W^{\nu,\mu,k}_{p,q}f(\omega))'}{f(\omega)}\right)^{\frac{1}{\delta}}\right\} = \mathcal{G}(\varkappa, s(\omega)) - 1,$$

where r and s are functions regular in $\mathfrak{D}$, with $r(0) = 0, |r(\zeta)| < 1$, $s(0) = 0, |s(\omega)| < 1$. Or, equivalently

$$\frac{1}{2}\left\{\frac{\zeta(W^{\nu,\mu,k}_{p,q}g(\zeta))'}{g(\zeta)} + \left(\frac{\zeta(W^{\nu,\mu,k}_{p,q}g(\zeta))'}{g(\zeta)}\right)^{\frac{1}{\delta}}\right\} = -1 + L_0(\varkappa) + L_1(\varkappa)r(\zeta) + L_2(\varkappa)r^2(\zeta) + \cdots, \tag{2.7}$$

and

$$\frac{1}{2}\left\{\frac{\omega(W^{\nu,\mu,k}_{p,q}f(\omega))'}{f(\omega)} + \left(\frac{\omega(W^{\nu,\mu,k}_{p,q}f(\omega))'}{f(\omega)}\right)^{\frac{1}{\delta}}\right\} = -1 + L_0(\varkappa) + L_1(\varkappa)s(\omega) + L_2(\varkappa)s^2(\omega) + \cdots. \tag{2.8}$$

On account of (1.4), (2.7) and (2.8), we obtain

$$\frac{1}{2}\left\{\frac{\zeta(W^{\nu,\mu,k}_{p,q}g(\zeta))'}{g(\zeta)} + \left(\frac{\zeta(W^{\nu,\mu,k}_{p,q}g(\zeta))'}{g(\zeta)}\right)^{\frac{1}{\delta}}\right\} = 1 + L_1(\varkappa)r_1\zeta + [L_1(\varkappa)r_2 + L_2(\varkappa)r_1^2]\zeta^2 + \cdots, \tag{2.9}$$

and

$$\frac{1}{2}\left\{\frac{\omega(W^{\nu,\mu,k}_{p,q}f(\omega))'}{f(\omega)} + \left(\frac{\omega(W^{\nu,\mu,k}_{p,q}f(\omega))'}{f(\omega)}\right)^{\frac{1}{\delta}}\right\} = 1 + L_1(\varkappa)s_1\omega + [L_1(\varkappa)s_2 + L_2(\varkappa)s_1^2]\omega^2 + \cdots. \tag{2.10}$$

If $|r(\zeta)| = |r_1\zeta + r_2\zeta^2 + r_3\zeta^3 + ...| < 1$, $\zeta \in \mathfrak{D}$, and if $|s(\omega)| = |s_1\omega + s_2\omega^2 + s_3\omega^3 + \cdots| < 1$, $\omega \in \mathfrak{D}$, then we known that

$$|r_i| \leq 1 \quad \text{and} \quad |s_i| \leq 1\,(i \in \mathbb{N}). \tag{2.11}$$

From (2.9) and (2.10), it can be inferred that

$$\left(\frac{1+\tau}{2\tau}\right)Md_2 = L_1(\varkappa)r_1, \tag{2.12}$$

$$\left(\frac{1+\tau}{2\tau}\right)(Nd_3 - Md_2^2) + \left(\frac{1-\tau}{4\tau^2}\right)M^2 d_2^2 = L_1(\varkappa)r_2 + L_2(\varkappa)r_1^2, \tag{2.13}$$

$$-\left(\frac{1+\tau}{2\tau}\right)Md_2 = L_1(\varkappa)s_1, \tag{2.14}$$

and

$$\left(\frac{1+\tau}{2\tau}\right)(N(2d_2^2 - d_3) - Md_2^2) + \left(\frac{1-\tau}{4\tau^2}\right)M^2 d_2^2 = L_1(\varkappa)s_2 + L_2(\varkappa)s_1^2. \tag{2.15}$$

From (2.12) and (2.14), we have

$$r_1 = -s_1 \tag{2.16}$$

and also

$$\left(\frac{(1+\tau)^2}{2\tau^2}\right)M^2 d_2^2 = (r_1^2 + s_1^2)(L_1(\varkappa))^2. \tag{2.17}$$

If we add (2.13) and (2.15), then we obtain

$$\left[\left(\frac{1+\tau}{\tau}\right)(N-M) + \left(\frac{1-\tau}{2\tau^2}\right)M^2\right]d_2^2 = L_1(\varkappa)(r_2+s_2) + L_2(\varkappa)(r_1^2+s_1^2). \tag{2.18}$$

Substituting the value of $(r_1^2 + s_1^2)$ from (2.17) in (2.18), we get

$$d_2^2 = \frac{2\tau^2(L_1(\varkappa))^3(r_2+s_2)}{[(2\tau(\tau+1)(N-M) + (1-\tau)M^2)(L_1(\varkappa))^2 - (1+\tau)^2 M^2 L_2(\varkappa)]}, \tag{2.19}$$

which produces (2.1), when applied (2.11). After deducting (2.15) from (2.13) and then applying (2.16), we get

$$d_3 = d_2^2 + \frac{\tau L_1(\varkappa)(r_2 - s_2)}{(1+\tau)N}. \tag{2.20}$$

Then in view of (2.17), (2.20) becomes

$$d_3 = \frac{2\tau^2(L_1(\varkappa))^2(r_1^2+s_1^2)}{(1+\tau)^2 M^2} + \frac{\tau L_1(\varkappa)(r_2-s_2)}{(1+\tau)N},$$

which produces (2.2), when applied (2.11). For $\xi \in \mathbb{R}$, we obtain in view of (1.4) from (2.19) and (2.20):

$$|d_3 - \xi d_2^2| = |L_1(\varkappa)|\left|\left(B(\xi,\varkappa) + \frac{\tau}{(1+\tau)N}\right)r_2 + \left(B(\xi,\varkappa) - \frac{\tau}{(1+\tau)N}\right)s_2\right|,$$

where

$$B(\xi,x) = \frac{2\tau^2(1-\xi)L_1^2(\varkappa)}{[(2\tau(\tau+1)(N-M) + M^2(1-\tau))L_1^2(\varkappa) - M^2(1+\tau)^2 L_2(\varkappa)]}.$$

Clearly,

$$|d_3 - \xi d_2^2| \le \begin{cases} \frac{2\tau|L_1(\varkappa)|}{(1+\tau)N}; & 0 \le |B(\xi,\varkappa)| \le \frac{\tau}{(1+\tau)N} \\ 2|B(\xi,\varkappa)||L_1(\varkappa)|; & |B(\xi,\varkappa)| \ge \frac{\tau}{(1+\tau)N}. \end{cases}$$

This produces (2.3), where J is the same as in (2.4) on using $L_1(\varkappa) = m(\varkappa)$, $L_2(\varkappa) = m^2(\varkappa) + 2n(\varkappa)$. Thus, Theorem 2.1 has been demonstrated. □

Remark 2.2. The results of Altınkaya and Yalçın [3, Theorem 2.1 and Theorem 3.1] are obtained by allowing $\mu = 1, \nu = 0,\ n(x) = 1$, and $m(x) = x$ in Theorem 2.1.

Corollary 2.3. Let us assume that $\nu = 1-\mu$ in Theorem 2.1. Then the upper bounds of $|d_2|, |d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for any function $g \in S_{\Sigma,p,q}^{\tau,k}(\varkappa,\mu)$ are given by (2.1), (2.2), and (2.3), respectively, with $M = M_1 = 2(1+\mu([2]_{p,q}-1)^k - 1)$, and $N = N_1 = 3(1+\mu([3]_{p,q}-1)^k - 1)$. For J in (2.4), M, *and* N are to be substituted with M_1, *and* N_1, respectively.

Corollary 2.4. Let us assume that $\nu = l+1-\mu$ in Theorem 2.1. Then the upper bounds of $|d_2|, |d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for any function $g \in T^{\tau,k}_{\Sigma,p,q}(\varkappa, l, \mu)$ are given by (2.1), (2.2), and (2.3), respectively, with

$$M = M_2 = \left(2\left(\frac{l+1+\mu([2]_{p,q}-1)}{l+1}\right)^k - 1\right), \qquad \text{and} \qquad N = N_2 = \left(3\left(\frac{l+1+\mu([3]_{p,q}-1)}{l+1}\right)^k - 1\right).$$

For J in (2.4), M and N are to be substituted with M_2 and N_2, respectively.

When $\tau = 1$, the following would result from Theorem 2.1.

Corollary 2.5. Let $\mu \geq 0$, ν a real number satisfying $\nu + \mu > 0$ and $k \in \mathbb{N}$. If a function $g \in \mathfrak{R}^k_{\Sigma,p,q}(\varkappa, \nu, \mu)$, then

(i)
$$|d_2| \leq \frac{\sqrt{|m^3(\varkappa)|}}{\sqrt{|(N-M-M^2)m^2(\varkappa) - 2M^2 n(\varkappa)|}},$$

(ii)
$$|d_3| \leq \frac{m^2(\varkappa)}{M^2} + \frac{|m(\varkappa)|}{N},$$

and for $\xi \in \mathbb{R}$

(iii)
$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|m(\varkappa)|}{N}; & |1-\xi| \leq \mathbb{J}_1 \\ \frac{|m^3(\varkappa)|\,|1-\xi|}{|(N-M-M^2)m^2(\varkappa)-2M^2n(\varkappa)|}; & |1-\xi| \geq \mathbb{J}_1, \end{cases}$$

where (2.5) provides M, (2.6) provides N, and

$$\mathbb{J}_1 = \left|\frac{(N-M-M^2)m^2(\varkappa) - 2M^2 n(\varkappa)}{4Nm^2(\varkappa)}\right|.$$

Remark 2.6. We acquire the results of Altınkaya and Yalçın [3, Corollary 2.1 and Corollary 3.1] by taking $n(x) = 1, m(x) = x$, $\nu = 0$, and $\mu = 1$ in Corollary 2.5.

Corollary 2.7. Let us assume that $q \to 1^-$ and $p = 1$ in Theorem 2.1. Then the upper bounds of $|d_2|, |d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for any function $g \in \mathfrak{Y}^{\tau,k}_{\Sigma}(\varkappa, \nu, \mu)$, are given by (2.1), (2.2), and (2.3), respectively, with $M = M_3 = \left(2\left(\frac{\nu+2\mu}{\nu+\mu}\right)^k - 1\right)$, and $N = N_3 = \left(3\left(\frac{\nu+3\mu}{\nu+\mu}\right)^k - 1\right)$. For J in (2.4), M and N are to be substituted with M_3 and N_3, respectively.

Remark 2.8. If $k = 0$ in the set $\mathfrak{Y}^{\tau,k}_{\Sigma}(\varkappa, \nu, \mu)$, then we obtain a subset $\mathfrak{Q}^{\tau}_{\Sigma}(\varkappa), 0 < \tau \leq 1$, which is the collection of members of $g \in \Sigma$ that satisfy

$$\frac{1}{2}\left\{\frac{\zeta g'(\zeta)}{g(\zeta)} + \left(\frac{\zeta g'(\zeta)}{g(\zeta)}\right)^{\frac{1}{\tau}}\right\} \prec \mathcal{G}(\varkappa, \zeta) - 1,\ \zeta \in \mathfrak{D}$$

and

$$\frac{1}{2}\left\{\frac{\omega f'(\omega)}{f(\omega)} + \left(\frac{\omega f'(\omega)}{f(\omega)}\right)^{\frac{1}{\tau}}\right\} \prec \mathcal{G}(\varkappa, \omega) - 1,\ \omega \in \mathfrak{D}.$$

Corollary 2.9. Let $0 < \tau \leq 1$. If a function $g \in \mathfrak{Q}^{\tau}_{\Sigma}(\varkappa)$, then

(i)
$$|d_2| \leq \frac{2\tau\sqrt{|m^3(\varkappa)|}}{\sqrt{|\tau(\tau-1)m^2(\varkappa) - 2(1+\tau)^2 n(\varkappa)|}},$$

(ii)

$$|d_3| \leq \frac{4\tau^2 m^2(\varkappa)}{(1+\tau)^2} + \frac{\tau|m(\varkappa)|}{1+\tau},$$

and for $\xi \in \mathbb{R}$

(iii)

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{\tau|m(\varkappa)|}{1+\tau}; & |1-\xi| \leq \left|\dfrac{\tau(\tau-1)m^2(\varkappa) - 2(1+\tau)^2 n(\varkappa)}{4\tau(\tau+1)m^2(\varkappa)}\right| \\ \frac{4\tau^2|m^3(\varkappa)|\,|1-\xi|}{|\tau(\tau-1)m^2(\varkappa)-2(1+\tau)^2 n(\varkappa)|}; & |1-\xi| \geq \left|\dfrac{\tau(\tau-1)m^2(\varkappa) - 2(1+\tau)^2 n(\varkappa)}{4\tau(\tau+1)m^2(\varkappa)}\right|. \end{cases}$$

Remark 2.10. We obtain the results of Altınkaya and Yalçın [4, Corollaries 1 and 3] by taking $\tau = 1$ in Corollary 2.9. These results are also stated in [29].

3 Conclusions

This study establishes upper bounds on $|d_2|$ and $|d_3|$ for functions in subfamily of Σ related to $(m, n)-$Lucas polynomials. Moreover, the Fekete-Szegö functional $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$ has been identified for functions in these subfamilies. By adjusting the parameters in Theorem 2.1, a few implications have been brought to light. Relevant connections to the current research are also discovered. Nevertheless, this paper does not address all of the significant subclasses of Σ that exist in the literature. For example, authors [26, 28, 42] have examined various subclasses involving (p, q)-operators introduced in (p, q)-calculus. Functions from this class may be studied with interesting results due to their symmetry properties. Topics about fuzzy differential subordination, and fuzzy differential subordination may be added in future research. By defining the (p, q)-analogue of the Swamy operator defined for p-valent functions [35], the results obtained in this paper could be extended. It is recommended that the interested reader review these papers and the associated references.

References

[1] A. Akgül, *(p,q)-Lucas polynomial coefficient inequalities of the bi-univalent function class*, Turk. J. Math. **43** (2019), 2170–2176.

[2] F.M. Al-Oboudi, *On univalent functions defined by a generalized Salagean operator*, Int. J. Math. Math. Sci. **27** (2004), 1429–1436.

[3] Ş. Altınkaya and S. Yalçın, *Lucas polynomials and applications to an unified class of bi-univalent functions equipped with (p,q)-derivative operators*, TWMS J. Pure Appl. Math. **11** (2020), no. 1, 100–108.

[4] Ş. Altınkaya and S. Yalçın, *On the (p,q)-Lucas polynomial coefficient bounds of the bi-univalent function class*, Bol. Soc. Mat. Mex. **25** (2019), no. 3, 567–575.

[5] Ş. Altınkaya and S. Yalçın, *Certain classes of bi-univalent functions of complex order associated with quasi-subordination involving (p, q)- derivative operator*, Kragujevac J. Math. **44** (2020), no. 4, 639–649.

[6] S. Araci, U. Duran, M. Acikgoz, and H.M. Srivastava, *A certain (p, q)-derivative operator and associated divided differences*, J. Inequal. Appl. **301** (2016), 1–8.

[7] M. Arik, Demircan, T. Turgut, L. Ekinci, and M. Mungan, *Fibonacci oscillators*, Z. Phys. CParticles Fields **55** (1992), 89–95.

[8] D.A. Brannan and T.S. Taha, *On some classes of bi-univalent functions*, Proc. Int. Conf. Math. Anal. Appl., Kuwait, 1985, Math. Anal. Appl., 1988, pp. 53–60.

[9] D.A. Brannan and J.G. Clunie, *Aspects of contemporary complex analysis*, Proc. NATO Adv. Study Inst. University of Durhary, Newyork, Academic Press, 1979.

[10] G. Brodimas, A. Jannussis, and R. Mignani, *Two-Parameter Quantum Groups*, Dipartimento di Fisica Universit di Roma "La Sapienza" I.N.F.N.-Sezione di Roma, 1991.

[11] J.D. Bukweli-Kyemba and M.N. Hounkonnou, *Quantum deformed algebras: Coherent states and special functions*, arXiv preprint arXiv:1301.01162013.

[12] S. Bulut, *Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions*, C R Acad. Sci. Paris Sér I. **352** (2014), 479–484.

[13] M. Çağlar, E. Deniz, and H.M. Srivastava, *Second Hankel determinant for certain subclasses of bi-univalent functions*, Turk. J. Math. **41** (2017), 694–706.

[14] A. Cătaş, *On certain class of p-valent functions defined by new multiplier transformations*, Proc. Int. Symp. Geom. Function Theory Appl., August, 20-24, 2007, TC Istanbul Kultur Univ., Turkey, 2007, pp. 241–250.

[15] R. Chakrabarti and R. Jagannathan, *A (p, q)-oscillator realization of two-parameter quantum*, J. Phys. A: Math. Gen. **24** (1991), no. 13, L711.

[16] L.-I. Cotîrlă, *New classes of analytic and bi-univalent functions*, AIMS Mathematics, 6(10) (2021) 10642–10651.

[17] U. Duran, M. Acikgoz, and S. Araci, *A study on some new results arising from (p, q)-calculus*, TWMS J. Pure Appl. Math. **11** (2020), no. 1, 57–71.

[18] P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259. Springer-Verlag, New York, 1983.

[19] M. Fekete and G. Szegö, *Eine bemerkung über ungerade schlichte funktionen*, J. Lond. Math. Soc. **89** (1933) 85–89.

[20] B.A. Frasin and M.K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. **24** (2011), 1569–1573.

[21] F.H. Jackson, *On q-functions and a certain difference operator*, Earth Envir. Sci. Trans. Royal Soc. Edinburgh **46** (1909), no. 2, 253–281.

[22] R. Jagannathan and K.S. Rao, *Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series*, Proc. Int. Conf. Number Theory Math. Phys., Srinivasa Ramanujan Centre, Kumbakonam, India, 20-21 December, 2005.

[23] G. Lee and M. Asci, *Some properties of the (p, q)-Fibonacci and (p, q)-Lucas polynomials*, J. Appl. Math. **2012** (2012), 1–18.

[24] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** (1967), 63–68.

[25] N. Magesh, C. Abirami, and Ş. Altınkaya, *Initial bounds for certain classes of bi-univalent functions defined bythe (p,q)-Lucas polynomials*, TWMS J. App. and Eng. Math. **11** (2021), no. 1, 282–288.

[26] D. Malyalı, *(p,q)-Hahn difference operator*, MSc. Thesis, Eastern Mediterranean University, 2020.

[27] SK. Mohapatra and T. Panigrahi, *Coefficient estimates for bi-univalent functions defined by (P, Q) analogue of the Salagean differential operator related to the Chebyshev polynomials*, J. Math. Fund. Sci. **53** (2021), no. 1, 49–66.

[28] A. Motamednezhad and S. Salehian, *New sublasses of Bi-univalent functions by (p; q)-derivative operator*, Honam Math. J. **41** (2019), no. 2, 381–390.

[29] G. Murugusundaramoorthy and S. Yalçın, *On λ-pseudo bi-starlike functions related (p,q)-Lucas polynomials*, Libertas Math. **39** (2019), no. 2, 79–88.

[30] P.N. Sadjang, *On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas*, arXiv preprint 1309.3934v1, 22 August 2013.

[31] C. Selvaraj, G. Thirupathi, and E. Umadevi, *Certain classes of analytic functions involving a family of generalized differential operators*, Transyl. J. Math. Mech. **9** (2017), no. 1, 51–61.

[32] H.M. Srivastava, A.K. Mishra, and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **23** (2010), 1188–1192.

[33] H.M. Srivastava, N. Raza, E.S.A. AbuJarad, G. Srivastava, and M.H. AbuJarad, *Fekete-Szegö inequality for classes of (p, q)-Starlike and (p, q)-convex functions*, Rev. Real Acad. Cien. Exactas, Fís. Natur. Serie A, Mat. **113** (2019), no. 4, 3563–3584.

[34] S.R. Swamy, *Inclusion properties of certain subclasses of analytic functions*, Int. Math. Forum **7** (2012), no. 36, 1751–1760.

[35] S.R. Swamy, *Inclusion properties for certain subclasses of analytic functions defined by a generalized multiplier transformation*, Int. J. Math. Anal. **6** (2012), no. 32, 1553–1564.

[36] S.R. Swamy, P.K. Mamatha, N. Magesh, and J. Yamini, *Certain subclasses of bi-univalent functions defined by Sălăgean operator associated with the* (p,q)*- Lucas polynomials*, Adv. Math.: Sci. J. **9** (2020), no. 8, 6017–6025.

[37] S.R. Swamy, J. Nirmala, and Y. Sailaja, *Some special families of holomorphic and Al-Oboudi type bi-univalent functions associated with (m,n)-Lucas polynomials involving modified sigmoid function*, Asian J. Math. Math. Sci. **17** (2021), no. 1, 1–16.

[38] S.R. Swamy and A.K. Wanas, *A comprehensive family of bi-univalent functions defined by* (m,n)*-Lucas polynomials*, Bol. Soc. Mat. Mex. **28** (2022), no. 2, 34.

[39] S.R. Swamy, A.K. Wanas, and Y. Sailaja, *Some special families of holomorphic and Sălăgean type bi-univalent functions associated with* (m,n)*-Lucas polynomials*, Commun. Math. Appl. **11** (2020), no. 4, 563–574.

[40] D.L. Tan, *Coefficient estimates for bi-univalent functions*, Chin. Ann. Math. Ser. A. **5** (1984), 559–568.

[41] A. Tuncer, A. Ali, and M. Syed Abdul, *On Kantorovich modification of (p, q)-Baskakov operators*, J. Inequal. Appl. **98** (2016), 1–8.

[42] S.P. Vijayalakshmi, T.V. Sudharsan, and T. Bulboac, *Symmetric Toeplitz determinants for classes defined by post quantum operators subordinated to the limacon function*, Stud. Univ. Babeş-Bolyai Math. **69** (2024), no. 2, 299—316.

[43] M. Wachs and D. White, *(p,q)-Stirling numbers and set partition statistics*, J. Combin. Theory Ser. A **56** (1991), no. 1, 27–46.