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On a class of Schrödinger-Kirchhoff-Poisson systems

Meysam Soluki[a,*], Ghasem Alizadeh Afrouzi[a], Seyed Hashem Rasouli[b]

[a] *Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran*
[b] *Department of Mathematics, Faculty of Basic Science, Babol Noshirvani University of Technology, Babol, Iran*

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Abstract

This article discusses the existence and multiplicity of solutions for the following Schrödinger-Kirchhoff-Poisson system:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2\right)\Delta u+\lambda\phi u=m(x)|u|^{q-2}u+f(x,u), & x\in\Omega,\\ -\Delta\phi=u^2, & x\in\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^3$, $a\geq 0$,$b>0$ and $\lambda>0$ is a parameter, $1<q<2$ and $f(x,u)$ is linearly bounded in u at infinity. Under some suitable assumptions on m and f, we prove the existence and multiplicity of solutions via variational methods.

Keywords: Schrödinger-Kirchhoff-Poisson systems, Combined nonlinearity, Variational methods
2020 MSC: 35A15, 35J05, 35J10

1 Introduction

In this paper, we investigate the existence and multiplicity of solutions for the following Schrödinger-Kirchhoff-Poisson system

$$\begin{cases} -\left(a+b\displaystyle\int_\Omega |\nabla u|^2\right)\Delta u+\lambda\phi u=m(x)|u|^{q-2}u+f(x,u), & x\in\Omega,\\ -\Delta\phi=u^2, & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^3$, $a\geq 0$,$b>0$ and $\lambda>0$ is a parameter, $1<q<2$ and $f(x,u)$ is linearly bounded in u at infinity that satisfying some conditions we will precise later. When $a=1$, $b=0$ and $m\equiv 1$, the problem (1.1) reduces to a Schrödinger-Poisson system like as follows:

$$\begin{cases} -\Delta u+\phi u=f(x,u), & x\in\Omega,\\ -\Delta\phi=u^2, & x\in\Omega. \end{cases} \tag{1.2}$$

*Corresponding author
Email addresses: meysam.soluki1986@gmail.com (Meysam Soluki), afrouzi@umz.ac.ir (Ghasem Alizadeh Afrouzi), s.h.rasouli@nit.ac.ir (Seyed Hashem Rasouli)

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System (1.2) is related to the nonlinear parabolic Schrödinger-Poisson system:

$$\begin{cases} i\dfrac{\partial \psi}{\partial t} - \Delta\psi + \phi(x)\psi = |\psi|^{p-2}\psi, & x \in \Omega, \\ -\Delta\phi = |\psi|^2, \ \lim\limits_{|x|\to\infty} \phi(x) = 0, & x \in \Omega. \end{cases} \tag{1.3}$$

The first equation in (1.3) is called the Schrödinger equation, which describes quantum (non-relativistic) particles interacting with the electromagnetic field generated by the motion. an interesting class of Schrödinger equations is where the potential $\phi(x)$ is determined by the charge of wave function itself, that is when the second equation in (1.3) (Poisson equation) holds. For more details about the physical relevance of the Schrödinger-Poisson system, we refer to [5, 9, 17].

System (1.2) has been extensively studied after the basic work of Benci and Fortunato [9]. Many important about existence and nonexistence of solutions, multiplicity of solutions, least energy solutions, radial and non-radial solutions, and so on, have been obtained. See for instance [1, 2, 5, 6, 7, 10, 11, 12, 18, 19, 20, 22, 24].

On the other hand, considering just the first equation in (1.2) with the potential equal to zero, we have the problem

$$\begin{cases} -(a + b\displaystyle\int_\Omega |\nabla u|^2)\Delta u = g(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

The Kirchhoff model [16] is a mathematical model used to study small transverse vibrations of an elastic string. A stationary and N-dimensional version of this model can be represented by considering the effect of the change in length during the vibrations. As the length of the string varies during the vibrations, the tension changes with time and depends on the L^2 norm of the gradient of the displacement u. The model includes variables such as $a = P_0/h$, and $b = E/2L$, where L is the length of the string, h is the area of cross-section, E is the Young modulus of the material and P_0 is the initial tension. This model is called nonlocal because of the presence of the term $\int_\Omega |\nabla u|^2 dx$, which implies that the equation in the text is no longer a pointwise identity. This phenomenon causes some mathematical difficulties, making the study of such a class of problems particularly interesting. Some existence and multiplicity results on Kirchhoff-type problems can be found in various papers, including [3, 4, 7, 14, 15, 27] and their references, which represents the stationary and N-dimensional version of the Kirchhoff model [16] for small transverse vibrations of an elastic string by considering the effect of the changing in the length during the vibrations. In fact, since the length of the string is variable during the vibrations, the tension changes with the time and depends of the L^2 norm of the gradient of the displacement u. More precisely, we have $a = P_0/h$ and $b = E/2L$, where L is the length of the string, h is the area of cross-section, E is the Young modulus of the material and P_0 is the initial tension. Problem (1.4) is called nonlocal because of the presence of the term $\int_\Omega |\nabla u|^2 dx$ which implies that the equation in (1.4) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties which makes the study of such a class of problem particularly interesting. Some existence and multiplicity results on Kirchhoff type problems can be found in [3, 4, 7, 14, 15, 27] and the references therein.

Recently, Schrödinger-Kirchhoff-Poisson systems (equivalently Schrödinger-Kirchhoff problems) like (1.1) have great attention of mathematical community. In [8], the authors studied the following Schrödinger-Kirchhoff-Poisson system

$$\begin{cases} -(a + b\displaystyle\int_\Omega |\nabla u|^2)\Delta u + \phi u = f(x, u), & x \in \Omega, \\ -\Delta\phi = u^2, & x \in \Omega, \\ \phi = u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded smooth domain of $\mathbb{R}^3$, and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. They proved that problem (1.5) has at least three solutions: one positive, one negative and one which changes its sign. Furthermore, in case f is odd with respect to u, the authors obtained unbounded sequence of sign-changing solutions. Shao and Chen [23] studied the following Schrödinger-Kirchhoff-Poisson system:

$$\begin{cases} -(a + b\displaystyle\int_\Omega |\nabla u|^2)\Delta u + \lambda\phi u = \eta f(x, u) + u^5, & x \in \Omega, \\ -\Delta\phi = u^2, & x \in \Omega, \\ \phi = u = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

where $a \geq 0$, $b > 0$ and $\eta, \lambda > 0$, $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain and with the help of the variational methods, the existence of a non-trivial solution was obtained. In [21], We consider the following nonlinear Schrödinger-Kirchhoff-Poisson system:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\Delta u + \phi u = \mu g(x,u) + \lambda f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, \qquad \lim_{|x|\to\infty}\phi(x) = 0, & x \in \mathbb{R}^3, \end{cases}$$

where $a, b > 0$. We prove the existence of infinitely many solutions with high energy by using the Fountain theorem. Motivated by the above works, we study the existence and multiplicity of solutions for the problem (1.1). Before stating our main results, we give the following assumptions on m and f.

(H1) $m(x) \in L^{\frac{2}{2-q}}(\Omega)$;

(H2) $m(x) > 0$ on Ω;

(H3) $f(x,u) \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $f(x,u)u \geq 0$ for all $(x,u) \in \Omega \times \mathbb{R}$ and

$$\lim_{u\to 0}\frac{f(x,u)}{u} = 0, \text{ uniformly in } x \in \Omega;$$

(H4) There exists $C > 0$ such that

$$\left|\frac{f(x,u)}{u}\right| \leq C, \text{ for all } x \in \Omega, u \in \mathbb{R} \text{ and } u \neq 0.$$

Throughout this paper, $C > 0$ will be used indiscriminately to denote a suitable positive constant whose value may change from line to line and we will use $o(1)$ for a quantity which goes to zero. Moreover, we use $|.|_p$ to denote the usual norm on $L^p(\Omega)$ for $1 < p < +\infty$. Our main results reads as follows.

Theorem 1.1. Suppose that $a > 0$, $b \geq 0$ and $(H1) - (H4)$ hold. Then there exists $M > 0$ such that for every m with $|m|_{\frac{2}{2-q}} < M$ and $\lambda > 0$, problem (1.1) has a nontrivial solution at negative energy.

Theorem 1.2. Under the assumptions of Theorem 1.1, there exists $\lambda^* > 0$ such that for every $\lambda > \lambda^*$, problem (1.1) has a nontrivial solution at negative energy.

Theorem 1.3. Let $(H1) - (H4)$ hold, and suppose further that $f(x,u)$ is odd in u. Then there exists $\overline{\lambda}$ such that for every $\lambda > \overline{\lambda}$, problem (1.1) has infinitely many solutions at negative energy.

The reminder of this paper is organized as follows. In section 2, we present a suitable variational framework for our problem. In section 3, we prove Theorems 1.1-1.2. Finally the proof of Theorem 1.3 will be given in section 4.

2 Preliminaries

Let us fix some notations:

(i) $H^1(\Omega)$ is the usual Sobolev space with the scalar product and norm

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v dx, \qquad \|u\|^2 = \int_\Omega |\nabla u|^2 dx.$$

(ii) Let $D^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_D = \int_\Omega |\nabla u|^2 dx.$$

The following result is well known (see.e.g. [13, 17, 26])

Lemma 2.1. [13] For any $u \in H^1(\Omega)$, there exists a unique $\phi_u \in D^{1,2}(\Omega)$ such that

$$-\Delta\phi_u = u^2. \tag{2.1}$$

Moreover, ϕ_u has the following properties:
(i) there exists $c > 0$ such that $\|\phi_u\| \le \|u\|^2$ and

$$\int_\Omega |\nabla\phi_u|^2 dx = \int_\Omega \phi_u u^2 dx \le c\|u\|^4;$$

(ii) $\phi_u \ge 0$ and $\phi_{tu} = t^2\phi_u$, for all $t > 0$;
(iii) if $u_n \rightharpoonup u$ in H^1, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\Omega)$ and

$$\lim_{n\to\infty} \int_\Omega \phi_{u_n} {u_n}^2 dx = \int_\Omega \phi_u u^2 dx.$$

We mean by a weak solution of (1.1), a function $u \in H^1(\Omega)$ such that

$$(a + b\|u\|^2) \int_\Omega \nabla u \nabla v dx + \lambda \int_\Omega \phi_u u v dx = \int_\Omega m(x)|u|^{q-2} u v dx + \int_\Omega f(x,u) v dx,$$

for all $v \in H^1(\Omega)$. Let us consider the functional $I : H^1(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{4}\int_\Omega \phi_u u^2 dx - \frac{1}{q}\int_\Omega m(x)|u|^q dx - \int_\Omega F(x,u) dx,$$

where $F(x,u) = \int_0^u f(x,t)dt$. Moreover, it is known that I is a C^1 functional with derivative given by

$$\langle I'(u), v\rangle = (a + b\|u\|^2) \int_\Omega \nabla u \nabla v dx + \lambda \int_\Omega \phi_u u v dx - \int_\Omega m(x)|u|^{q-2} u v dx - \int_\Omega f(x,u) v dx,$$

Clearly, critical points of I are weak solutions of problem (1.1).

Definition 2.2. We say a C^1 functional I satisfies Palais-Smale condition $\big($Cerami condition$\big)$ if any sequence $\{u_n\} \subset H^1(\Omega)$ such that

$$I(u_n) \text{ being bounded, } I'(u_n) \to 0, \quad \text{as } n \to 0 \tag{2.2}$$

$$\Big(I(u_n) \text{ being bounded, } (1 + \|u_n\|)I'(u_n) \to 0, \text{ as } n \to 0\Big)$$

admits a convergent subsequence, and such a sequence is called a palais-Smale sequence $\big($Cerami sequence$\big)$.

Lemma 2.3. Assume that $(H1), (H3)$ and $(H4)$ hold. Then any Cerami sequence of I is bounded in $H^1(\Omega)$.

Proof . Let $\{u_n\}$ be a Cerami sequence of I. By contradiction, let $\|u_n\| \to \infty$. By definition of Cerami sequence we have

$$\frac{\langle I'(u_n), u_n\rangle}{\|u_n\|^4} = o(1),$$

that is

$$o(1) = b + \lambda \int_\Omega \frac{\phi_{u_n} u_n^2}{\|u_n\|^4} dx - \int_\Omega \frac{m(x)|u_n|^q}{\|u_n\|^4} dx - \int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|^4} dx. \tag{2.3}$$

By Sobolev and Hölder inequalities, we have

$$\int_\Omega m(x)|u_n|^q dx \le |m|_{\frac{2}{2-q}} |u_n|_2^q \le C|m|_{\frac{2}{2-q}} \|u_n\|^q. \tag{2.4}$$

Hence

$$\int_\Omega \frac{m(x)|u_n|^q}{\|u_n\|^4}dx \to 0. \tag{2.5}$$

By $(H4)$, we get that

$$\int_\Omega \frac{|f(x,u_n)u_n|}{\|u_n\|^4}dx = \int_\Omega \left|\frac{f(x,u_n)}{u_n}\right|\frac{u_n^2}{\|u_n\|^4}dx \le \frac{C}{\|u_n\|^2} \to 0. \tag{2.6}$$

By Lemma 2.1, we have

$$\int_\Omega \phi_{u_n}u_n^2 dx \ge 0,$$

which is a contradiction, because $b > 0$. Thus $\{u_n\}$ is bounded in $H^1(\Omega)$ and the proof is completed. □

Lemma 2.4. Under the assumptions of Lemma 2.3, any Cerami sequence of I has a convergent subsequence in $H^1(\Omega)$.

Proof . Let $\{u_n\}$ be a Cerami sequence of I. We show that $\{u_n\}$ possesses a strong convergent subsequence. Since $\{u_n\}$ is bounded in $H^1(\Omega)$ $\big($Lemma 2.3$\big)$, we may assume that for some $u \in H^1(\Omega)$, up to a subsequence, $u_n \rightharpoonup u$ in $H^1(\Omega)$. By the fact that the embedding $H^1(\Omega) \hookrightarrow L^p_{loc}(\Omega)$ is compact for $p \in [2,6)$, it is easy to see that

$$u_n \to u \quad \text{in} \quad L^p(\Omega), \quad p \in [2,6). \tag{2.7}$$

Since $\langle I'(u_n), u\rangle = o(1)$ and $\langle I'(u_n), u_n\rangle = o(1)$,

$$o(1) = a\int_\Omega |\nabla u_n|\nabla u dx + b\|u_n\|^2\int_\Omega |\nabla u_n||\nabla u|dx - \lambda\int_\Omega \phi_{u_n}u_n u dx - \int_\Omega m(x)|u_n|^{q-2}u_n u dx - \int_\Omega f(x,u_n)u dx$$

and

$$o(1) = a\int_\Omega |\nabla u_n|\nabla u_n dx + b\|u_n\|^2\int_\Omega |\nabla u_n||\nabla u_n|dx - \lambda\int_\Omega \phi_{u_n}u_n u_n dx - \int_\Omega m(x)|u_n|^{q-2}u_n u_n dx - \int_\Omega f(x,u_n)u_n dx.$$

Hence, we have

$$\int_\Omega \phi_{u_n}u_n(u_n-u)dx = o(1), \tag{2.8}$$

$$\int_\Omega m(x)|u_n|^{q-1}(u_n-u)dx = o(1), \tag{2.9}$$

and

$$\int_\Omega f(x,u_n)(u_n-u)dx = o(1). \tag{2.10}$$

In fact, by Hölder inequality and (2.7), we have

$$\int_\Omega \phi_{u_n}u_n(u_n-u)dx \le |\phi_{u_n}|_6|u_n|_3|u_n-u|_2 \le C|u_n-u|_2 \to 0. \tag{2.11}$$

Similarly,

$$\begin{aligned}\left|\int_\Omega m(x)|u_n|^{q-1}(u_n-u)dx\right| &\le \int_\Omega |m(x)||u_n|^{q-1}|u_n-u|dx \\ &\le |m|_{\frac{2}{2-q}}|u_n|_2^{q-1}|u_n-u|_2 \to 0.\end{aligned} \tag{2.12}$$

By $(H4)$, we can see

$$\begin{aligned}\left|\int_\Omega f(x,u_n)(u_n-u)dx\right| &\le \int_\Omega \left|\frac{f(x,u_n)}{u_n}\right||u_n||u_n-u|dx \\ &\le C|u_n|_2|u_n-u|_2 \to 0.\end{aligned} \tag{2.13}$$

Thus, by (2.11), (2.12) and (2.13) we have

$$(a+b\|u_n\|^2)\Big(\int_\Omega (\nabla u_n - \nabla u)\nabla u_n\Big) \to 0 \tag{2.14}$$

We have to prove that

$$\|u_n\| \to \|u\|.$$

By definition of the norm

$$\begin{aligned} \|u_n - u\|^2 &= \int_\Omega (\nabla u_n - \nabla u)(\nabla u_n - \nabla u)dx \\ &= \int_\Omega (\nabla u_n - \nabla u)\nabla u_n dx - \int_\Omega (\nabla u_n - \nabla u)\nabla u dx. \end{aligned} \tag{2.15}$$

Thus,

$$\int_\Omega (\nabla u_n - \nabla u)\nabla u_n dx = \|u_n - u\|^2 + \int_\Omega (\nabla u_n - \nabla u)\nabla u dx. \tag{2.16}$$

Moreover,

$$\begin{aligned} \int_\Omega (\nabla u_n - \nabla u)\nabla u_n dx &= \int_\Omega |\nabla u_n|^2 - \nabla u_n \nabla u dx \\ &= \|u_n\|^2 - \int_\Omega \nabla u_n \nabla u dx. \end{aligned} \tag{2.17}$$

Hence, we have

$$2\int_\Omega (\nabla u_n - \nabla u)\nabla u_n dx = \|u_n - u\|^2 + \|u_n\|^2 - \|u\|^2. \tag{2.18}$$

By (2.14)-(2.18), we obtain

$$\frac{a + b\|u_n\|^2}{2}\|u_n - u\|^2 + \frac{a + b\|u_n\|^2}{2}\Big(\|u_n\|^2 - \|u\|^2\Big) \to 0. \tag{2.19}$$

Since $a \geq 0$ and $b > 0$, we obtain

$$\|u_n - u\|^2 \to 0,$$

and

$$\|u_n\|^2 - \|u\|^2 \to 0.$$

Therefore, it is easy to see that $\|u_n\| \to \|u\|$ in $H^1(\Omega)$. Therefore, $u_n \to u$ in $H^1(\Omega)$ and the proof is completed. □

3 Existence and multiplicity results

In this section, under the assumptions on m and f, we give the proof of Theorems 1.1-1.2. Clearly, by $(H3)$ and $(H4)$, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(x,u)| \leq \varepsilon|u| + C_\varepsilon |u|^{p-1}, \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}. \tag{3.1}$$

and

$$|F(x,u)| \leq \varepsilon u^2 + C_\varepsilon |u|^p, \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}, \tag{3.2}$$

for some $p \in (2,6)$.

Lemma 3.1. Suppose that $a > 0$, $b \geq 0$ and $(H1) - (H4)$ hold. Then

(i) There exists $M > 0$ and $\rho_1 > 0$ such that for all m with $|m|_{\frac{2}{2-q}} < M$,

$$I(u) > 0, \text{ for } u \in H^1(\Omega) \text{ with } \|u\| = \rho_1.$$

(ii) There exist $\lambda* > 0$ and $\rho_2 > 0$ such that for all $\lambda > \lambda*$,

$$I(u) > 0, \text{ for } u \in H^1(\Omega) \text{ with } \|u\| = \rho_2.$$

Proof . (i) By Sobolev inequality and (2.4) and (3.2) we have

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{4}\int_\Omega \phi_u u^2 dx - \frac{1}{q}\int_\Omega m(x)|u|^q dx - \int_\Omega F(x,u)dx \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{4}\int_\Omega \phi_u u^2 dx - \frac{1}{q}|m|_{\frac{2}{2-q}}\|u\|^q - \varepsilon\int_\Omega u^2 dx - C_\varepsilon \int_\Omega |u|^p dx \\ &\geq C_1\|u\|^2 - C_2|m|_{\frac{2}{2-q}}\|u\|^q - C_\varepsilon\|u\|^p \\ &\geq \left(C_1 - C_2|m|_{\frac{2}{2-q}}\|u\|^{q-2} - C_\varepsilon\|u\|^{p-2}\right)\|u\|^2. \end{aligned} \tag{3.3}$$

Let

$$J(t) = C_1 - C_2|m|_{\frac{2}{2-q}} t^{q-2} - C_\varepsilon t^{p-2}, \text{ for } t > 0.$$

Since $1 < q < 2 < p$, the function $J(t)$ achieves its maximum on $(0,\infty)$ at $t_0 > 0$. Moreover, there exists $M > 0$ such that for $|m|_{\frac{2}{2-q}} < M$, we have

$$\max_{t\in(0,\infty)} J(t) = J(t_0) > 0.$$

By $\rho_1 = t_0$, the proof will be completed.

(ii) As in [17], by equation (2.1) we have

$$\begin{aligned} \sqrt{\lambda}\int_\Omega |u|^3 dx &= \sqrt{\lambda}\int_\Omega \nabla\phi_u \nabla|u| dx \\ &\leq \frac{1}{2}\int_\Omega (|\nabla|u||^2 + \lambda|\nabla\phi_u|^2)dx \\ &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda\phi_u u^2)dx. \end{aligned} \tag{3.4}$$

Thus,

$$\lambda\int_\Omega \phi_u u^2 dx \geq 2\sqrt{\lambda}\int_\Omega |u|^3 - \int_\Omega |\nabla u|^2 dx. \tag{3.5}$$

By (2.4), (3.2) and (3.5) with $p = 3$ for λ large enough, we obtain

$$\begin{aligned} I(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{4}\int_\Omega \phi_u u^2 dx - C|m|_{\frac{2}{2-q}}\|u\|^q - \int_\Omega \varepsilon u^2 + C_\varepsilon|u|^3 dx \\ &\geq \left(\frac{a}{2} - \varepsilon - \frac{1}{4}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 + \left(\frac{\sqrt{\lambda}}{2} - C_\varepsilon\right)\int_\Omega |u|^3 dx - C|m|_{\frac{2}{2-q}}\|u\|^q \\ &\geq \left(\frac{a}{2} - \varepsilon - \frac{1}{4}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C|m|_{\frac{2}{2-q}}\|u\|^q. \end{aligned} \tag{3.6}$$

Since $q < 2$, if we choose ρ_2 large enough, then the conclusion holds. The proof is completed. □

Proof .[Theorem 1.1] By Lemma 3.1 (i), we define

$$\overline{B}_{\rho_1} = \{u \in H^1(\Omega)\ : \|u\| \leq \rho_1\}, \qquad \partial B_{\rho_1} = \{u \in H^1(\Omega)\ : \|u\| = \rho_1\}.$$

Then we have

$$I\big|_{\partial B_{\rho_1}} > 0. \tag{3.7}$$

Clearly $I \in C^1(\overline{B}_\rho, \mathbb{R})$, hence I is lower semicontinuous and bounded from below on $\overline{B}_\rho$. Let

$$c_1 = \inf\{I(u) : u \in \overline{B}_\rho\} > -\infty.$$

By $(H2)$, we can choose $v \in C_0^\infty(\Omega)$. Since $m(x) > 0$ on Ω and $1 < q < 2$, it is easy to obtain

$$I(tv) < 0, \quad \text{for } t \text{ small}.$$

Thus $c_1 < 0$. Now by (3.6), Lemma 2.4 and Ekeland's variational principle, c_1 can be achieved at some inner point $u_1 \in \overline{B}_\rho$ and u_1 is a critical point of I. $\square$

By applying 3.1(ii), we can prove Theorem 1.2 using the same argument as in the proof of Theorem 1.1. The details are omitted.

4 Existence of infinitely many solutions

In this section, we prove the existence of infinitely many solutions using the critical point theorem when $f(x, u)$ is odd in u.

Proposition 4.1. [25] Assume X is a reflexive Banach space, $I \in C^1(X, \mathbb{R})$ satisfies the (PS) condition and is even and bounded from below, $I(0) = 0$. If for any $k \in \mathbb{N}$, there exist a k-dimensional subspaces X^k and $\rho_k > 0$, such that

$$\sup_{X^k \bigcap S_{\rho_k}} I < 0,$$

where $S_{\rho_k} = \{u \in X \ : \ \|u\| = \rho_k\}$. Then I has a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$.

Proof . [Theorem 1.3] By Lemma 2.4, I satisfies the Cerami condition. Using (3.6), it is easy to see that I is coercive in E and bounded from below for large λ. In order to apply Proposition 4.1, for any $n \in \mathbb{N}$, it suffices to fined a subspace E_n and $\rho_n > 0$ such that

$$\sup_{E_n \bigcap S_{\rho_n}} I < 0.$$

In fact, for any $n \in \mathbb{N}$, we fined n linearly independent functions $e_1, ..., e_n \in C_0^\infty(\Omega)$, and define $E_n := span\{e_1, ..., e_n\}$. Since $m(x) > 0$ in Ω, we can choose

$$\|u\|_{q,m} = \Big(\int_\Omega m(x)|u|^q dx\Big)^{\frac{1}{q}},$$

as an equivalent norm in E_n. Using the fact that all the norms on E_n are equivalent, for $u \in E_n$, similar to (3.3) and by Lemma 2.1 (i), we have

$$\begin{aligned} I(u) &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{4}\int_\Omega \phi_u u^2 dx - \|u\|_{q,m}^q \\ &\leq C_1\|u\|^2 + \Big(b + \lambda C\Big)\|u\|^4 - \|u\|_{q,m}^q \\ &< 0, \end{aligned}$$

for $\|u\| = \rho_n$ small since $q < 2$. The proof is completed. $\square$

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