Int. J. Nonlinear Anal. Appl. In Press, 1–12 ISSN: 2008-6822 (electronic) http://dx.doi.org/10.22075/ijnaa.2024.34744.5196



Principally π -Baer rings

Somayeh Moradia, Hamid Haj Seyyed Javadia,*, Ahmad Moussavib

^aDepartment of Mathematics and Computer Science, Shahed University, Tehran, Iran ^bDepartment of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran, Iran

(Communicated by Madjid Eshaghi Gordji)

Abstract

We say a ring is right principally π -Baer (or simply right $p.\pi$ -Baer) if an idempotent generates the right annihilator of every projection invariant principal left ideal. The class of right $p.\pi$ -Baer rings includes the von Neumann regular rings (and hence right p.p-rings) and all π -Baer rings. This class of rings is closed under direct products. The behavior of the right $p.\pi$ -Baer condition is investigated concerning various constructions and extensions. Moreover, we extend a theorem of Kist for commutative p.p-rings to right $p.\pi$ -Baer rings for which every prime ideal contains a unique minimal prime ideal without using topological arguments.

Keywords: p.p ring, Baer ring, quasi-Baer ring, principally quasi-Baer ring, π -Baer ring, $p.\pi$ -Baer ring, semicentral idempotent

2020 MSC: 16N60

1 Introduction

Throughout this paper, all rings are associative with identity, and all modules are unital. A ring R is called (quasi-)Baer if the right annihilator of each (right ideal) nonempty subset of R is generated by an idempotent. In [12], Kaplansky introduced Baer rings to abstract various properties of AW^* -algebras and von Neumann algebras. In [9], Clark defined a ring to be quasi-Baer if the left annihilator of each ideal is generated, as a left ideal, by an idempotent. He then uses the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra.

A ring R is called right (left) p.p if every principal right (left) ideal is projective (equivalently, if the right (left) annihilator of any element of R is generated (as a right (left) ideal) by an idempotent of R. The ring R is called p.p if it is both right and left p.p. Birkenmeier et al. [7] initiated the concept of principally quasi-Baer rings. A ring R is called right principally quasi-Baer (or simply right p.q-Baer) if an idempotent generates the right annihilator of a principal right ideal. Equivalently, R is right p.q-Baer if R modulo the right annihilator of any principal right ideal is projective. A ring R is called p.q-Baer if it is both right and left p.q-Baer rings includes all biregular rings, all quasi-Baer rings and all abelian p.p rings. Further work on quasi-Baer and p.q-Baer rings appears in [4, 5, 6, 7, 15].

Recall from [2] that the left (right) ideal I of R is said projection invariant if for each $e = e^2 \in R$, $Ie \subseteq I(eI \subseteq I)$. If R is an abelian ring, then every one-sided ideal is an invariant projection. Note that every ideal of R is projection

*Corresponding author

Email addresses: somayeh.moradi@shahed.ac.ir (Somayeh Moradi), h.s.javadi@shahed.ac.ir (Hamid Haj Seyyed Javadi), moussavi.a@modares.ac.ir (Ahmad Moussavi)

invariant, but not conversely. In [2], Birkenmeier et al. define a ring to be π -Baer if the right annihilator of every projection invariant left ideal (as a right ideal) is generated by an idempotent of R. Like the quasi-Baer case, the π -Baer condition is left-right symmetric.

Shahidikia et al. in [16] introduced the concept of generalized π -Baer rings. A ring R is called generalized right π -Baer, if for any projection invariant left ideal I of R, the right annihilator of I^n is generated by an idempotent, for some positive integer n, depending on I.

In this paper, we introduce the notion of right (left) $p.\pi$ -Baer rings. Ring R is right $p.\pi$ -Baer if the right annihilator of each projection invariant principal left ideal is generated by an idempotent. Left case may be defined analogously. If ring R is both right and left $p.\pi$ -Baer then we say R is $p.\pi$ -Baer. The class of right $p.\pi$ -Baer is closed with respect to direct products. If R is an abelian ring, then the $p.\pi$ -Baer and p.p conditions agree.

A ring R [8] is right π -extending if every projection invariant right ideal of R is essential in a direct summand of R_R . We say that a ring R is right principally π -extending if every projection invariant principal right ideal of R is essential in a direct summand of R_R . We demonstrate the connection between the principally π -extending property and the $p.\pi$ -Baer property. Recall that an idempotent $e \in R$ is left (right) semicentral if xe = exe (ex = exe), for all $x \in R$ and is denoted by $S_l(R)$ ($S_r(R)$). If xe = ex = exe for all $x \in R$, then $e = e^2 \in R$ is called a central idempotent and is denoted by B(R). Let R be a ring, then $Z_l(R)(Z_r(R)), r_R(X)(l_R(X)), I(R), P(R)$ and N(R) denote the left singular ideal of R (the right singular ideal of R), right annihilator of X in R (left annihilator of X in R) for a nonempty subset X of R, subring of R is generated by idempotents, the prime radical and the set of nilpotent elements of R, respectively.

2 The Preliminaries and Basic Concepts

Definition 2.1. A principal left (right) ideal I of R is called projection invariant if $Ie \subseteq I$ ($eI \subseteq I$) for all $e = e^2 \in R$.

Definition 2.2. A ring R (with unity) is projection invariant principally right Baer (denoted right principally π -Baer or right p. π -Baer) if for each projection invariant principal left ideal I there exists $c = c^2 \in R$ such that $r_R(I) = cR$

Proposition 2.3. A ring R is right $p.\pi$ -Baer if and only if for every projection invariant principal left ideal I of R, there exists $e \in S_l(R)$ such that $r_R(I) = eR$.

Proof. Let R is a right $p.\pi$ -Baer and I be a projection invariant principal left ideal of R. By definition there is an idempotent $e \in R$ such that $r_R(I) = eR$. Let $c = c^2 \in R$. Since I is projection invariant principal left ideal, $Ic \subseteq I$ and hence, $eR = r_R(I) \subseteq r_R(Ic)$. Thus Ice = 0. Then $ce \in r_R(I) = eR$, hence ce = ece. By [10, Proposition 1], $e \in S_l(R)$. The converse is obvious. \Box

Lemma 2.4. [2, Lemma 2.3] The following conditions are equivalent:

- (i) R is abelian.
- (ii) Every one-sided ideal is projection invariant.
- (iii) Every right ideal generated by an idempotent is projection invariant.
- (iv) Every right ideal generated by an idempotent is an ideal.

Recall from [1] that R is said to satisfy the *IFP* (insertion of factors property) or is semicommutative if right annihilator of any element (subsets) of R is an ideal of R. Note that for every R ring, reduced condition implies IFP and IFP condition implies abelian; but the converse is not hold.

Proposition 2.5. The following conditions are equivalent:

- (i) R is domain.
- (ii) R is π -Baer and 0 and 1 are the only idempotents of R.
- (iii) R is $p.\pi$ -Baer and 0 and 1 are the only idempotents of R.
- (iv) R is right $p.\pi$ -Baer and 0 and 1 are the only idempotents of R.

Proof . (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) are clear.

 $(iv) \Rightarrow (i)$ Let xy = 0 for $x, y \in R$. Then $y \in r_R(Rx)$. Since 0 and 1 are the only idempotents of R, by Lemma 2.4, Rx is projection invariant principal left ideal of R, so $r_R(Rx) = eR$. If e = 0, then y = 0 and if e = 1, then x = 0. Thus, R is a domain. \Box

Proposition 2.6. [7, Proposition 1.7] The following conditions are equivalent:

(i) R is a right p.q-Baer ring.

- (ii) The right annihilator of any finitely generated right ideal is generated by an idempotent.
- (iii) The right annihilator of any principal ideal is generated by an idempotent.
- (iv) The right annihilator of any finitely generated ideal is generated by an idempotent.

Proposition 2.7. If R has the IFP, then the following conditions are equivalent:

- (i) R is a right $p.\pi$ -Baer ring.
- (ii) The right annihilator of any finitely generated projection invariant left ideal is generated by an idempotent.
- (iii) The right annihilator of any principal ideal is generated by an idempotent.
- (iv) The right annihilator of any finitely generated ideal is generated by an idempotent.

Proof. (i) \iff (ii) Clearly, (ii) implies (i). Assume that (i) holds and $I = \sum_{i=1}^{n} Rx_i$. Since any Rx_i is projection invariant principal left ideal of R, there exists $e_i = e_i^2 \in S_l(R)$ such that $r_R(Rx_i) = e_iR$. Since each e_i is central, there is an idempotent $e \in R$ such that $r_R(I) = r_R(\sum_{i=1}^{n} Rx_i) = \bigcap_{i=1}^{n} e_iR = eR$.

The equivalences (i) \iff (iii) and (ii) \iff (iv) follow from the fact that $r_R(I) = r_R(IR)$, where I is any projection invariant left ideal of R. \Box

Proposition 2.8. (i) Assume that R = I(R). If R be right p.q-Baer ring, then R is right p. π -Baer.

(ii) Let R be a ring with IFP. R is right $p.\pi$ -Baer if and only if R is right p.q-Baer.

Proof. (i) Assume R = I(R) and I be a projection invariant principal left ideal of R, then I is a principal ideal. By Proposition 2.6, $r_R(I) = eR$.

(ii) Assume R be right $p.\pi$ -Baer and I be a principal right ideal of R. Then RI is a principal ideal. By Proposition 2.7, $r_R(I) = r_R(RI) = eR$.

Conversely, assume I be a projection invariant principal left ideal of R, then IR is a principal ideal of R. Proposition 2.6 yields, $r_R(I) = r_R(IR) = eR$. \Box

The following example shows that there exists a right p.q-Baer ring which is not right $p.\pi$ -Baer.

Example 2.9. Let

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z}, a \equiv d, b \equiv 0 \text{ and } c \equiv 0 \pmod{2} \right\}.$$

R is a prime ring. Then is right p.q-Baer ring, but *R* is not domain. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are the only idempotents of *R*. Therefore by Proposition 2.5, *R* is not right *p.π*-Baer ring.

Proposition 2.10. (i) Let R is a right p.p ring. Then R is right $p.\pi$ -Baer.

(ii) Let R is an abelian right $p.\pi$ -Baer, then R is right p.p.

Proof. (i) Let I be a projection invariant principal left ideal of R. Then there exists $x \in R$ such that I = Rx. Since R is right p.p ring, $r_R(I) = r_R(Rx) = r_R(x) = eR$, for some $e = e^2 \in S_l(R)$.

(ii) Let $x \in R$. Then Rx is a projection invariant principal left ideal of R. Thus $r_R(x) = r_R(Rx) = eR$, for some $e \in S_l(R)$. \Box

Corollary 2.11. Let R is an abelian ring. Then the following conditions are equivalent:

(i) R is right p.p.

(ii) R is p.p.

(iii) R is right $p.\pi$ -Baer.

Proof . (i) \Rightarrow (ii) It follows from [7, Proposition 1.14].

(ii) \Rightarrow (iii) and (iii) \Rightarrow (i) these implications follow from Proposition 2.10. \Box

Proposition 2.12. If R be right $p.\pi$ -Baer ring, then the following are equivalent:

(i) R is abelian.

(ii) R is semicommutative.

(iii) R is reduced.

Proof. (i) \Rightarrow (ii) Assume that R is an abelian right $p.\pi$ -Baer ring. Let $x \in R$, then Rx is a projection invariant principal left ideal of R. Thus $r_R(x) = r_R(Rx) = eR$, where $e \in S_l(R)$. Hence by [7, Lemma 1.1], r(x) is an ideal.

(ii) \Rightarrow (iii) Let $x^2 = 0$ for some $x \in R$. Then $Rx^2 = 0$ and so $x \in r_R(Rx) = eR$, where e is a central idempotent. Then x = ex = xe = 0. Hence, R is a reduced ring.

(iii) \Rightarrow (i) The proof is routine. \Box

Proposition 2.13. An abelian right $p.\pi$ -Baer ring is semiprime.

Proof. It follows from Proposition 2.12 and [13, Example 10.17]. \Box

It is clear that every π -Baer ring is a right $p.\pi$ -Baer ring. Examples 2.14 and 2.16 show that there are right $p.\pi$ -Baer rings that are not π -Baer.

Example 2.14. Let F be a fild and $F_n = F$ for $n = 1, 2, \cdots$. The ring

$$R = \begin{pmatrix} \prod_{n=1}^{\infty} F_n & \bigoplus_{n=1}^{\infty} F_n \\ \bigoplus_{n=1}^{\infty} F_n & < \bigoplus_{n=1}^{\infty} F_n, 1 > \end{pmatrix}$$

is a subring of the 2 × 2 matrix ring over the ring $\prod_{n=1}^{\infty} F_n$, where $\langle \bigoplus_{n=1}^{\infty} F_n, 1 \rangle$ is the *F*-algebra generated by $\bigoplus_{n=1}^{\infty} F_n$ and $\lim_{n=1}^{\infty} F_n$. The ring *R* is a regular ring (hence p.p). Thus *R* is a right *p*. π -Baer ring. But by [7, Example 1.6] the ring *R* is not right p.q-Baer. Therefore is not quasi-Baer and by [2, Theorem 2.1], *R* is not π -Baer ring.

Lemma 2.15. [7, Lemma 1.4] Let T be a ring with unity such that |T| > 1, and let $S = \prod_{\lambda \in \Lambda} T_{\lambda}$, where $T_{\lambda} = T$ and Λ is an infinite indexing set. If R is the subring of S generated by $\bigoplus_{\lambda \in \Lambda} T_{\lambda}$ and either $1 \in S$ or $\{f : \Lambda \to T \mid f \text{ is a constant function}\}$, then R is not quasi-Baer. Moreover if T is a right p.q-Baer ring, then so is R.

Example 2.16. (i) Let T in Lemma 2.15 be a commutative regular ring. Then R is p.p, hence R is right $p.\pi$ -Baer ring that is not π -Baer, as is not quasi-Baer.

(ii) Let F be a filed and $F_i = F$ for $i = 1, 2, \cdots$. Let $R = \prod_{i=1}^{\infty} F_i$ and S be the subring of R generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_R , then S is commutative p.p. hence right p. π -Baer. But S is not quasi-Baer, so is not π -Baer.

Proposition 2.17. Let R be a left Noetherian and right $p.\pi$ -Baer ring with IFP. Then R is π -Baer.

Proof. Let *I* be a projection invariant left ideal of *R*. Since *R* is left Noetherian, *I* is a finitely generated projection invariant left ideal. By Proposition 2.7, $r_R(I) = eR$. Hence *R* is π -Baer. \Box

Proposition 2.18. The center of $p.\pi$ -Baer ring is a p.p ring.

Proof. Let Z(R) be the center of R and $a \in Z(R)$. Then $r_R(Ra) = eR$ with $e \in S_l(R)$ and $l_R(aR) = Rf$ with $f \in S_r(R)$, as R is a $p.\pi$ -Baer ring. Since $r_R(Ra) = l_R(aR)$, we have e = f. Hence $e \in B(R)$. Consequently, $r_{Z(R)}(a) = r_R(a) \cap Z(R) = eZ(R)$. Therefore, Z(R) is a p.p ring. \Box

Corollary 2.19. Let S be a subring of R and $I(R) \subseteq S$. If X is a projection invariant principal left (right) ideal of S, then RX(XR) is a projection invariant principal left (right) ideal of R. Moreover, if R is right (left) $p.\pi$ -Baer, then S is right (left) $p.\pi$ -Baer.

Proof. Assume $I(R) \subseteq S$ and X is a projection invariant principal left ideal of S. Then $(RX)e = R(Xe) \subseteq RX$ for all $e = e^2 \in R$. So RX is a projection invariant principal left ideal of R. Since R is right $p.\pi$ -Baer, $r_R(RX) = eR$ for some $e = e^2 \in R$. It follows that $r_S(X) = eS$, so S is right $p.\pi$ -Baer. \Box

The following example shows that a subring of a right $p.\pi$ -Baer ring need not be right $p.\pi$ -Baer.

Example 2.20. Let $R = \mathbb{Z} \oplus \mathbb{Z}$, then by Theorem 2.24, R is a right $p.\pi$ -Baer ring. Consider the subring $S = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b(modp)\}$, where p is a prime number. The only idempotents of S are (0,0) and (1,1). For projection invariant principal left ideal S(0,p), we have $r_S(S(0,p)) = (p,0)S$, that dose not contain a nonzero idempotent of S. Hence S is not right $p.\pi$ -Baer ring.

Proposition 2.21. A ring R is right $p.\pi$ -Baer if and only if whenever I is a projection invariant principal left ideal of R there exists $e \in S_r(R)$ such that $I \subseteq Re$ and $r_R(I) \cap Re = (1-e)Re$.

Proof. Assume R is right $p.\pi$ -Baer and I is a projection invariant principal left ideal of R. Then there exists $c \in S_l(R)$ such that $r_R(I) = cR$. Hence $I \subseteq R(1-c)$. Let e = 1-c, then $I \subseteq Re$ and $e \in S_r(R)$. Now $r_R(I) \cap Re = (1-e)R \cap Re = (1-e)Re$. Conversely, let I be a projection invariant principal left ideal of R and assume there exists $e \in S_r(R)$ such that $I \subseteq Re$ and $r_R(I) \cap Re = (1-e)Re$. Let $\alpha \in r_R(I)$. Then $\alpha = e\alpha + (1-e)\alpha$. So $\alpha e = e\alpha e + (1-e)\alpha e$. But $\alpha e \in r_R(I) \cap Re$, hence $\alpha e = (1-e)\alpha e$ and so $e\alpha e = 0$. Since $e \in S_r(R)$, $0 = e\alpha e = e\alpha$ and $\alpha = (1-e)\alpha \in (1-e)R$. Hence $r_R(I) \subseteq (1-e)R$. Thus R is right $p.\pi$ -Baer. \Box

Observe that Proposition 2.21 is the right $p.\pi$ Baer analogue of characterization of a right p.q-Baer ring in [7, Proposition 1.9]. However, contrary to the right p.q-Baer case in [7, Corollary 1.10], following example shows that I + (1 - e)Re is not essential in Re.

Example 2.22. Let R be the free ring $\mathbb{Z} < x, y >$. Since R is domain, R is right $p.\pi$ -Baer. Take I = Rx. Since 0 and 1 are the only idempotents of R, I is a projection invariant principal left ideal of R. Then $r_R(I) \cap Re = 0 = (1-e)Re$, and I + (1-e)Re = I is not essential in R.

Proposition 2.23. Let R be a right $p.\pi$ -Baer ring. If every essential left ideal of R is an essential extension of a projection invariant principal left ideal of R, then R is left nonsingular.

Proof. Let $0 \neq x \in Z_l(R)$. Then $l_R(x) \leq e^{ess} R$. By hypothesis, there exists a projection invariant principal left ideal I of R such that $I \leq e^{ess} l_R(x)$. So $I \leq e^{ess} R$. Hence Ix = 0 implies that $x \in r_R(I) = eR$, because R is right $p.\pi$ -Baer. It is clear that $I \subseteq l_R(r_R(I)) = R(1-e)$. Since I is essential in R, e = 0 and hence x = 0, a contradiction. \Box

If R be an abelian right $p.\pi$ -Baer ring, then R is right nonsingular, since every right p.p ring is right nonsingular.

Theorem 2.24. Let $R = \prod_{i \in K} R_i$. Then R is a right $p.\pi$ -Baer ring if and only if R_i is a right $p.\pi$ -Baer ring for each $i \in K$.

Proof. Let π_i , be ith projection homomorphism of $R = \prod_{i \in K} R_i$. Assume that each R_i is a right $p.\pi$ -Baer ring and I is a projection invariant principal left ideal of R. Then there exists $e_i = e_i^2 \in R_i$ such that $r_{R_i}(\pi_i(I)) = e_i R_i$. Now put $e = (e_i)_{i \in K} \in R$. Clearly, $\pi_i(e) = e_i$ for each $i \in K$. Thus $r_R(I) = eR$ and hence R is a right $p.\pi$ -Baer ring.

Conversely, assume R is a right $p.\pi$ -Baer ring, then each projection invariant principal left ideal I_i of R_i can consider as a projection invariant principal left ideal of R, since R is a right $p.\pi$ -Baer ring, $r_R(I_i) = eR$. If put $e_i = \pi_i(e)$, then we have $r_{R_i}(I_i) = e_i R_i$. Therefore R_i is a right $p.\pi$ -Baer ring. \Box

Definition 2.25. A ring R is called principally left (right) π -extending if every projection invariant principal left (right) ideal of R is essential in a left (right) ideal is generated by an idempotent of R.

Theorem 2.26. Let R be a left nonsingular ring. Then R is principally left π -extending if and only if R is right $p.\pi$ -Baer and $_RI \leq ^{ess} l_R(r_R(I))$ for every projection invariant principal left ideal I of R.

Proof. Assume $_RR$ is principally left π -extending and I be a projection invariant principal left ideal of R. Then there is $e = e^2 \in R$ such that $_RI \leq^{ess} Re$. Since $Z(_RR) = 0$, $r_R(I) = r_R(Re) = (1 - e)R$. So R is right $p.\pi$ -Baer. Hence $l_R(r_R(I)) = l_R((1 - e)R) = Re$. Thus, $_RI \leq^{ess} l_R(r_R(I))$. Conversely, assume R is right $p.\pi$ -Baer and $_RI \leq^{ess} l_R(r_R(I))$ for every projection invariant principal left ideal I of R. Since R is right $p.\pi$ -Baer, $r_R(I) = eR$. Now, $l_R(r_R(I)) = l_R(eR) = R(1 - e) = Rc$, for some $c = c^2 \in R$. Therefore $_RI \leq^{ess} Rc$. Then $_RR$ is principally left π -extending. \Box

Lemma 2.27. [2, Lemma 2.4] Assume R_R is nonsingular. Then the following conditions are equivalent:

- (i) Every closed right ideal of R is a right annihilator.
- (ii) R is right cononsingular.
- (iii) For each $I_R \leq R_R, I_R \leq^{ess} r_R(l_R(I))$.

Corollary 2.28. Let R is right nonsingular and right cononsingular. Then R is left $p.\pi$ -Baer if and only if R_R is principally right π -extending.

Proof. This follows from Lemma 2.27 and Theorem 2.26. \Box

Note: If R be a right $p.\pi$ -Baer ring with $S_l(R) = B(R)$, then R have no nonzero nilpotent projection invariant principal left ideal. Assume $I \neq 0$ be nilpotent projection invariant principal left ideal. Then $I^n = 0$ and $I^{n-1} \neq 0$ for some positive integer n. Since R is right $p.\pi$ -Baer, $r_R(I) = eR$ for some $e = e^2 \in R$. But $I^{n-1} \subseteq r_R(I) = eR$. Then $eI^{n-1} = I^{n-1}$, hence we obtain $I^{n-1} = eI^{n-1} = I^{n-1}e = 0$, a contradiction.

For a ring R and (R, R)-bimodule M, the trivial extension of R by M is the ring $T(R, M) = R \bigoplus M$ with the addition componentwise and the following multiplication:

$$(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2),$$

where $a_1, a_2 \in R$ and $m_1, m_2 \in M$. Ring T is isomorphism to the ring of matrices $\begin{pmatrix} a & m \\ 0 & a \end{pmatrix}$, where $a \in R$ and $m \in M$, see [11, 15].

Theorem 2.29. Let R be an abelian ring and define

$$R_n := \{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \},$$

with n a positive integer ≥ 2 . If R_n be a right p. π -Baer ring, then R is right p. π -Baer.

Proof. Let I = Ra be projection invariant principal left ideal of R, where $a \in R$. Consider

$$A = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in R_n$$

Since R is an abelian ring, by [11, Lemma 2], R_n is an abelian ring, hence R_nA is a projection invariant principal left ideal of R_n . Then, $r_{R_n}(R_nA) = eR_n$ for some $e = e^2 \in R_n$. So there exists $f = f^2 \in R$ such that $e = \begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}$. It is clear that $\begin{pmatrix} af & 0 & \cdots & 0 \\ 0 & af & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & af \end{pmatrix} = 0$, so $fR \subseteq r_R(Ra)$. Next let $b \in r_R(Ra)$, then Rab = 0. Clearly, $\begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{pmatrix} \in R_n$ is contained in $r_{R_n}(R_nA) = eR_n$, so $b = fb \in fR$. Therefore R is right $p.\pi$ -Baer. \Box The following example shows that the converse assertion fails.

Example 2.30. Let $R = \mathbb{Z}$. Since R is domain, R is right $p.\pi$ -Baer. Let $R_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be a projection invariant principal left ideal of R_2 , then for $a, b \in R$,

$$r_{R_2}\left(R_2\begin{pmatrix}0&1\\0&0\end{pmatrix}\right) = r_{R_2}\left(\begin{pmatrix}a&b\\0&a\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\right) = r_{R_2}\left(\begin{pmatrix}0&a\\0&0\end{pmatrix}\right) = \left\{\begin{pmatrix}0&d\\0&0\end{pmatrix} \mid d \in \mathbb{Z}\right\},$$

is not generated by an idempotent. Let $\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$ R_2 for $f = f^2 \in R$. Then $\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} fa & fb \\ 0 & fa \end{pmatrix}$. Hence d = 0, a contradiction. Thus R_2 is not right $p.\pi$ -Baer.

3 $p.\pi$ -Baer Matrix Rings

In this section, we characterize right $p.\pi$ -Baer 2-by-2 generalized triangular matrix rings. Throughout this section, $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ denotes a 2-by-2 generalized upper triangular matrix ring where A and B are rings and M is a (A, B)-bimodule.

Theorem 3.1. Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be an abelian 2-by-2 generalized upper triangular matrix ring where A and B are rings and M is a (A, B)-bimodule. Then R is right $p.\pi$ -Baer if and only if the following conditions are satisfied:

- (i) A and B are right $p.\pi$ -Baer rings.
- (ii) For every $a \in A$, $m \in M$ and $b \in B$ there exist $e = e^2 \in A$, $k \in M$ and $f = f^2 \in B$ such that
- (a) $r_A(Aa) = eA$ and ek + kf = k.
- (b) $f \in r_B(Bb) \cap r_B(Aak + Am + Mb)$.
- (c) If $\delta \in B$ and $y \in M$ such that $Bb\delta = 0$ and $Aay + (Am\delta + Mb\delta) = 0$, then $\delta \in fB$ and $y \in eM + kB$.

Proof. Let *I* be projection invariant principal left ideal of *A*, then there exist $a \in A$, such that I = Aa. Clearly, $K = \begin{pmatrix} Aa & 0 \\ 0 & 0 \end{pmatrix}$ is projection invariant principal left ideal of *R*. Since *R* is right *p*. π -Baer,

$$r_R\left(\begin{pmatrix}Aa & 0\\0 & 0\end{pmatrix}\right) = \begin{pmatrix}e & k\\0 & f\end{pmatrix}\begin{pmatrix}A & M\\0 & B\end{pmatrix} = \begin{pmatrix}eA & eM + kB\\0 & fB\end{pmatrix}$$

where $e = e^2$, $f = f^2$ and ek + kf = k. Thus $r_A(Aa) = eA$ and hence A is right $p.\pi$ -Baer. Similarly, B is right $p.\pi$ -Baer. For each $a \in A, m \in M$ and $b \in B$, ideal $K = R \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \{ \begin{pmatrix} \alpha a & \alpha m + xb \\ 0 & \beta b \end{pmatrix} | \alpha \in A, x \in M, \beta \in B \}$ is a projection invariant principal left ideal of R. Let $\begin{pmatrix} \alpha a & \alpha m + xb \\ 0 & \beta b \end{pmatrix} \in R \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$, then

$$\begin{pmatrix} \alpha a & \alpha m + xb \\ 0 & \beta b \end{pmatrix} \begin{pmatrix} e & k \\ 0 & f \end{pmatrix} = \begin{pmatrix} \alpha a e & \alpha a k + (\alpha m + xb)f \\ 0 & \beta b f \end{pmatrix} = 0.$$

Thus, $r_A(Aa) = eA$, $f \in r_B(Bb)$ and $f \in r_B(Am + Mb + Aak)$, because k = ek + kf, so Aak = Aaek + Aakf, then Aak = Aakf. Let $\delta \in B$ and $y \in M$ such that $Bb\delta = 0$ and $Aay + (Am\delta + Mb\delta) = 0$. Then

$$\begin{pmatrix} Aa & Am + Mb \\ 0 & Bb \end{pmatrix} \begin{pmatrix} 0 & y \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence $\begin{pmatrix} 0 & y \\ 0 & \delta \end{pmatrix} \in r_R(R \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) = \begin{pmatrix} eA & eM + kB \\ 0 & fB \end{pmatrix}$, then $\delta \in fB$ and $y \in eM + kB$.

Conversely, assume A and B are right $p.\pi$ -Baer rings and there exist $e = e^2 \in A, k \in M$ and $f = f^2 \in B$ such that conditions (a), (b) and (c) hold. From condition (a) follows that $\begin{pmatrix} e & k \\ 0 & f \end{pmatrix}$ is an idempotent. Also, it is clear that

 $\begin{pmatrix} e & k \\ 0 & f \end{pmatrix} R \subseteq r_R(R \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}). \text{ Now assume } \begin{pmatrix} \gamma & y \\ 0 & \delta \end{pmatrix} \in r_R(R \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}), \text{ then for } \alpha \in A, x \in M \text{ and } \beta \in B, \text{ we have } \begin{pmatrix} \alpha a & \alpha m + xb \\ 0 & \beta b \end{pmatrix} \begin{pmatrix} \gamma & y \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \text{ Therefore } Aa\gamma = 0, Aay + (Am + Mb)\delta = 0 \text{ and } \beta b\delta = 0. \text{ By condition } (c), \delta \in fB \text{ and } y \in eM + kB. \text{ Therefore } \begin{pmatrix} \gamma & y \\ 0 & \delta \end{pmatrix} \in \begin{pmatrix} e & k \\ 0 & f \end{pmatrix} R. \text{ Then } R \text{ is right } p.\pi\text{-Baer. } \Box$

4 $p.\pi$ -Baer Polynomial Extensions

In this section, we investigate the behavior of the $p.\pi$ -Baer condition with respect to polynomial extension R[x](R[[x]]).

Lemma 4.1. [2, Lemma 4.1] Let $e(x) \in R[x](R[[x]])$, where $e(x) = e_0 + e_1x + \dots + e_nx^n(e(x) = e_0 + e_1x + \dots)$.

(i) $e(x) = (e(x))^2$ if and only if $e_k = \sum_{i+j=k} e_i e_j$, for $0 \le k$ and $0 \le i, j \le k$. Hence, e(x) = 0 if and only if $e_0 = 0$.

(ii) If $e(x) = e_0 + e_1 x + \dots + e_n x^n = (e(x))^2$, then $\sum_{i=0}^n e_i = (\sum_{i=0}^n e_i)^2 \in R$.

(iii) Assume that $e(x) = (e(x))^2$. Then,

(a) $e_i \in I(R)$, for all $i \ge 0$. Hence, $I(R[x]) \subseteq I(R)[x]$ and $I(R[[x]]) \subseteq I(R)[[x]]$.

(b) $e_i, e_0e_i, e_ie_0 \in N(R)$, for all $i \ge 1$ where N(R) denotes the subring of R (not necessarily with unity) generated by the nilpotent elements of R.

(iv) Let I be a right ideal of R. Then I is (principal) projection invariant if and only if I[x] (respectively, I[[x]]) is a (principal) projection invariant right ideal in R[x] (respectively, R[[x]]).

Theorem 4.2. Let R[x](R[[X]]) be a right $p.\pi$ -Baer ring, then R is a right $p.\pi$ -Baer.

Proof. Assume that I = Ra is a projection invariant principal left ideal of R. By Lemma 4.1 I[x] = R[x]a is a projection invariant principal left ideal of R[x]. Thus $r_{R[x]}(R[x]a) = e(x)R[x]$. We claim that $r_R(Ra) = e_0R$. By hypothesis R[x]ae(x) = 0, then $ae_0 = 0$ and hence $e_0R \subseteq r_R(Ra)$. Now let $b \in r_R(Ra)$, then R[x]ab = 0 and so $b \in r_{R[x]}(R[x]a) = e(x)R[x]$. Hence b = e(x)g(x), where $g(x) \in R[x]$. So e(x)b = b, hence $b = e_0b \in e_0R$. Therefore, $r_R(Ra) = e_0R$, hence R is a right $p.\pi$ -Baer ring. \Box

Theorem 4.3. Let R be a ring with $S_l(R) = B(R)$. If R be right p. π -Baer ring, then R[x] is right p. π -Baer.

Proof. Assume I[x] = R[x]p(x) be a projection invariant principal left ideal of R[x], where $p(x) = c_0 + c_1x + \cdots + c_nx^n \in R[x]$. By Lemma 4.1, Rc_i is a projection invariant principal left ideal of R. Then there exists $e_i \in S_l(R)$ such that $r_R(Rc_i) = e_iR$, for $i = 0, 1, \cdots, n$. Let $e = e_0e_1 \cdots e_n$. Then $e \in S_l(R)$ and $eR = \bigcap_{i=0}^n r_R(Rc_i)$. Hence $eR[x] \subseteq r_{R[x]}(R[x]p(x))$.

It is clear that $r_{R[x]}(R[x]p(x)) \subseteq r_{R[x]}(Rp(x))$. Now, let $\alpha(x) \in r_{R[x]}(Rp(x))$ and $g(x) = b_0 + \cdots + b_k x^k \in R[x]$. Then

$$g(x)p(x)\alpha(x) = b_0p(x)\alpha(x) + \dots + b_k x^k p(x)\alpha(x) = b_0p(x)\alpha(x) + \dots + b_k p(x)\alpha(x) x^k = 0$$

Hence $\alpha(x) \in r_{R[x]}(R[x]p(x))$. So $r_{R[x]}(R[x]p(x)) = r_{R[x]}(Rp(x))$. Now let $\alpha(x) = a_0 + a_1x + \cdots + a_mx^m \in r_{R[x]}(Rp(x))$, then $Rp(x)\alpha(x) = 0$. For any $b \in R$, we have the following system of equations:

(0) $bc_0a_0 = 0$

(1) $bc_0a_1 + bc_1a_0 = 0$

(2) $bc_0a_2 + bc_1a_1 + bc_2a_0 = 0$

(3) $bc_0a_3 + bc_1a_2 + bc_2a_1 + bc_3a_0 = 0$

:

 $(k) \ \Sigma_{i+j=k} b c_i a_j = 0$

 $(n+m) \ bc_n a_m = 0.$

From equation (0), $a_0 \in r_R(Rc_0) = e_0R$, where $e_0 \in S_l(R)$. Take $b = se_0$ in equation (1), where $s \in R$, then $se_0c_0a_1 + se_0c_1a_0 = 0$. But $se_0c_0a_1 = 0$, as e_0 is central idempotent. So $bc_1a_0 = 0$. Hence $a_0 \in r_R(Rc_1) = e_1R$

where $e_1 \in S_l(R)$. Thus $a_0 \in e_0e_1R$. So $bc_0a_1 = 0$, and $a_1 \in r_R(Rc_0) = e_0R$. In equation (2), take $b = se_0e_1$. Then $se_0e_1c_0a_2 + se_0e_1c_1a_1 + se_0e_1c_2a_0 = 0$. But $se_0e_1c_0a_2 = se_0e_1c_1a_1 = 0$. Hence $se_0e_1c_2a_0 = bc_2a_0 = 0$. So $a_0 \in e_0e_1e_2R$. Equation (2) simplifies to

$$(2)' bc_0 a_2 + bc_1 a_1 = 0.$$

Take $b = se_0$ in equation (2)'. Then $se_0c_0a_2 + se_0c_1a_1 = 0$ and so, $bc_1a_1 = se_0c_1a_1 = 0$, as $se_0c_0a_2 = 0$. Thus $a_1 \in r_R(Rc_1) = e_1R$. Then $a_1 \in e_0e_1R$, and $bc_0a_2 = 0$, so $a_2 \in r_R(Rc_0) = e_0R$. We have $a_0 \in e_0e_1e_2R$, $a_1 \in e_0e_1R$ and $a_2 \in e_0R$. Now in equation (3), take $b = se_0e_1e_2$. Continuing this procedure yields $a_i \in eR$ for all $i = 0, 1, \dots, n$. Hence $\alpha(x) \in eR[x]$. Thus R[x] is right $p.\pi$ -Baer. \Box

5 Minimal Prime Ideals of $p.\pi$ -Baer Rings

In this section, we investigate the condition, every prime ideal contains a unique minimal prime ideal, in a right $p.\pi$ -Baer ring. Let P be a prime ideal of a ring R. O(P) and $\overline{O}(P)$ be defined as

 $O(P) = \{a \in R | aRs = 0, \text{ for some } s \in R \setminus P\} \text{ and } \overline{O}(P) = \{x \in R | x^n \in O(P), \text{ for some } n \in \mathbb{N}\}.$

In [6] Birkenmeier et al. show that if $\overline{O}(P) = P$ for every minimal prime ideal P of a right p.q-Baer ring R, then every prime ideal of R contains a unique minimal prime ideal. In this section, we derive some conditions which ensure that a prime ideal $P = \overline{O}(P)$ in the class of ablian right $p.\pi$ -Baer rings. Let N(R) denote the set of nilpotent elements of a ring R. A ring is called 2-primal if P(R) = N(R). Every commutative ring is 2-primal. It is clear that $O(P) \subseteq P$ and $O(P) \subseteq \overline{O}(P)$. Since $\overline{O}(P)/O(P)$ is the set of all nilpotent elements in the ring R/O(P), the condition that $P = \overline{O}(P)$ gives us important information about the ring R/O(P). For any subset A of R, let supp(A) be the set of all prime ideals P such that $A \notin P$. If $A = \{a\}$, we write supp(a). For any $P \in Spec(R)$, one can show that $\{supp(s)|s \in R\}$ forms a base on Spec(R). This induced topology on Spec(R) is called the hull-kernel topology on Spec(R). From Dauns and Hofmann, Spec(R) is a compact space. Let $\kappa(R) = \bigcup R/O(P)$ be the disjoint union of rings R/O(P) with $P \in Spec(R)$. For any $a \in R$, let $\hat{a} : Spec(R) \to \kappa(R)$ be defined by $\hat{a}(P) = a + O(P)$. Then $\kappa(R)$ is a sheaf of rings over Spec(R) with the topology on $\kappa(R)$ generated by a base $\{\hat{a}(Supp(s))|a, s \in R\}$, and \hat{a} is a global section for $a \in R$. We mean a sheaf representation whose base space is Spec(R) and whose stalks are the quotients R/O(P).

Lemma 5.1. Let P be a prime ideal of a ring R with IFP. Then $O(P) = \bigcup_{s \in R \setminus P} l_R(sR)$.

Proof. Assume $x \in O(P)$. Then there is $s \in R \setminus P$ such that xRs = 0. Thus xs = 0 and hence xsR = 0. Then $x \in l_R(sR) \subseteq \bigcup_{s \in R \setminus P} l_R(sR)$. Now let $x \in \bigcup_{s \in R \setminus P} l_R(sR)$, then there is $s \in R \setminus P$ such that $x \in l_R(sR)$. So xsR = 0, hence xs = 0. Since R satisfies IFP, xRs = 0, which implies that $x \in O(P)$. \Box

Proposition 5.2. Let P be a prime ideal of ring R and $B_P = \{e \in R | e \text{ is a central idempotent such that } e \notin P\}$. Then B_P is a denominator set.

Proof. It is clear that $1 \in B_P$. B_P is multiplicatively closed. Assume $e, f \in B_P$ and $ef \in P$. Observe that $(ef)^2 = ef$. So $eRf \subseteq P$, a contradiction. Hence $ef \notin P$. Now let $e \in B_P$ and $a \in R$ such that ea = 0. Then ae = ea = 0. \Box

Lemma 5.3. Let P be a prime ideal of an abelian right $p.\pi$ -Baer ring R. Then $O(P) = \{a \in R | aRe = 0, e \in R \setminus P\} = T$

Proof. Let $x \in O(P)$. Then xRs = 0 for some $s \in R \setminus P$. Since Rxs = 0, $s \in r_R(Rx) = eR$. So s = es and thus $e \notin P$. Now we have xeR = 0 and since e is central, then xRe = 0, hence $x \in T$. The reverse containment is obvious. \Box

Lemma 5.4. Let P be a prime ideal of an abelian right $p.\pi$ -Baer ring R. Then $O(P) = \sum Rf$, where $f \in P$.

Proof. Let $a \in O(P)$. Then by Lemma 5.3 there exists idempotent $e \in R \setminus P$, such that aRe = 0. So ae = 0 and hence $a = a(1-e) \in R(1-e) \subseteq \sum R(1-e)$. Therefore $O(P) \subseteq \sum R(1-e)$. Now let e be idempotent of R such that $e \notin P$. It is clear that (1-e)Re = 0, then $(1-e) \in O(P)$, hence $R(1-e) \subseteq O(P)$. Therefore $\sum R(1-e) \subseteq O(P)$. Set 1-e = f, then $O(P) = \sum Rf$, where the sum is taken for all central idempotents $f \in R$ such that $f \in P$. \Box

Lemma 5.5. Let P and Q be prime ideals of an abelian right $p.\pi$ -Baer ring R, such that $Q \subseteq P$. Then O(P) = O(Q).

Proof. By definition it is clear that $O(P) \subseteq O(Q)$. From Lemma 5.4, $O(Q) \subseteq O(P)$, thus O(Q) = O(P).

Theorem 5.6. Let R be an abelian right $p.\pi$ -Baer ring such that for any minimal prime ideal B of R, $\overline{O}(B) = B$. Then the following hold:

- (i) R is 2-primal ring.
- (ii) Every prime ideal of R contains a unique minimal prime ideal.
- (iii) For every prime ideal P of R, the ring R/O(P) is 2-primal and P(R/O(P)) is a completely prime ideal.

Proof. (i) From Proposition 2.12 ring R is reduced. So, R is 2-primal.

(ii) Assume Q be a prime ideal of R which contains minimal prime ideals P_1 and P_2 . Lemma 5.5 yields that $O(P_1) = O(Q)$ and $O(P_2) = O(Q)$. So $P_1 = \overline{O}(P_1) = \overline{O}(P_2) = P_2$.

(iii) Let Q/O(P) be a prime ideal of R/O(P). By Lemma 5.4, $O(P) \subseteq O(Q)$. Let B be the unique minimal prime ideal of R contained in P. From Lemma 5.5, we have $B = \overline{O}(B) = \overline{O}(P) \subseteq \overline{O}(Q) = K \subseteq Q$, where K is the unique minimal prime ideal of R contained in Q. Therefore $B = K = \overline{O}(P)$. Then $\overline{O}(P)/O(P)$ is the unique minimal prime ideal of R/O(P). Consequently, $N(R/O(P)) = \overline{O}(P)/O(P) = P(R/O(P))$. Thus R/O(P) is 2-primal. From [3, Lemma 2.1] B is completely prime in R. Then P(R/O(P)) is completely prime ideal in R/O(P), as $B = \overline{O}(P)$. \Box

Proposition 5.7. Let P be a prime ideal of an abelian right p. π -Baer ring R with R = I(R). Then P is a left essential ideal of R or P = eR for some idempotent $e \in R$.

Proof. Let *P* be a prime ideal of *R* and it is not left essential ideal. Then there is a nonzero projection invariant principal left ideal Ra, where $a \in R$, such that $P \cap Ra = 0$. So $P \subseteq r_R(Ra) = eR$. But $RaeR = 0 \subseteq P$. Hence $eR \subseteq P$. So P = eR. \Box

Proposition 5.8. Let P be a prime ideal of an abelian right $p.\pi$ -Baer ring R with R = I(R). Then either

- (i) O(P) is left essential in P; or
- (ii) O(P) = eR.

Proof. Let O(P) is not left essential in P. Then there is a nonzero projection invariant principal left ideal Ra of R such that $Ra \subseteq P$ and $O(P) \cap Ra = 0$. So $O(P) \subseteq r_R(Ra) = eR$. But RaeR = 0 or aRe = 0. If $e \notin P$, then $a \in O(P)$, a contradiction. Hence $e \in P$, then $eR \subseteq P$. It is clear that $eR(1 - e) = 0 \subseteq P$. Then $e \in O(P)$. \Box

A ring R [3] satisfies the CZ2 condition if whenever $(xy)^n = 0$ with $x, y \in R$ and n a positive integer, then $x^m R y^m = 0$ for some positive integer m.

Corollary 5.9. If R is an abelian right $p.\pi$ -Baer and P is a prime ideal of R, then O(P) is the unique minimal prime ideal of R contained in P and O(P) is completely prime.

Proof. It is clear that *R* is reduced. Hence *R* is 2-primal and satisfies CZ2 condition. By [[3], Theorem 2.3(ii)], for any minimal prime ideal *Q* of *R*, $\overline{O}(Q) = Q$. From Theorem 5.6, there exists a unique minimal prime ideal *B* of *R* such that $B \subseteq P$. By Lemma 5.5 O(B) = O(P). So $B = \overline{O}(B) = \overline{O}(P)$. Since *R* is reduced, for every prime ideal *P* of *R*, $O(P) = \overline{O}(P)$, then O(P) is a minimal prime ideal, and hence by [[3], Proposition 1.1 (iii)], O(P) is a completely prime ideal. \Box

Proposition 5.10. Let R be an abelian right $p.\pi$ -Baer ring. If N(R) is an ideal, then $\overline{O}(P)$ is an ideal of R for every prime ideal P of R.

Proof. Let $x, y \in \overline{O}(P)$. So there exist positive integers n, m such that $x^n, y^m \in O(P)$. From Lemma 5.3 there exist central idempotents $e, f \in R \setminus P$, such that $x^n Re = 0$ and $y^m Rf = 0$. Let $r \in R$, then $x^n re = 0 \in N(R)$ and $y^m rf = 0 \in N(R)$. Since N(R) is completely semiprime, $xre \in N(R)$. So there exists a positive integer k such that $(xre)^k = 0 = (xr)^k e$. Hence $(xr)^k Re = (xr)^k eR = 0$. So $xr \in \overline{O}(P)$. Similarly $rx \in \overline{O}(P)$. Now we have $(x - y)ef = xef - yef \in N(R)$, as N(R) is ideal. So there exists a positive integer t such that $[(x - y)ef]^t = 0 = (x - y)^t ef$. Therefore $(x - y)^t Ref = 0$, as ef is a central idempotent. We claim that $ef \notin P$. Assume $ef \in P$. Then $Ref \subseteq P$, hence $\overline{O}(P)$ is an ideal of R. \Box

Theorem 5.11. Let R be an abelian right p. π -Baer ring. Then for every prime ideal P of R, $R[B_P^{-1}] \simeq R/O(P)$.

Proof. It is clear $\{a \in R | aRe = 0 \text{ for some } e \in B_P\} \subseteq \{a \in R \mid ae = 0, \text{ for some } e \in B_P\}$. Now let $a \in R$ such that ae = 0, for some $e \in B_P$. Then a = a(1-e). So aRe = a(1-e)Re = a(1-e)eR = 0. Thus $\{a \in R | aRe = 0 \text{ for some } e \in B_P\} = \{a \in R | ae = 0 \text{ for some } e \in B_P\} = \{a \in R | ae = 0 \text{ for some } e \in B_P\} = \{a \in R | ae = 0 \text{ for some } e \in B_P\} = O(P)$. Since $R[B_P^{-1}]$ is the right ring of fractions with the denominator set B_P , there is a ring homomorphism $\phi : R \to R[B_P^{-1}]$ and $R[B_P^{-1}] = \{\varphi(a)\varphi(e)^{-1} | a \in R \text{ and } e \in B_P\}$. Also, since $O(P) = \{a \in R | ae = 0, for some e \in B_P\}$, it is clear that the kernel of φ is O(P). For any $e \in B_P$, $\varphi(e)$ is an idempotent in $R[B_P^{-1}]$ which is invertible. Thus $\varphi(e) = 1$ for any $e \in B_P$. So $R[B_P^{-1}] = \varphi(R)$ that is a homomorphic image of R with the kernel O(P). So $R[B_P^{-1}] \simeq R/O(P)$. \Box

Lemma 5.12. [4, Lemma 3.1] A ring R has a representation as a subdirect product of the rings R/O(P), where P ranges through all prime ideals of R.

Lemma 5.13. Let R be an abelian right $p.\pi$ -Baer ring. Then R is a subdirect product of R/O(P), where P ranges through all minimal prime ideals.

Proof. Assume *P* be a minimal prime ideal and *Q* be a prime ideal of *R* containing *P*. By Lemma 5.5, O(P) = O(Q). So $\bigcap_{P \in Spec(R)} O(P) = \bigcap_{P \in MinSpec(R)} O(P)$. The result follows by Lemma 5.12. \Box

Lemma 5.14. [14, Lemma 3.9] Let R be a ring with a nontrivial central idempotent e. Then for every prime ideal P of R, $O(P) \neq 0$.

Corollary 5.15. Let R be an abelian right $p.\pi$ -Baer ring which is not prime. Then for prime ideal P of R, $O(P) \neq 0$.

Proof. Since R is not prime, there exists a nonzero element $a \in R$ such that $r_R(Ra) \neq 0$. Also, since R is an abelian, Ra is projection invariant principal left ideal, so $r_R(Ra) = eR \neq 0$. Therefore by Lemma 5.14, $O(P) \neq 0$. \Box

Theorem 5.16. Let R be an abelian right $p.\pi$ -Baer ring such that $O(P) \neq 0$ for any minimal prime ideal P of R. Then R has a nontrivial representation as a subdirect product of the rings of fractions $R[B_P^{-1}]$, where P ranges through all minimal prime ideals.

Proof. Theorem 5.11 and Lemma 5.13, yield the result. \Box

Proposition 5.17. Let R be an abelian left $p.\pi$ -Baer ring. If P and Q are prime ideals of R, then we have the following results:

- (i) $O(P) = \{x \in R | x = xe, \text{ for some central idempotent } e \in P\}.$
- (ii) If $Q \subseteq P$, then O(P) = O(Q).
- (iii) $R[B_P^{-1}] \simeq R/O(P).$
- (iv) If for any minimal prime ideal B of R, $\overline{O}(B) = B$, then:
 - (a) R is 2-prime.
 - (b) Every prime ideal of R contains a unique minimal prime ideal.
 - (c) $\overline{O}(P)$ is the unique minimal prime ideal of R contained in P.

Proof. (i) Let $x \in O(P)$. Then there is $s \in R \setminus P$ such that xRs = 0. Thus, $x \in l_R(sR) = Re$, where $e = e^2 \in R$. Therefore, x = xe. Clearly, $e \in P$. Then, $O(P) \subseteq \{x \in R | x = xe$, for some central idempotent $e \in P\}$. Now, assume x = xe for some idempotent $e \in P$. Then x(1 - e) = 0, so xR(1 - e) = 0. Since $(1 - e) \notin P$, we have $x \in O(P)$. The proof is complete.

The proofs of (ii), (iii) and (iv) are similar to the proofs of Lemma 5.5, Theorems 5.6 and 5.11, respectively. \Box

References

- [1] H.E. Bell, Near rings in wich each element is a power of itself, Bull. Aust. Math. Soc. 2 (1973), no. 2, 363–368.
- [2] G.F. Birkenmeier, Y. Kara, and A. Tercan, π -Baer rings, J. Alg. Appl. 16 (2018), no. 11, 1–19.
- [3] G.F. Birkenmeier, J.Y. Kim, and J.K. Park, A characterization of minimal prime ideals, Glasgow Math. J. 40 (1998), 223–236.
- [4] G.F. Birkenmeier, J.Y. Kim, and J.K. Park, A sheaf representation of quasi-Baer rings, J. Pure Appl. Alg. 146 (2000), 209–223.
- [5] G.F. Birkenmeier, J.Y. Kim, and J.K. Park, On polynomial extensions of principally quasi-Baer rings, Kyungpook Math. J. 40 (2000), 247–253.
- [6] G.F. Birkenmeier, J.Y. Kim, and J.K. Park, Prime ideals of principally quasi-Baer rings, Acta Math. Hung. 98 (2003), no. 3, 217–225.
- [7] G.F. Birkenmeier, J.Y. Kim, and J.K. Park, Principally quasi-Baer rings, Commun. Alg. 29 (2001), no. 2, 639–660.
- [8] G.F. Birkenmeier, A. Tercann, and C.C. Yucel, Projection invariant extending rings, J. Alg. Appl. 15 (2016), no. 6, 1650121, 11 pp.
- [9] W.E. Clark, Twisted matrix units semigroup algebras, Duke Math. J. 34 (1997), 417–424.
- [10] H.E. Heathely and R.P. Tucci, Central and semicentral idempotents, Kyungpook Math. J. 40 (2000), 255–258.
- [11] C. Huh, H.K. Kim, and Y. Lee, p.p rings and generalized p.p rings, J. Pure Appl. Alg. 167 (2002), 37–52.
- [12] I. Kaplanski, Rings of Operators, Benjamin, New York. 1968.
- [13] T.Y. Lam, A First Course in Noncommutative Rings, Graduate Texts in Mathematics, Springer, New York, 2000.
- [14] A. Majidinya and A. Moussavi, Weakly principally quasi-Baer rings, J. Alg. Appl. 15 (2016), no. 1, 1–19.
- [15] A. Moussavi, H. Haj Seyyed Javadi, and E. Hashemi, Generalized quasi-Baer rings, Commun. Alg. 33 (2005), 2115–2129.
- [16] A. Shahidikia, H. Haj Seyyed Javadi, and A. Moussavi, Generalized π -Baer rings, Turk. J. Math. 44 (2020), 2021–2040.