Int. J. Nonlinear Anal. Appl. In Press, 1–10
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.32062.4755

Laplace optimized decomposition method for solving fractional order logistic growth in a population

Abubker Ahmed[a,b]

[a] University of Science & Technology, College of Engineering, Sudan

[b] AlMughtaribeen University, College of Engineering, Department of General Sciences, Sudan

(Communicated by Javad Damirchi)

Abstract

In this paper, we introduce a semi-analytical method called the Laplace optimized decomposition method, abbreviated as LODM, for solving a model of a nonlinear ordinary differential equation describing the growth of population, the so-called Logistic equation with the fractional-order type, using the Caputo fractional derivative sense. The proposed technique combines the Laplace transform (LT) with a new technique called the optimized decomposition method (ODM). The results obtained by this method have been compared with those obtained by other methods. Finally, we demonstrate our numerical results with the help of tables and figures.

Keywords: Laplace optimized decomposition method, Caputo fractional derivative, fractional differential equations, logistic equation
2020 MSC: 34A08, 81Q05

1 Introduction

Fractional calculus is a specialisation of applied analysis that deals with derivatives of arbitrary (real or complex) order. The differential equations that involve fractional order have been widely used to model various phenomena in many scientific fields [23, 33, 40, 43, 44, 55]. In the literature, numerous operators of arbitrary order have been proposed; the most famous and most widely used are Riemann-Liouville (RL), Caputo derivative (CD) [38], Caputo-Fabrizio derivative (CFD) [21], and Atangana-Baleanu (AB) [15]. Applications of these fractional derivatives have been investigated by many researchers in various fields of science and engineering. (See [6, 15, 16, 27, 28, 29, 35, 36, 39, 51]). The difficulty of finding exact solutions to fractional differential equations is a major challenge for scientists and mathematicians, especially for phenomena that are modelled in the form of non-linear equations. Researchers have presented numerous numerical and analytical techniques, like the variational iteration method (VIM) [31], the differential transform method (DTM) [54], the Homotopy analysis method (HAM) [1, 42], the Homotopy perturbation method (HPM) [32], the Hussein-Jassim method (HJM) [34], the residual power series method (RPSM) [2, 9], the numerical inverse Laplace transform methods [8], the predictor corrector method [24, 48] the Daftardar-Jafari method (DJM) [19, 22] Laplace transform method [3],[37], and so on. Among these techniques in the first attempt, George Adomian introduced a semi-analytical technique called the Adomian decomposition method (ADM) in the 1980s. After that, it has been used to find approximate solutions to the nonlinear fractional differential equations [4, 5, 52]. Recently,

Email address: abobaker633@gmail.com (Abubker Ahmed)

Received: October 2023 *Accepted:* August 2024

Zaid Odibat was introduced and developed an effective decomposition method, called the optimized decomposition method (ODM), to produce analytically approximate solutions for nonlinear ordinary or partial differential equations. The principal concept of the ODM is the linear approximation of a nonlinear term, which is used to decompose the solution in an infinite series form. For more details, see [47, 49]. Additionally, M. Laoubi et al. [41] modified and extended the optimized decomposition method for use in the treatment of nonlinear fractional differential equations. Banan Maayah et al. [45] provide an analytical solution for a fractional order model of dengue fever disease under the Caputo-Fabrizio derivative by using the Laplace optimized decomposition method (LODM). The nonlinear ordinary differential equation describing the growth of population, the so-called logistic equation model, was first studied by Pierre Verhulst in 1938 [20]. Which have many applications in different fields of science, such as biology [56], medicine [59], economy [58], and data security in optical networks [26]. In recent years, researchers have used logistic equations to study the evolution of the COVID-19 pandemic [53], [50]. In the literature, different versions and generalizations of the logistic equation model with fractional-order type have been considered and discussed (see [12, 13, 18, 25, 30, 46, 57]). Recently, Area et al. [11] studied the Λ-fractional logistic differential equation in the Λ-space. The authors of the article [17] have presented an efficient computational technique based on the reproducing kernel theory for approximating the solutions of the logistic differential equation of fractional order. Alshammari et al. [10] established a numerical solution of a logistic equation with fractional order using the residual power series method. The fractional Euler's method is presented to obtain the approximate solution of the fractional logistic equation [60]. Ahmed [7] developed a new application of the Laplace transform method (LTM) and used the series expansion of the dependent variable for solving the fractional logistic growth model in a population and fractional prey-predator models. The fractional logistic ordinary differential equation has the form

$$^{c}D_t^{\alpha} y(t) = \rho y(t)(1-y(t)), \quad t>0, \quad \rho>0, \quad 0<\alpha\leq 1, \tag{1.1}$$

subject to the initial condition

$$y(0)=\mu, \quad \mu>0. \tag{1.2}$$

In particular, if we put $\alpha=1$, in equation (1.1), we have the following classical logistic differential equation

$$\frac{dy}{dt} = \rho y(t)(1-y(t)), \quad t>0, \tag{1.3}$$

has an exact solution in the form

$$y(t) = \frac{\mu}{\mu+(1-\mu)e^{-\rho t}}\ , \tag{1.4}$$

The main objective of this work is to find an approximate solution for the model of nonlinear differential equations describing the growth of population, called the logistic equation, for the fractional-order model using the Caputo fractional derivative sense. For that purpose, the Laplace Optimized Decomposition method (LODM) is utilized to obtain numerical results. Further, we compare our results with those obtained by other methods.

The rest of the paper is organized as follows: in Section 2, we recall some fundamental definitions given. In Section 3, the formulation of the proposed technique is described. In section 4, some numerical applications and discussion are given.

2 Preliminaries

This section contains some fundamental concepts and definitions that, in order to be needed throughout this manuscript, are recollected from [34, 14]

Definition 2.1. If $f(t)\in C([a,b])$, and $a<t<b$, then the Riemann-Liouville fractional integral operator of order $\alpha>0$, is defined as

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau, \quad \alpha>0\,, \tag{2.1}$$

where Γ is the well-known gamma function. In addition, some properties of the Riemann-Liouville fractional integral can be found in [34].

Definition 2.2. [34] The fractional Caputo derivative of $f(t)$, is defined as

$$^{c}_{a}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau, \quad \text{for} \quad m-1<\alpha\leq m, m\in N. \tag{2.2}$$

In addition, some properties of the Caputo fractional derivative can be found in [34].

Definition 2.3. Let $y(t)$ be piecewise continuous function is defined for $t > 0$. The Laplace transform of $y(t)$ is defined in [14] as

$$L[y(t)] = \int_0^\infty exp(-st)y(t)dt = Y(s). \tag{2.3}$$

The Laplace transform of the Caputo fractional derivative of order $m-1 < \alpha \leq m$, is given as

$$L[{}^cD_t^\alpha y(t)] = s^\alpha L[y(t)] - \sum_{i=1}^{m} y^{(i-1)}(0)s^{\alpha-i}. \tag{2.4}$$

3 Laplace Optimized Decomposition Method (LODM)

In this section, we will describe the basic steps of the (LODM) to solve fractional differential equations. To achieve this goal, we consider the following nonlinear fractional order differential equation of the form

$${}^cD_t^\alpha y(t) = \chi[y(t)] + f(t), \qquad t > 0, \quad 0 < \alpha \leq 1, \tag{3.1}$$

with the initial condition $y(0) = \mu$, the function $y(t)$ is an analytical function, ${}^cD_t^\alpha(.)$ is the Caputo fractional derivative, χ indicates the nonlinear operator, and $f(t)$ is a known function. Now, applying the Laplace transform to both sides of (3.1) and using the initial condition, we get

$$L[y(t)] = \frac{\mu}{s} + \frac{1}{s^\alpha}\left(L[f(t)] \ + \ L[\ \chi[y(t)]\ ]\ \right). \tag{3.2}$$

Applying the inverse Laplace transform to (3.2), we get

$$y(t) = \mu + L^{-1}\left[\frac{1}{s^\alpha}\left(L[f(t)] \ + \ L[\ \chi[y(t)]\ ]\ \right)\right]. \tag{3.3}$$

The Laplace optimized decomposition method suggests that the solution y(t) be expressed by the decomposition series

$$y(t) = \sum_{k=0}^{\infty} y_k(t), \tag{3.4}$$

and the nonlinear terms $\chi[y(t)]$ is represented by

$$\chi[y(t)] = \sum_{k=0}^{\infty} P_k(t), \tag{3.5}$$

where $k \geq 0$ such that $y_k(t)$ are the components of $y(t)$ that will be determined recursively, and $P_k(t)$ are called the Adomian polynomials that represent the nonlinear $\chi[y(t)]$ and can be determined from the relation

$$P_k(t) = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k}\chi\left[\sum_{k=0}^{\infty} \lambda^k y_k(t) \right]\right]_{\lambda=0}, \quad k \geq 0. \tag{3.6}$$

Inserting (3.4) and (3.5) into (3.3), we get

$$\sum_{k=0}^{\infty} y_k(t) = \mu + L^{-1}\left[\frac{1}{s^\alpha}\left(L[f(t)] + L\left[\sum_{k=0}^{\infty} P_k(t) \right]\right)\right]. \tag{3.7}$$

Consequently, the components of $y(t)$ can be elegantly determined by using the recursive iteration relation

$$\begin{cases} y_0(t) = \mu + L^{-1}\left[\frac{1}{s^\alpha}L[f(t)]\right], \\ y_1(t) = L^{-1}\left[\frac{1}{s^\alpha}L[P_0(t)]\right], \\ y_2(t) = L^{-1}\left[\frac{1}{s^\alpha}L[P_1(t) + \zeta(y_1(t))]\right], \\ \quad \vdots \\ y_{k+1}(t) = L^{-1}\left[\frac{1}{s^\alpha}L[P_k(t) + \zeta(y_k(t)) - \zeta(y_{k-1}(t))]\right], \quad k \geq 2, \end{cases} \tag{3.8}$$

where

$$\zeta = \left. \frac{\frac{\partial}{\partial y}\varphi\left({}^cD_t^\alpha y(t), y(t)\right)}{\frac{\partial}{\partial {}^cD_t^\alpha y}\varphi\left({}^cD_t^\alpha y(t), y(t)\right)}\right|_{t=0} = -\left.\frac{\partial}{\partial y}\varphi\left({}^cD_t^\alpha y(t), y(t)\right)\right|_{t=0}, \tag{3.9}$$

such that we assume that the function $\varphi\left({}^cD_t^\alpha y(t), y(t)\right) = {}^cD_t^\alpha y(t) - \chi[y(t)]$ can be linearized by a first-order Taylor series expansion at $t = 0$. Solving $\varphi\left({}^cD_t^\alpha y(0), y(0)\right) = 0$ thus, the Taylor series expansion of the function $\varphi\left({}^cD_t^\alpha y(t), y(t)\right)$ near (Y, μ) where $Y = {}^cD_t^\alpha y(0)$ and $\mu = y(0)$ is

$$\varphi\left({}^cD_t^\alpha y(t), y(t)\right) \approx {}^cD_t^\alpha y + \frac{\partial\varphi}{\partial y(t)}(Y, \mu)y(t)\,. \tag{3.10}$$

4 Numerical Applications and Discussion

In this section, we consider the fractional logistic differential equation, then the (LODM) is applied in order to obtain the approximate solutions.

Example 4.1. Consider the following fractional-order logistic differential equation [10]

$${}^cD_t^\alpha y(t) = \frac{1}{4}y(t)(1 - y(t)), \quad t > 0, \quad 0 < \alpha \leq 1 \tag{4.1}$$

with the initial condition

$$y(0) = \frac{1}{3}, \tag{4.2}$$

In particular, if we put $\alpha = 1$, in equation (4.1), the exact solution given by

$$y(t) = \frac{1}{1 + 2e^{-\frac{1}{4}t}}, \tag{4.3}$$

In view of (3.3), we have

$$y(t) = \frac{1}{3} + L^{-1}\left[\frac{1}{s^\alpha}\left(\frac{1}{4}L\left[y(t) - y^2(t)\right]\right)\right], \tag{4.4}$$

Linearizing the function $\varphi\left({}^cD_t^\alpha y(t), y(t)\right) = {}^cD_t^\alpha y(t) - \frac{1}{4}y(t) + \frac{1}{4}y^2(t)$, near the point (Y, μ), we obtain the linear approximation

$$\varphi\left({}^cD_t^\alpha y(t), y(t)\right) \approx {}^cD_t^\alpha y(t) + \zeta y(t), \tag{4.5}$$

where

$$\zeta = \frac{\partial}{\partial y}\left.\left(\varphi\left({}^cD_t^\alpha y(t), y(t)\right)\right)\right|_{t=0} = -\frac{1}{12},$$

Assume the series solution has the form

$$y(t) = \sum_{k=0}^{\infty} y_k(t). \tag{4.6}$$

The Adomian polynomials P_k for $y - y^2$ have been calculated before and given by

$$P_k(t) = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[\sum_{k=0}^{\infty}\lambda^k y_k(t) - \left(\sum_{k=0}^{\infty}\lambda^k y_k(t)\right)^2\right]_{\lambda=0}, \quad k \geq 0. \tag{4.7}$$

According to equation (3.8) we obtain

$$\begin{cases} y_0(t) = \frac{1}{3}, \\ y_1(t) = \frac{0.0555555556}{\Gamma(\alpha+1)}t^\alpha \ , \\ y_2(t) = 0\ , \\ y_3(t) = \frac{0.0046296296}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{0.0007716049\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}, \\ \quad\vdots \end{cases} \tag{4.8}$$

Using equation (4.6), we obtain the 4th order approximation of $y(t)$ is given by

$$\begin{aligned} y_{LODM}(t) &= y_0(t)+\sum_{k=1}^{3} y_k(t) \\ &= \frac{1}{3}+\frac{0.0555555556}{\Gamma(\alpha+1)}t^{\alpha}+\frac{0.0046296296}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{0.0007716049\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}\,. \end{aligned} \tag{4.9}$$

Table 1 shows our numerical results for the approximate solution of example 4.1 (4−term solution), for $\alpha = 1$ then we compared our results with those obtained by exact solution, the fractional residual power series method (FRPSM) [10], the variational iteration method (VIM) [17], and the reproducing kernel Hilbert space method (RKHSM) [17]. We present the comparison of absolute errors in Table 2. Further, in Table 3, we show the approximate solutions that are obtained by (LODM) of example 4.1 at different values of α. Fig. **??** shows the (LODM) solutions for different values of α.

Table 1: Comparison of numerical results that are obtained for example 4.1, by using (LODM) when $\alpha = 1$, by other methods

t	$y_{EXACT}(t)$	$y_{VIM}(t)$	$y_{LODM}(t)$	$y_{RKHSM}(t)$	$y_{FRPSM}(t)$
0.0	0.3333333333	0.3333333333	0.3333333333	0.3333333333	0.3333333333
0.3	0.3502029635	0.3502013889	0.3502013889	0.3502029364	0.3502029634
0.5	0.3616644631	0.3616576645	0.3616576646	0.3616644354	0.3616644609
0.8	0.3791524531	0.3791275720	0.3791275720	0.3791524268	0.3791524170
1.0	0.3909913152	0.3909465021	0.3909465021	0.3909912851	0.3909911774

Table 2: Comparison of absolute errors that are obtained for example 4.1, by using (LODM) when $\alpha = 1$, by other methods.

t	$y_{VIM}(t)$	$y_{LODM}(t)$	$y_{RKHSM}(t)$	$y_{FRPSM}(t)$
0.0	0.0000000	0.0000000	0.0000000	0.0000000
0.3	1.5746×10^{-6}	1.5746×10^{-6}	2.71×10^{-8}	1.00×10^{-8}
0.5	6.7986×10^{-6}	6.7985×10^{-6}	2.77×10^{-8}	2.15×10^{-8}
0.8	2.4881×10^{-5}	2.4881×10^{-5}	2.63×10^{-8}	3.61×10^{-8}
1.0	4.4813×10^{-5}	4.4813×10^{-5}	3.01×10^{-8}	1.378×10^{-7}

Table 3: The numerical results that are obtained for example 4.1, by using (LODM) for different values of α

t	$\alpha = 1$		$\alpha = 0.85$	$\alpha = 0.65$	$\alpha = 0.45$	$\alpha = 0.25$
	y_{EXACT}	y_{LODM}	y_{LODM}	y_{LODM}	y_{LODM}	y_{LODM}
0	0.3333333333	0.3333333333	0.3333333333	0.3333333333	0.3333333333	0.3333333333
1	0.3909913152	0.3909465021	0.3943093549	0.3972550045	0.3971513174	0.3957052888
2	0.4518627619	0.4516460905	0.4468422902	0.4374762338	0.4219673513	0.4096353649
3	0.5142093777	0.5138888889	0.4970289020	0.4723955481	0.4396160378	0.4196395483
4	0.5761168848	0.5761316872	0.5443372321	0.5023843989	0.4508447948	0.4264196003
5	0.6357240312	0.6368312757	0.5874596345	0.5262626057	0.4546193716	0.4293356014
6	0.6914384540	0.6944444444	0.6248156990	0.5423439370	0.4493672542	0.4273726390
7	0.7420886558	0.7474279835	0.6546974071	0.5487172994	0.4332888578	0.4193656765
8	0.7869860422	0.7942386831	0.6753269548	0.5433534429	0.4044699739	0.4040768925
9	0.8259012891	0.8333333333	0.6848843257	0.5241539546	0.3609314751	0.3802285841
10	0.8589810787	0.8631687243	0.6815220860	0.4889767401	0.3006543670	0.3465192358

Example 4.2. Consider the fractional logistic differential equation [10]

$${}^{c}D_t^{\alpha} y(t)=\frac{1}{2}y(t)(1-y(t)),\quad t>0,\quad 0<\alpha\le 1, \tag{4.10}$$

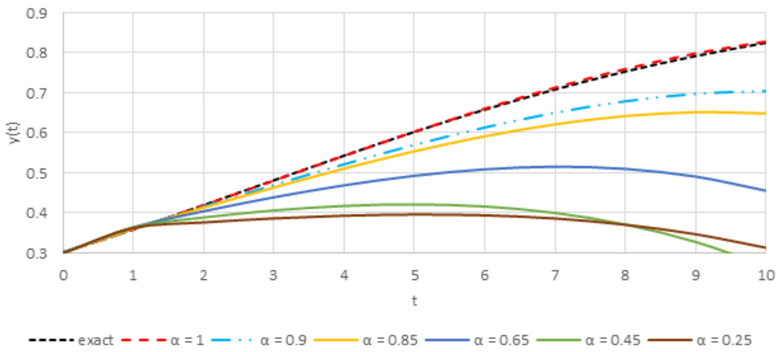

Figure 1: Plots of the approximate solutions for example 4.1 that are obtained by using (LODM) for different values of α

with the initial conditions

$$y(0) = \frac{1}{2}, \tag{4.11}$$

In particular, if we put $\alpha = 1$, in equation (4.10), the exact solution given by

$$y(t) = \frac{1}{1 + e^{-\frac{1}{2}t}}. \tag{4.12}$$

According of our method and equation (3.8), the components of the Laplace optimized decomposition series are given as follows

$$\begin{cases} y_0(t) = \frac{1}{2}, \\ y_1(t) = \frac{0.125}{\Gamma(\alpha+1)}t^{\alpha} \quad , \\ y_2(t) = \frac{0.0625}{\Gamma(2\alpha+1)}t^{2\alpha} \quad , \\ y_3(t) = -\frac{0.0625}{\Gamma(2\alpha+1)}t^{2\alpha} \quad + \quad \frac{0.03125}{\Gamma(3\alpha+1)}t^{3\alpha} \quad - \quad \frac{0.015625\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}, \\ \quad \vdots \end{cases} \tag{4.13}$$

and so on. Therefore, we obtain the 4th order approximation of $y(t)$ is given by

$$\begin{aligned} y_{LODM} &= y_0(t) + \sum_{k=1}^{3} y_k(t) \\ &= \frac{1}{2} + \frac{0.0625}{\Gamma(2\alpha+1)}t^{\alpha} \quad + \quad \frac{0.03125}{\Gamma(3\alpha+1)}t^{3\alpha} \quad - \quad \frac{0.015625\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}. \end{aligned} \tag{4.14}$$

Table 4 shows our numerical results for the approximate solution of example 4.2 (4−term solution), for $\alpha = 1$ then we compared our results with those obtained by exact solution, the (FRPSM) [10], the (RKHSM) [17], and the optimal homotopy asymptotic method (OHAM) [17]. We present the comparison of absolute errors in Table 5. Further, in Table 6, we show the approximate solutions that are obtained by (LODM) of example 4.2 at different values ofα. Fig. 2 shows the (LODM) solutions for different values of α.

Table 4: Comparison of numerical results that are obtained for example 4.2, by using (LODM), when $\alpha = 1$, by other methods.

t	$y_{EXACT}(t)$	$y_{LOMD}(t)$	$y_{RKHSM}(t)$	$y_{FRPSM}(t)$	$y_{OHAM}(t)$
0.0	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
0.3	0.5374298453	0.5375000000	0.5374297936	0.5374298457	0.5374288935
0.5	0.5621765009	0.5625000000	0.5621764494	0.5621765137	0.5621790838
0.8	0.5986876601	0.6000000000	0.5986876097	0.5986880000	0.5986911508
1.0	0.6224593312	0.6250000000	0.6224592820	0.6224609375	0.6224603770

Table 5: Comparison of absolute errors that are obtained for example 4.2, by using (LODM) when $\alpha = 1$, by other methods.

t	$y_{LODM}(t)$	$y_{RKHSM}(t)$	$y_{FRPSM}(t)$	$y_{OHAM}(t)$
0.0	0.000000	0.000000	0.00000	0.000000
0.3	7.02E-05	5.17E-08	3.56E-10	9.52E-07
0.5	0.000323	5.15E-08	1.28E-08	2.58E-06
0.8	0.001312	5.04E-08	3.4E-07	3.49E-06
1.0	0.002541	4.92E-08	1.61E-06	1.05E-06

Table 6: The numerical results that are obtained for example 4.2, by using (LODM) for different values of α

t	$\alpha = 1$		$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.6$
	y_{EXACT}	y_{LODM}	y_{LODM}	y_{LODM}	y_{LODM}
0.0	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
0.2	0.5249791875	0.5250000000	0.5305379320	0.5370462089	0.5532907788
0.4	0.5498339973	0.5500000000	0.5570168301	0.5645750909	0.5809535487
0.6	0.5744425168	0.5750000000	0.5822051605	0.5895059340	0.6037159692
0.8	0.5986876601	0.6000000000	0.6066454399	0.6130229215	0.6241412937
1.0	0.6224593312	0.6250000000	0.6306016932	0.6356847031	0.6433635662

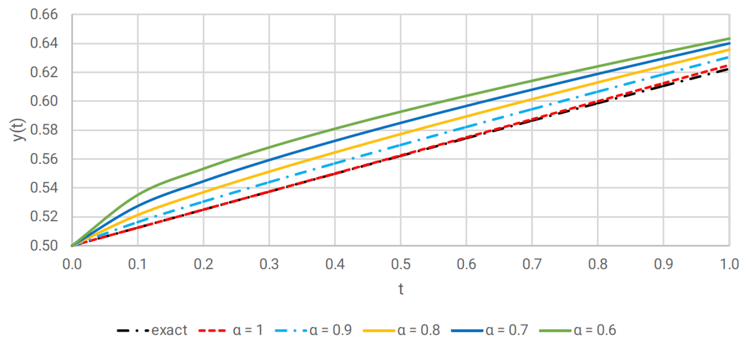

Figure 2: Plots of the approximate solutions for example 4.2 that are obtained by using (LODM) for different values of α

Comment 1. Based on Tables 2 and 5 and comparing the absolute errors of the methods, there is not much difference between the numerical approximation results of the methods. Furthermore, the "RKHSM", "FRPSM" and "OHAM" methods have achieved better results.

5 Conclusion

In this study, an analytical solution for a nonlinear fractional ordinary differential equation describing the growth of population is provided under the Caputo fractional derivative. A comparison between our method and other methods is presented. As a result, the LODM studied in this work is an efficient and powerful technique for obtaining accurate analytic approximate solutions to fractional-order logistic differential equations. In the future, we will explore applying the aforementioned technique to fuzzy differential equations and complex dynamical systems, such as infectious disease models.

Acknowledgements

I would like to extend my sincerest appreciation to Dr. Mohamed Khabir, Department of Mathematics, University of Science & Technology, Sudan, the editor and the reviewers for their time spent on reviewing my manuscript and their comments, which paved the way to improve my study.

References

[1] K.S. Aboodh and A. Ahmed, *On the application of homotopy analysis method to fractional differential equations*, J. Faculty Sci. Technol. **7** (2020), 1–18.

[2] I. Abu Irwaq, M. Alquran, M. Ali, I. Jaradat, and M.S.M. Noorani, *Attractive new fractional-integer power series method for solving singular perturbed differential equations involving mixed fractional and integer derivatives*, Results Phys. **20** (2021), 103780.

[3] B. Acay, R. Ozarslan, and E. Bas, *Fractional physical models based on falling body problem*, AIMS Math. **5** (2020), 2608–2628.

[4] G. Adomian, *A review of the decomposition method in applied mathematics*, J. Math. Anal. Appl. **135** (1988), 501–544.

[5] G. Adomian, *A review of the decomposition method and some recent results for nonlinear equations*, Comput. Math. Appl. **21** (1991), 101–127.

[6] A. Ahmadian, S. Salahshour, and M. Salimi, *A robust numerical approximation of advection diffusion equations with nonsingular kernel derivative*, Phys. Scripta **96** (2021), no. 12, Article ID 124015.

[7] A. Ahmed, *Laplace transform method for logistic growth in a population and predator models with fractional order*, Open J. Math. Sci. **7** (2023), 239–245.

[8] S. Ahmad, K. Shah, T. Abdeljawad, and B. Abdalla, *On the approximation of fractal fractional differential equations using numerical inverse Laplace transform methods*, CMES **135** (2023), no. 3.

[9] M. Alquran, F. Yousef, F. Alquran, T.A. Sulaiman, and Y.A. Dualwave, *Solutions for the quadratic-cubic conformable-Caputo time-fractional Klein-Fock-Gordon equation*, Math. Comput. Simul. **185** (2021), 62–76.

[10] S. Alshammari, M. Al-Smadi, M. Al Shammari, I. Hashim, and M.A. Alias, *Advanced analytical treatment of fractional logistic equations based on residual error functions*, Int. J. Differ. Equ. **2019** (2019), Article ID 7609879, 1–11.

[11] I. Area, K.A. Lazopoulos, and J.J. Nieto, $\Gamma-$*fractional logistic equation*, Prog. Fract. Differ. Appl. **9** (2023), 345–350.

[12] I. Area and J.J. Nieto, *Fractional-order logistic differential equation with Mittag–Leffler type kernel*, Fractal Fractional **5** (2021), no. 4, 273.

[13] I. Area and J.J. Nieto, *Power series solution of the fractional logistic equation*, Physica A **573** (2021), 125947.

[14] A. Atangana and A. Akgül, *On solutions of fractal fractional differential equations*, Discrete Continuous Dyn. Syst. Ser. S **14** (2021), no. 10, 3441–3457.

[15] A. Atangana and D. Baleanu, *New fractional derivatives with non-local and nonsingular kernel: Theory and application to heat transfer model*. Therm. Sci. **20** (2016), 763–769.

[16] A. Atangana, E. Bonyah, and A.A. Elsadany, *A fractional order optimal 4D chaotic financial model with Mittag-Leffler law*, Chinese J. Phys. **65** (2020), 38-–53.

[17] N. Attia, A. Akgul, D. Seba, and A. Nour, *On solutions of fractional logistic differential equations*, Progr. Fract. Differ. Appl. **9** (2023), 351–362.

[18] C. Balzotti, M. D'Ovidio, and P. Loreti, *Fractional SIS epidemic models*, Fractal Fractional **4** (2020), 18 pages.

[19] S. Bhalekar and V. Daftardar-Gejji, *Solving fractional-order logistic equation using a new iterative method*, Int. J. Diff. Equ. **2012** (2012), Article ID 975829, 12 pages.

[20] F. Brauer, C. Castillo-Chavez, and Z. Feng, *Mathematical Models in Epidemiology*, Springer-Verlag, New York, 2019.

[21] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Prog. Fractional Diff. Appl. **1** (2015), no. 2, 73–85.

[22] V. Daftardar-Gejji and H. Jafari, *An iterative method for solving nonlinear functional equations*, J. Math. Anal. Appl. **316** 2006, 753–763.

[23] L. Debnath, *Recent applications of fractional calculus to science and engineering*, Int. J. Math. Math. **54** (2003), 3413–3442.

[24] K. Diethelm, N. Ford, and A. Freed, *A predictor-corrector approach for the numerical solution of fractional differential equations*, Nonlinear Dyn. **29** (2002), 3–22.

[25] M. D'Ovidio and P. Loreti, *Solutions of fractional logistic equations by Euler's numbers*, Physical A **506** (2018), 1081–1092.

[26] J.D. do Nascimento, R.L.C. Damasceno, G.L. de Oliveira, and R.V. Ramos, *Quantum-chaotic key distribution in optical networks: From secrecy to implementation with logistic map*, Quantum Inf. Process. **17** (2018), 329.

[27] V.P. Dubey, S. Dubey, D. Kumar, and J. Singh, *Computational study of fractional model of atmospheric dynamics of carbon dioxide gas*, Chaos Solitons Fractals **142** (2021), 279–312.

[28] V.P. Dubey, D. Kumar, and S. Dubey, *A modified computational scheme and convergence for fractional order hepatitis E virus model*, Advanced Numerical Methods for Differential Equations, CRC Press, 2021, pp. 279–312.

[29] V.P. Dubey, J. Singh, A.M. Alshehri, S. Dubey, and D. Kumar, *Numerical investigation of fractional model of phytoplankton-toxic Phytoplankton-Zooplankton system with convergence analysis*, Int. J. Biomath. **15** (2022), no. 4, 2250006.

[30] A.M.A. El-Sayed, A.E.M. El-Mesiry, H.A.A. and El-Saka, *on the fractional-order logistic equations*, Appl. Math. Lett. **20** 2007 817-823.

[31] J.H. He, *A new approach to nonlinear partial differential equations*, Commun. Nonlinear Sci. Numer. Simul. **2** (1997), 203–205.

[32] J.H. He, *Homotopy perturbation method: A new nonlinear analytical technique*. Appl. Math. Comput. **135** (2003), 73–79.

[33] M. Ichise, Y. Nagayanagi, and T. Kojima, *An analog simulation of non-integer order transfer functions analysis of electrode process*, J. Electroanal. Chem. Interfacial Electrochem. **33** (1971), 253–265.

[34] H.K. Jassim and M. Abdulshareef Hussein, *A new approach for solving nonlinear fractional ordinary differential equations*, Mathematics. **11** (2023), no. 7, 1565.

[35] R. Kamal, Kamran, G. Rahmat, A. Ahmadian, N.I. Arshad, and S. Salahshour, *Approximation of linear one dimensional partial differential equations including fractional derivative with non-singular kernel*, Adv. Diff. Equ. **1** (2021), 317—415.

[36] Kamran, A. Ali, and J.F. Gómez-Aguilar, *A transform based´ local RBF method for 2D linear PDE with Caputo-Fabrizio derivative*, Comptes Rendus Math. **358** (2020), no. 7, 831–842.

[37] S. Kazem, *Exact solution of some linear fractional differential equations by Laplace transform*, Int. J. Nonlinear Sci. **16** (2013), no. 1, 3–11.

[38] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, The organization, Elsevier, Amsterdam, Netherlands, 2006.

[39] I. Koca, *Modelling the spread of Ebola virus with Atangana Baleanu fractional operators*, Eur. Phys. J. Plus **133** (2018), 100–111.

[40] R.C. Koeller, *Applications of fractional calculus to the theory of viscoelasticity*, J. Appl. Mech. **51** (1984), 299–307.

[41] M. Laoubi, Z. Odibat, and B. Maayah, *Effective optimized decomposition algorithms for solving nonlinear fractional differential equations*, J. Comput. Nonlinear Dyn. **18** (2023), no. 2, 021001.

[42] S.J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman & Hall/CRC Press: Boca Raton, FL, USA, 2003.

[43] F. Liu, V. Anh, and I. Turner, *Numerical solution of the space fractional Fokker-Planck equation*, J. Comput. Appl. Math. **166** (2004), 209–219.

[44] J. Lu and G. Chen, *A note on the fractional-order Chen system*, Chaos Solitons Fractals **27** (2006), no. 3, 685–688.

[45] B. Maayah, S. Bushnaq, and A. Moussaoui, *Numerical solution of fractional order SIR model of dengue fever*

disease via Laplace optimized decomposition method, J. Math. Comput. Sci. **32** (2024), 86–93.

[46] J.J. Nieto, *Solution of a fractional logistic ordinary differential equation*, Appl. Math. Lett. **123** (2022), 107568.

[47] Z. Odibat, *An Optimized decomposition method for nonlinear ordinary and partial differential equations*, Phys. A **541** (2020), 13 pages.

[48] Z. Odibat, *A universal predictor–corrector algorithm for numerical simulation of generalized fractional differential equations*, Nonlinear Dyn. **105** (2021), no. 3, 2363–2374.

[49] Z. Odibat, *The optimized decomposition method for a reliable treatment of IVPs for second order differential equations*, Phys. Scr. **96** (2021).

[50] E. Pelinovsky, A. Kurkin, O. Kurkina, M. Kokoulina, and A. Epifanova, *Logistic equation and COVID-19*, Chaos Solitons Fractals **140** (2020), 110241.

[51] X. Qiang, Kamran, A. Mahboob, and Y.M. Chu, *Numerical approximation of fractional-order Volterra integrodiferential equation*, J. Funct. Spaces **2020** (2020), Article ID 8875792, 12 pages.

[52] D. Rani and V. Mishra, *Modification of Laplace Adomian decomposition method for solving nonlinear Volterra integral and integro-differential equations based on Newton Raphson formula*, Eur. J. Pure Appl. Math. **11** (2018), 202–214.

[53] T. Saito and K. Shigemoto, *A logistic curve in the SIR model and its application to deaths by COVID-19 in Japan*, Eur. J. Appl. Sci. **10** (2022), no. 5.

[54] K. Shah, T. Abdeljawad, F. Jarad, and Q. Al-Mdallal, *On nonlinear conformable fractional order dynamical system via differential transform method*, CMES **136** (2023), no. 2, 1457–1472.

[55] H.H. Sun, A.A. Abdelwahad, and B. Onaral, *Linear approximation of transfer function with a pole of fractional order*, IEEE Trans. Autom. Control. **29** (1984), 441–444.

[56] H.R. Thieme, *Mathematics in Population Biology*, Princeton Series in Theoretical and Computational Biology, Princeton University Press, 2003.

[57] V. Tarasov, *Exact solutions of Bernoulli and logistic fractional differential equations with power law confidents*, Mathematics **8** (2020), no. 12, 2231.

[58] V.V. Tarasova and V.E. Tarasov, *Logistic map with memory from economic model*, Chaos Solitons Fractals **95** (2017), 84–91.

[59] C.A. Valentim Jr, J.A. Oliveira, S.A. Rabi, and N.A. David, *Can fractional calculus help improve tumor growth models?*, J. Comput. Appl. Math. **379** (2020), 112964.

[60] D. Vivek, K. Kanagarajan, and S. Harikrishnan, *Numerical solution of fractional-order logistic equations by fractional Euler's method*, IJRASET **4** (2016), 775–780.