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An existence theorem for a general class of weakly singular integral equations in Banach spaces

Manochehr Kazemi, Hamid Reza Sahebi*

Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran

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Abstract

The purpose of this paper is devoted to establish the existence of a solution for a general class of nonlinear integral equations with weakly-singular terms. Our technique is based on the P-theorem associated with the Hausdorff M.N.C. Furthermore, we provide an example to demonstrate the practicality of the result obtained.

Keywords: existence of solution, measure of noncompactness, $\mathbb{NIE}$s, P-theorem, Banach space
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1 Introduction

One of the most commonly used methods is the concept of noncompactness measure (M.N.C). The root of this concept goes back to the famous work of Kuratowski [13]. This method plays a vital role in the publications of research [2]. In 1955, an extension of this direction was introduced by Italian mathematician Darbo [4]. He studied the existence of fixed points for condensing operators, generalising the Schauder fixed point theorem and the Banach contraction principle. After this pioneering work, the number of research papers dealing with Darbo fixed point theorem has increased considerably in recent years [1, 3, 5, 6, 7, 12, 14, 16, 19, 20, 22]. In 2016, by the assistance of the measure of noncompactness and Petryshyn's fixed point theorem ($\mathbf{P-theorem}$), Kazemi and Ezzati established that the sublinear conditions in the Darbo fixed point theorem are an additional condition [10]. We employ the idea of Kazemi and Ezzati to the existence of solutions for weakly singular integral equations as follows:

$$x(t) = g\big(t, (x(\beta_i(t)))_{i=1}^{s}\big) + f\big(t, (x(\alpha_j(t)))_{j=1}^{m}\big) \int_0^t ln|t-\tau| u\big(t, \tau, (x(\gamma_k(t)))_{k=1}^{n}\big) d\tau, \tag{1.1}$$

where

$$(x(\beta_i(t)))_{i=1}^{s} := \big(x(\beta_1(t)), x(\beta_2(t)), \dots, x(\beta_s(t))\big).$$

The paper is structured as follows. In Section 2, we collect some definitions, lemmas and theorems, which are essential to prove our main results. In Section 3, we establish and prove a new existence theorem by utilising $\mathbf{P-theorem}$ for $\mathbb{NIE}s$ (1.1). In Section 4, we also give an example to support our main theorem. Finally, in section 5, the paper concludes.

*Corresponding author
Email addresses: univer-ka@yahoo.com (Manochehr Kazemi), sahebi.aiau.ac.ir@gmail.com (Hamid Reza Sahebi)

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2 Auxiliary facts and notations

In this section, we will review several definitions and theorems, providing additional facts to enhance understanding.

- $\mathbb{X}$: Real Banach space;
- $\bar{B}_{\varphi_0}$: Closed ball at center 0 with radius φ_0 ;
- $\partial\bar{B}_{\varphi_0}$: Sphere in $\digamma$ around 0 with radius $\varphi_0 > 0$;

Definition 2.1. [13] Let A is a bounded subset of a Banach space $\mathbb{X}$, and

$$\chi(A) = \inf\{\varepsilon > 0 : A \text{ maybe covered by finitely multiple sets of diameter } \leq \varepsilon\} \tag{2.1}$$

is called the Kuratowski M.N.C.

Definition 2.2 ([9]). The Hausdroff M.N.C is defined as

$$\mu(A) = \inf\{\varepsilon > 0 : \exists \text{ a finite } \varepsilon\text{-net for } A \text{ in } \mathbb{X}\}. \tag{2.2}$$

These M.N.C are mutually alike as follows

$$\mu(A) \leq \chi(A) \leq 2\mu(A)$$

for any bounded set $A \subset \mathbb{X}$.

Theorem 2.3 ([18]). Let A , $\tilde{A} \in \mathbb{X}$ and $\lambda \in \mathbb{R}$. Then

(i) $\mu(A) = 0$ if and only if A is relatively-compact;
(ii) $A \subseteq \tilde{A}$ implies $\mu(A) \leq \mu(\tilde{A})$;
(iii) $\mu(\bar{A}) = \psi(ConvM) = \mu(A)$;
(iv) $\mu(A \cup \tilde{A}) = \max\{\mu(A), \mu(\tilde{A})\}$;
(v) $\mu(\lambda A) =\mid \lambda \mid \mu(A)$;
(vi) $\mu(A + \tilde{A}) \leq \mu(A) + \mu(\tilde{A})$.

In the pursuing, we will operate in the space $C([0, d])$ with the usual norm

$$\|x\| = \sup\{|x(\varphi)| : \varphi \in [0, d]\}.$$

Identify that the modulus of continuity of a function $x \in C([0, d])$ is defined as

$$\omega(x, \varepsilon) = \sup\{|x(\varphi) - x(\tilde{\varphi})| : |\varphi - \tilde{\varphi}| \leq \varepsilon\}.$$

Theorem 2.4. [11] In $C([0, d])$, the M.N.C (2.2) is equivalent to

$$\mu(A) = \lim_{\varepsilon \to 0} \sup_{x \in A} \omega(x, \varepsilon) \tag{2.3}$$

for all bounded sets $A \subset C([0, d])$.

Definition 2.5. Assume that $T : \mathbb{X} \to \mathbb{X}$ be a continuous mapping of $\mathbb{X}$. T is called a k-set contraction if for all $H \subset \mathbb{X}$ with H bounded, $T(H)$ is bounded and $\chi(TH) \leq k\chi(H), 0 < k < 1$. If

$$\chi(TH) < \chi(H), \text{ for all } \chi(H) > 0,$$

then T is called densifying (or condensing) map [17].

Now, we recall the well known theorem of Petryshyn's.

Theorem 2.6. [18], Assume that $T : \bar{B}_{\varphi_0} \to \mathbb{X}$ be a densifying mapping which satisfies the boundary condition,

$$T(x) = kx, \text{ for some x in } \partial B_{\varphi_0} \text{ with } k \leq 1, \tag{2.4}$$

then the set of fixed points of T in $\bar{B}_{\varphi_0}$ is non-empty.

3 Main results based on $P-$theorem

In this section, we consider the following assumptions to verify the existence of a solution for functional $\mathbb{NIE}s$ (1.1).

$W1)$: Let $g \in C([0,d] \times \mathbb{R}^s, \mathbb{R})$, $f \in C([0,d] \times \mathbb{R}^m, \mathbb{R})$ and the following functions are continuous:

$$\beta_i : [0,d] \to [0,d], \; for \; 1 \leq i \leq s,$$
$$\alpha_j : [0,d] \to [0,d], \; for \; 1 \leq j \leq m,$$
$$\gamma_k : [0,d] \to [0,d], \; for \; 1 \leq k \leq n,$$

$W2)$: There exists nonnegative constants q_i, λ_j for $1 \leq i \leq s$, $1 \leq j \leq m$ such that

$$\Big| g(t, x_1, x_2, \ldots, x_s) - g(t, y_1, y_2, \ldots, y_s) \Big| \leq \sum_{i=1}^{s} q_i \big| x_i - y_i \big|$$
$$\Big| f(t, x_1, x_2, \ldots, x_m) - f(t, y_1, y_2, \ldots, y_m) \Big| \leq \sum_{j=1}^{m} \lambda_j \big| x_j - y_j \big|.$$

$W2)$: There exist nonnegative φ_0 such that $\sup \Big\{ g + fMd \ln d \Big\} \leq \varphi_0$, with $\Big(\sum_{i=1}^{s} q_i + M \sum_{j=1}^{m} \lambda_j \Big) < 1$, where

$$M = \sup \Big\{ \big| u(t, \tau, x_1, x_2, \ldots, x_n) \big| ; t, \tau \in [0,d], x_i \in [-\varphi_0, \varphi_0] \; \forall 1 \leq i \leq n \Big\}.$$

Theorem 3.1. Assuming (W1)-(W3) hold, the $\mathbb{NIE}$ (1.1) has at least one solution in $\mathbb{X} = C([0,d])$.

Proof . Define the operator $\nabla : \bar{B}_{\phi_0} \to \mathbb{X}$ as follows:

$$(\nabla x)(t) = g\big(t, (x(\beta_i(t)))_{i=1}^{s}\big) + f\big(t, (x(\alpha_j(t)))_{j=1}^{m}\big) \int_0^t \ln |t - \tau| u\big(t, \tau, (x(\gamma_k(t)))_{k=1}^{n}\big) d\tau. \tag{3.1}$$

We divided the proof into several steps.

Step 1. The operator ∇ is continuous on the ball B_{φ_0}. Consider arbitrary $x, y \in B_{\varphi_0}$ and $\varepsilon > 0$ such that $\|x - y\| < \varepsilon$, we have

$$\begin{aligned}
|(\nabla x)(t) - (\nabla y)(t)| = & \Big| g\big(t, (x(\beta_i(t)))_{i=1}^{s}\big) + f\big(t, (x(\alpha_j(t)))_{j=1}^{m}\big) \int_0^t \ln |t - \tau| u\big(t, \tau, (x(\gamma_k(t)))_{k=1}^{n}\big) d\tau \\
& - g\big(t, (y(\beta_i(t)))_{i=1}^{s}\big) - f\big(t, (y(\alpha_j(t)))_{j=1}^{m}\big) \int_0^t \ln |t - \tau| u\big(t, \tau, (y(\gamma_k(t)))_{k=1}^{n}\big) d\tau \Big| \\
\leq & \Big| g\big(t, (x(\beta_i(t)))_{i=1}^{s}\big) - g\big(t, (y(\beta_i(t)))_{i=1}^{s}\big) \Big| + \Big| \Big[f\big(t, (x(\alpha_j(t)))_{j=1}^{m}\big) - f\big(t, (y(\alpha_j(t)))_{j=1}^{m}\big) \Big] \times \\
& \int_0^t \ln |t - \tau| u\big(t, \tau, (y(\gamma_k(t)))_{k=1}^{n}\big) d\tau \Big| \\
& + \Big| f\big(t, (x(\alpha_j(t)))_{j=1}^{m}\big) \int_0^t \ln |t - \tau| \Big[u\big(t, \tau, (x(\gamma_k(t)))_{k=1}^{n}\big) - u\big(t, \tau, (y(\gamma_k(t)))_{k=1}^{n}\big) \Big] d\tau \Big| \\
\leq & \Big| g\big(t, (x(\beta_i(t)))_{i=1}^{s}\big) - g\big(t, (y(\beta_i(t)))_{i=1}^{s}\big) \Big| + \Big| f\big(t, (x(\alpha_j(t)))_{j=1}^{m}\big) - f\big(t, (y(\alpha_j(t)))_{j=1}^{m}\big) \Big| \times \\
& \int_0^t \ln |t - \tau| \Big| u\big(t, \tau, (y(\gamma_k(t)))_{k=1}^{n}\big) d\tau \Big| \\
& + \Big| f\big(t, (x(\alpha_j(t)))_{j=1}^{m}\big) \Big| \int_0^t \ln |t - \tau| \Big| u\big(t, \tau, (x(\gamma_k(t)))_{k=1}^{n}\big) - u\big(t, \tau, (y(\gamma_k(t)))_{k=1}^{n}\big) \Big| d\tau \Big| \\
\leq & \sum_{i=1}^{s} q_i \big| x(\beta_i(t)) - y(\beta_i(t)) \big| + \sum_{j=1}^{m} \lambda_j \big| x(\alpha_j(t)) - y(\alpha_j(t)) \big| \times M \; d \ln d + \sup(f) \; d \ln d \; \omega(x, \varepsilon) \\
\leq & \left(\sum_{i=1}^{s} q_i + M \; d \ln d \sum_{j=1}^{m} \lambda_j \right) \|x - y\| + \sup(f) \; d \ln d \; \omega(x, \varepsilon),
\end{aligned}$$

where

$$\sup(f) =\Big\{ \big|f(t, x_1, x_2, \ldots, x_m)\big| : t \in [0, d], x_i \in [-\varphi_0, \varphi_0]\ for 1 \le i \le n \Big\},$$

$$\omega(x, \varepsilon) = \sup\Big\{ \big|u(t, \tau, x_1, x_2, \ldots, x_n) - u(t, \tau, y_1, y_2, \ldots, y_n)\big|; t, \tau \in [0, d], x_i, y_i \in [-\varphi_0, \varphi_0]\ for 1 \le i \le n,\ \|x - y\| \le \varepsilon \Big\}.$$

The uniformly continuously $u = u(t, \tau, x_1, x_2, \ldots, x_n)$ on $[0, d]^2 \times [-\varphi_0, \varphi_0]^n$ implies that $\omega(x, \varepsilon) \to 0$ as $\varepsilon \to 0$. Then, the operator ∇ is continuous on $\bar{B}_{\varphi_0}$.

Step 2. We show that the operator ∇ fulfils the condensing map in view of measure μ. For arbitrary $\varepsilon > 0$ and $x \in \Psi \subset \mathbb{X}$ is bounded set and $t_1, t_2 \in [0, d]$ such that $|t_2 - t_1| \le \varepsilon$, we obtain

$$\begin{aligned}
&\Big|(\nabla x)(t_2) - (\nabla x)(t_1)\Big| \\
=&\Big|g\big(t_2, (x(\beta_i(t_2)))_{i=1}^s\big) + f\big(t_2, (x(\alpha_j(t_2)))_{j=1}^m\big) \int_0^{t_2} \ln|t_2 - \tau| u\big(t_2, \tau, (x(\gamma_k(\tau)))_{k=1}^n\big) d\tau \\
&- g\big(t_1, (x(\beta_i(t_1)))_{i=1}^s\big) - f\big(t_1, (x(\alpha_j(t_1)))_{j=1}^m\big) \int_0^{t_1} \ln|t_1 - \tau| u\big(t_1, \tau, (x(\gamma_k(\tau)))_{k=1}^n\big) d\tau\Big| \\
&\le \Big|g\big(t_2, (x(\beta_i(t_2)))_{i=1}^s\big) - g\big(t_2, (x(\beta_i(t_1)))_{i=1}^s\big)\Big| + \Big|g\big(t_2, (x(\beta_i(t_1)))_{i=1}^s\big) - g\big(t_1, x_{i=1}^s(\beta_i(t_1))\big)\Big| + \\
&\Big|f\big(t_2, (x(\alpha_j(t_2)))_{j=1}^m\big) \int_0^{t_2} \ln|t - \tau_2| u\big(t_2, \tau, (x(\gamma_k(\tau)))_{k=1}^n\big) d\tau \\
&- f\big(t_1, (x(\alpha_j(t_1)))_{j=1}^m\big) \int_0^{t_1} \ln|t_1 - \tau| u\big(t_1, \tau, (x(\gamma_k(\tau)))_{k=1}^n\big) d\tau\Big| \\
&\le \Big|g\big(t_2, (x(\beta_i(t_2)))_{i=1}^s\big) - g\big(t_2, (x(\beta_i(t_1)))_{i=1}^s\big)\Big| + \Big|g\big(t_2, (x(\beta_i(t_1)))_{i=1}^s\big) - g\big(t_1, (x(\beta_i(t_1)))_{i=1}^s\big)\Big| + \\
&+ \Big\{\Big|f\big(t_2, (x(\alpha_j(t_2)))_{j=1}^m\big) - f\big(t_2, (x(\alpha_j(t_1)))_{j=1}^m\big)\Big| + \Big|f\big(t_2, (x(\alpha_j(t_1)))_{j=1}^m\big) - f\big(t_1, (x(\alpha_j(t_1)))_{j=1}^m\big)\Big|\Big\} \times \\
&\int_0^{t_2} \ln|t_2 - \tau| u\big(t_2, \tau, (x(\gamma_k(\tau)))_{k=1}^n\big) d\tau \\
&+ \Big|f\big(t_1, (x(\alpha_j(t_1)))_{j=1}^m\big) \int_0^{t_2} \ln|t_2 - \tau| \Big\{u\big(t_2, \tau, (x(\gamma_k(\tau)))_{k=1}^n\big) - u\big(t_1, \tau, (x(\gamma_k(\tau)))_{k=1}^n\big)\Big\} d\tau\Big| \\
&+ \Big|f\big(t_1, (x(\alpha_j(t_1)))_{j=1}^m\big) \int_0^{t_2} u\big(t_1, \tau, (x(\gamma_k(\tau)))_{k=1}^n\big) \Big(\ln(t_2 - \tau) - \ln(t_1 - \tau)\Big) d\tau\Big| \\
&+ \Big|f\big(t_1, (x(\alpha_j(t_1)))_{j=1}^m\big) \int_{t_1}^{t_2} \ln|t_1 - \tau| u\big(t_1, \tau, (x(\gamma_k(\tau)))_{k=1}^n\big) d\tau\Big| \\
&\le \sum_{i=1}^s q_i \big|x(\beta_i(t_2)) - x(\beta_i(t_1))\big| + \omega_g(\varepsilon) + \Big(\sum_{j=1}^m \lambda_j \big|x(\alpha_j(t_2)) - x(\alpha_j(t_1))\big| + \omega_f(\varepsilon)\Big) \times d \ln d\ M \\
&+ \sup(f)|t_2 \ln t_2| \omega_u(\varepsilon) + \sup(f) M |t_2 - t_1| + \sup(f) t_2 |t_2 - t_1| + \sup(f) |t_2 - t_1| \ln|t_2 - t_1|,
\end{aligned}$$

where

$$\omega_f(\varepsilon) = \sup\Big\{ \big|f(t_2, x_1, x_2, \ldots, x_m) - f(t_1, x_1, x_2, \ldots, x_m)\big| : t_2, t_1 \in [0, d], x_i \in [-\varphi_0, \varphi_0],\ \text{for } 1 \le i \le m,\ |t_2 - t_1| \le \varepsilon \Big\},$$

$$\omega_g(\varepsilon) = \sup\Big\{ \big|g(t_2, x_1, x_2, \ldots, x_s) - g(t_1, x_1, x_2, \ldots, x_s)\big| : t_2, t_1 \in [0, d], x_j \in [-\varphi_0, \varphi_0],\ \text{for } 1 \le j \le m,\ |t_2 - t_1| \le \varepsilon \Big\},$$

$$\begin{aligned}
\omega_u(\varepsilon) = \sup\Big\{ &\big|u(t_2, \tau, x_1, x_2, \ldots, x_n) - f(t_1, \tau, x_1, x_2, \ldots, x_n)\big| : t_2, t_1 \in [0, d], x_k \in [-\varphi_0, \varphi_0], \\
&\text{for } 1 \le k \le m,\ |t_2 - t_1| \le \varepsilon \Big\}.
\end{aligned}$$

From the above inequalities, we can conclude

$$\omega(\nabla x,\varepsilon)\leq\sum_{i=1}^{s}q_i\omega(x,\omega(\beta,\varepsilon))+\omega_g(\varepsilon)+\Big(\sum_{j=1}^{m}\lambda_j\omega(x,\omega(\alpha,\varepsilon))+\omega_f(\varepsilon)\Big)Md\ln d$$
$$+\sup(f)|t_2\ln t_2\big|\omega_u(\varepsilon)+\varepsilon\sup(f)M+\varepsilon\sup(f)t_2+\varepsilon\sup(f)\ln\varepsilon.$$

By taking supremum over Ψ and $\varepsilon\to 0$, we obtain

$$\mu(\nabla\Psi)\leq\big(\sum_{i=1}^{s}q_i+M\sum_{j=1}^{m}\lambda_j\big)\mu(\Psi). \tag{3.2}$$

Assumption (W3) together (2.5) implies that ∇ is a condensing map.

Step 3. Suppose that $x\in\partial\bar{m_{\varphi_0}}$ and if $\nabla x=\zeta x$ then we get $\|\nabla x\|=\zeta\|x\|=\zeta\varphi_0$. Thus, (W3) implies that

$$\Big|(\nabla x)(t)\Big|=g\big(t,(x(\beta_i(t)))_{i=1}^{s}\big)+f\big(t,(x(\alpha_j(t)))_{j=1}^{m}\big)\int_0^t\ln|t-\tau|u\big(t,\tau,(x(\gamma_k(t)))_{k=1}^{n}\big)d\tau \tag{3.3}$$
$$\leq\varphi_0,$$

therefore, $\|\nabla x\|\leq\varphi_0$ implies that $\zeta\leq 1$. for all $t\in[0,d]$, hence $\|Gz\|\leq t_0$, so this show that $k\leq 1$. $\square$

4 Application via illustrative example

In this section, we present an example of functional integral equations to illustrate the usefulness of our result.

Example 4.1. Consider the following $\mathbb{NIE}s$ in $\mathbb{X}=C([0,1])$, as follows

$$x(t)=\tfrac{t^4}{3(1+t^4)}\Big(x(\sqrt{t})+x(t^2)\Big)+\tfrac{t^2+x(\sqrt{\sin t})}{12(1+t)}\int_0^1\ln|t-\tau|\big(\sqrt[3]{x(\tau)}+\ln(1+|x(\tau^2)|)\big)d\tau. \tag{4.1}$$

Here,

- $\beta_1(t)=\sqrt{t}$, $\beta_2(t)=t^2$, $\alpha_1(t)=\sqrt{\sin t}$, $\gamma_1(\tau)=\tau$, and $\gamma_2(\tau)=\tau^2$,
- $g(t,x_1,x_2)=\frac{t^4}{3(1+t^4)}x_1+\frac{t^4}{3(1+t^4)}x_2$, $f(t,x_1)=\frac{t^2+x_1}{12(1+t)}$,
- $u(t,\tau,x_1,x_2)=\sqrt[3]{x_1}+\ln(1+|x_2|)$.

It is evident that

$$\Big|g(t,x_1,x_2)-g(t,y_1,y_2)\Big|=\Big|\tfrac{t^4}{3(1+t^4)}x_1+\tfrac{t^4}{3(1+t^4)}x_2-\tfrac{t^4}{3(1+t^4)}y_1-\tfrac{t^4}{3(1+t^4)}y_2\Big|$$
$$\leq\tfrac{1}{3}\big|x_1-y_1\big|+\tfrac{1}{3}\big|x_2-y_2\big|$$

and

$$\Big|f(t,x_1)-f(t,y_1)\Big|=\Big|\tfrac{t^2+x_1}{12(1+t)}-\tfrac{t^2+y_1}{12(1+t)}\Big|\leq\tfrac{1}{12}\Big|x_1-y_1\Big|.$$

Based on the above inequalities, it can be deduced that $q_1=\frac{1}{3}$, $q_2=\frac{1}{3}$, $\lambda_1=\frac{1}{12}$. In order to verify assumption (W3) observe that the inequality appearing in this assumption has the form

$$\tfrac{2}{3}\varphi_0+\tfrac{\varphi_0+1}{12}\big(\sqrt[3]{\varphi_0}+\ln(1+|\varphi_0|)\big)\leq\varphi_0.$$

It is easy to verify that the number $\varphi_0\in[0.3103,4.01835]$. Hence, all conditions of Theorem3.1 are fulfill, then the equation (4.1) has at least one solution in $C[0,1]$.

5 Conclusion

The central aim in the theory of integral equations revolves around the existence and uniqueness of solutions. Therefore, several researchers have shared their findings and methodologies in this field. In alignment with this, the authors of this paper present a new approach using measures of noncompactness and the P-theorem for a nonlinear weakly singular integral equation. This method offers several advantages over similar techniques, including fewer conditions and no need to confirm the operator's mapping of a closed convex subset onto itself. The outcomes of this research are diverse and noteworthy, making it intriguing and deserving of further investigation in subsequent studies.

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